1. Bloch-Lectures on Algebraic Cycles

Remark. For a given divisor D, $l(D)$ is the dimension of space of functions f such that $(f)+D$ is effective..

Remark. For Riemann surface X, given a cycle $D=\sum n_i p_i$, when does $D=(f)$. First we observe that a necessary condition is $D=\sum n_i(p_i-p_0)$. A necessary and sufficient condition is that

$$\sum n_i \int_{p_0}^{p_i} \omega = \int_\gamma \omega$$

for some $\gamma \in H_1(X,\mathbb{Z})$ and all $\omega \in \Gamma(X,\Omega_X^1)$ where ω can be written as $f(z)dz$ locally. We may also consider $\int_\gamma \in \Gamma(X,\Omega_X^1)^*$. Abel-Jacobi theorem says that we have map

$$A_0(X) \to \Gamma(X,\Omega_X^1)^*/H_1(X,\mathbb{Z}) := J(X)$$

is an isomorphism where $A_0(X) := \frac{\text{divisiors of degree 0}}{\text{divisors of functions}}$. (I have no idea about the surjectivity of this map).

Definition 1.1 (Chow group)**.**

$$CH^r(X) = z^r(X)/z^r_{rat}(X)$$
$$A^r(X) = z^r_{alg}(X)/z^r_{rat}(X)$$

Proposition 1.2. *$CH_0(X)/A_0(X) \cong \mathbb{Z}$ given by the degree map. $CH^1(X) \cong Pic(X)$ and if X has a rational point then $A^1(X) \cong$ k-points of the Picard variety of X.*

We will consider $A_0(X)$ of algebraic surface X with $P_g \neq 0$. Let X be a projective variety of dimension 2 over $\mathbb{C}$.

Definition 1.3. $q = \dim \Gamma(X,\Omega_X^1)$ and $P_g = \dim \Gamma(X,\Omega_X^2)$. Albanese of X, $\mathrm{Alb}(X) := \Gamma(X,\Omega_X^1)^*/H_1(X,\mathbb{Z})$ which is a complex torus. Fixing a point p_0, we have a map $\phi : X \to \mathrm{Alb}(X)$ given by $\phi(p) = \int_{p_0}^p$

Proposition 1.4. *With above hypothesis, we have*

1. *$\phi_n : Sym^n(X) \to Alb(X)$ is surjective when $n \gg 0$.*
2. *Albanese variety satisfies the universal property.*
3. *$z_0(X) \to Alb(X)$ factors through $CH_0(X) \to Alb(X)$. The induced map $A_0(X) \to Alb(X)$ is surjective and independent of choice of p_0.*

Lemma 1.5. *Let Y be a smooth projective variety, and then $T(Y) = \ker(A_0(Y) \to Alb(Y))$ is a divisible group.*

Example. Let E, F be elliptic curves. Fix a point $\eta \in E$ of order 2. Let $X = (F \times E)/\{1,\sigma\}$ where σ is a involution given by $\sigma(e,f) = (-f, e+\eta)$. Let $E' = E/\{1,\eta\}$. We have a natural projection $\rho : X \to E'$ with fiber F. We have $\Gamma(X,\Omega_X^1) = \Gamma(F \times E, \Omega^1_{F\times E})^{\{1,\sigma\}} \cong [\Gamma(F,\Omega^1) \oplus \Gamma(E,\Omega^1)]^{\{1,\sigma\}} = \mathbb{C}$. Therefore $\mathrm{Alb}(X)$ is a Riemann surface. We conclude that $\mathrm{Alb}(X) \cong E'$. Now we want to show $A_0(X) \cong E'$ and $T(X) = 0$. Note that $4T(X) = 0$ and $T(X)$ is divisible group and thus $T(X) = 0$. This surface is called hyperelliptic with $P_g = 0$.

Theorem 1.6 (Mumford, Bloch)**.** *When $P_g > 0$, $T(X)$ is enormous and $A_0(X)$ is not finite dimensional. We say $A_0(X)$ is finite dimensional if there exists a complete smooth curve C mapping to X such that $J(C) \to A_0(X)$ is surjective.*

Definition 1.7. We define Neron-Severi group of projective variety X by $\mathrm{NS}(S) = \mathrm{im}(c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}2\pi i))$. An alternative definition is $\mathrm{NS}(X) = CH^1(X)/A^1(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$

Proposition 1.8. *X is a smooth variety over k and Y is a variety over k. Then, writing $K = k(Y)$,*

$$CH^n(X_K) \cong \varinjlim_{U \subset Y \, open} CH^n(X \times_k U).$$

Let X be a smooth projective surface over an algebraically closed field k. Let $A_0(X) \subset CH_0(X)$ be the subgroup of cycles of degree 0. We say $A_0(X)$ is finite dimensinal if there exists a complete smooth curve C mapping to X such that the map $J(C) \to A_0(X)$ is surjective.

Lemma 1.9. *Let $k \subset K \subset K'$ be extensions of fields. Then the kernel of $CH^2(X_K) \to CH^2(X_{K'})$ is torsion.*

Proof. Need clarification. □

Proposition 1.10. *Let $\Omega \supset k$ be a universal domain. Assume $A_0(X_\Omega)$ is finite dimensional. Then there exist one-dimensional subschemes $C', C'' \subset X$ and a 2-cycle Gamma supported on $C' \times X) \cup (X \times C'')$ such that some non-zero multiple of the diagonal Δ on $X \times_k X$ is rationally equivalent to Γ. (Ω is the algebraic closure of $k(x_1, \dots, x_n, \dots)$.)*

Proof. □

2. Abelian Varieties

Remark. Give n a curve C over field k, $\mathrm{Pic}^0(C)$ is one to one corresponding to k-rational points of $J(C)$.

Lemma 2.1 (Rigidity Lemma)**.** *Let X, Y and Z be algebraic variety and suppose that X is complete. If $f : X \times Y \to Z$ is a morphism such that for some $y \in Y(k)$, $X \times y$ is mapped to a point $z \in Z(k)$ then f is completely determined by Y.*

Proof. An important ingredient of the proof is realizing that $X \to V$ is constant if X is complete and V is affine. □

Corollary 2.2. *When we fix identity $\mathrm{e} \in X(k)$, there is at most one structure of an abelian variety on X. All abelian varieties are commutative.*

Now we want to show that Abelian varieties are projective.

Theorem 2.3. *Let X and Y be varieties. Suppose X is complete. Let L and M be two line bundles on $X \times Y$. If for all closed points $y \in Y$ we have $L_y \cong M_y$ there exists a line bundle N on Y such that $L \cong M \otimes p^*N$ where p is the projection $X \times Y \to Y$.*

Proof. $L_y \otimes M_y^{-1}$ is trivial and X_y is complete, so we have $H^0(X_y, L_y \otimes M_y^{-1}) \cong k(y)$. By Chapter III Theorem 12.11 on Hartshorne, $p_*(L \otimes M^{-1})$ is a line bundle. We can show that $p^*p_*(L \otimes M^{-1}) \cong L \otimes M^{-1}$. □

Lemma 2.4. *Let X, Y be varieties, with X complete. For a line bundle L on $X \times Y$, the set $\{y \in Y | L_y$ is trivial$\}$ is closed in Y.*

Proof. We have

$$\{y \in Y | L_y \text{ is trivial}\} = \{y \in Y | h^0(L_y) > 0\} \cap \{y \in Y | h^0(L_y^{-1}) > 0\}$$

since X is complete. By semi-continuous property, we know the two sets on right hand side are closed in Y. □

Theorem 2.5 (Important!). *Let X and Y be complete varieties and let Z be a connected, locally noetherian scheme. Let $x \in X(k), y \in Y(k)$ and let z be a point of Z. If L is a line bundle on $X \times Y \times Z$ whose restriction to $\{x\} \times Y \times Z$, to $X \times \{y\} \times Z$ and to $X \times Y \times \{z\}$ is trivial then L is trivial.*

Several corollaries follow from this theorem. We write $I = \{i_1, \ldots, i_r\} \subset [n]$ and we define

$$p_I : X^n \to X$$

sending $(x_1, x_2, \ldots, x_n) \mapsto x_{i_1} + \cdots + x_{i_r}$.

Corollary 2.6 (Theorem of Cube). $\Theta := \bigotimes p_I^* L^{\otimes (-1)^{1+\#I}}$ *on X^3 is trivial.*

Corollary 2.7 (Theorem of the Square). *Let X be an abelian variety and let L be a line bundle on X. Then for all $x, y \in X(k)$,*

$$t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L$$

More generally, let T be a k-scheme and write L_T for the pull-back of L to X_T. Then

$$t_{x+y}^* L_T \otimes L_T \cong t_x^* L_T \otimes t_y^* L_T \otimes p_T^*((x+y)^L \otimes x^* L^{-1} \otimes y^* L^{-1})$$

for all $x, y \in X(T)$.

Corollary 2.8. *We get a homomorphism $\phi_L : X(k) \to Pic(X)$ given by $x \mapsto [t_x^* L \otimes L^{-1}]$.*

Proof. This follows from previous corollary. □

We define $[n] : X \to X$ by multiplication by n. We get the following corollary,

Corollary 2.9.

$$n^* L \cong L^{n(n+1)/2} \otimes (-1)^* L^{n(n-1)/2}$$

Proof. Applying Theorem of the square, we get $(n+1)^* L \cong n^* L^* \otimes (n-1)^* L \otimes L \otimes (-1)^* L^{-1}$ and apply induction. □

Remark. We note that for any line bundle $M = (M \otimes (-1)^* M) \otimes (M \otimes (-1)^* M^{-1}) = L_+ \otimes L_-$ and L_+ is symmetric and L_- is antisymmetric, i.e. $(-1)^* L_+ \cong L_+, (-1)^* L_- \cong L_-^{-1}$. By previous corollary, we get $n^* L_+ \cong L_+^{n^2}$ and $n^* L_- \cong L_-^n$.

Definition 2.10 (Mumford line bundle). Let L be a line bundle on an ableian variety X. On $X \times X$ we define $\Lambda(L)$ by

$$\Lambda(L) := m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}.$$

The restriction of $\Lambda(L)$ to $\{x\} \times X$ and $X \times \{x\}$ are both $t_x^* L \otimes L^{-1}$. In particular, $\Lambda(L)$ is trivial on $\{0\} \times X$ and on $X \times \{0\}$. We define $K(L) \subset X$ as the maximal closed subscheme such that $\Lambda|_{X \times K(L)}$ is trivial over $K(L)$, i.e. $\Lambda|_{X \times K(L)} \cong p^* M$ for some line bundle on $K(L)$.

Lemma 2.11. *We have $M = \mathcal{O}_{K(L)}$ and $\Lambda|_{X \times K(L)} \cong \mathcal{O}_{X \times K(L)}$.*

Proposition 2.12. *The scheme $K(L)$ is a subgroup scheme of X.*

Lemma 2.13. *If L is ample then $K(L)$ is finite group scheme.*

Proof. $K(L)^0_{red}$ has a trivial ample line bundle. Then $\dim(Y) = 0$. Thus $K(L)$ is finite. □

Proposition 2.14 (Very important!)**.** *X is an abelian variety over algebraically closed field k. Let $f : X \to Y$ be a morphism of k-varieties. For $x \in X$, let C_x denote the connected component of the fiber over $f(x)$ such that $x \in C_x$, and write F_x for the reduced scheme underlying C_x. Then F_0 is an abelian subvariety of X and $F_x = t_x(F_0) = x + F_0$ for all $x \in X(k)$.*

Now I plan to talk about the Picard schemes and dual varieties. I start with an analytic approach. I suppose that X is a smooth projective variety over $\mathbb{C}$. We have the first Chern map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. Note that $\mathrm{im}(c_1) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C}) := NS(X)$. Now we consider the short exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 1$$

which induces an exact of cohomologies

$$0 \to H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

We call $\mathrm{Pic}^0(X) := H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$ which picks out the topoligically trivial holomorphic line bundles on X. As a complex manifold, we have $\dim(\mathrm{Pic}^0(X)) = \frac{1}{2}b_1$ and $\dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega_X)$.
Now we define the relative Picard functor.

Definition 2.15. We define contravariant **Picard functor** as $\underline{\mathrm{Pic}}_{X/S} : \mathbf{Sch}_{/S} \to \mathbf{Ab}$ by $\underline{\mathrm{Pic}}_X(Y) = \{\text{line bundles on } X \times Y \text{ trivialized on } * \times Y\}$

Theorem 2.16 (Grothendieck)**.** *$\underline{Pic}_X$ is representable by the countable union of quasi-projective schemes.*

I also denote the scheme representing the Picard functor also as Pic_X. I want to show this is a group scheme which means I want to impose group structure on Pic_X. Denote $h_{\mathrm{Pic}_X} := \hom(-, \mathrm{Pic}_X)$. We have multiplication structure $m : h_{\mathrm{Pic}_X \times \mathrm{Pic}_X} = h_{\mathrm{Pic}_X} \times h_{\mathrm{Pic}_X} \to h_{\mathrm{Pic}_X}$ induced by the multiplication of $\underline{\mathrm{Pic}}_X$ since it is group valued. By Yoneda lemma, this multiplication structure is induced by a unique morphism $m : \mathrm{Pic}_X \times \mathrm{Pic}_X \to \mathrm{Pic}_X$. We denote Pic^0_X as the component of containing 0.

Definition 2.17. An alternative definition for **Picard functor** is $\underline{\mathrm{Pic}}_X$ is the sheafification of the functor $Y \mapsto \mathrm{Pic}(X \times_k Y)/\mathrm{Pic}(Y)$

Our goal is to understand the tangent space of Pic^0_X at $0 \in \mathrm{Pic}^0_X$.

Proposition 2.18. $T_0 Pic_X \cong H^1(X, \mathcal{O}_X)$

3. Kodaira Dimension

We denote ω_X as the canonical bundle of a projective variety.

Definition 3.1. The plurigenera of X is $P_m(X) = \dim_k H^0(X, \omega_X^{\otimes m})$.

Example. (1) $X = \mathbb{P}^n$, $\omega_X = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, and so $P_m(X) = 0$ for all $m \geq 0$.

(2) $X = C$, a smooth curve of genus g. We have $P_1(X) = g$. If $g = 1$ then $\omega_C \cong \mathcal{O}_c$ and in particular $P_m(C) = 1$ for all m. If $g \geq 2$, $\deg \omega_C^{\otimes m} = m(2g-2) > 2g-1$ for $m \geq 2$. We have $l(K - mK) = 0$ where K is the canoncial divisor. Thus, $P_m(C) = (2m-1)(g-1)$ for $m \geq 2$.

(3) $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree d. We have $\omega_X \cong \mathcal{O}_X(d-n-1)$. If $d \leq n$, then $P_m(X) = 0$ for all $m \geq 0$. If $d = n+1$, $\omega_X \cong \mathcal{O}_X$ and $P_m(X) = 1$ for all $m \geq 0$. If $d \geq n+2$, ω_X is very ample line bundle and we have $P_m(X) = \chi(X, \omega_X^{\otimes m}) = \frac{d(d-n-1)}{(n-1)!}m^{n-1} + O(m^{n-2})$. Note that in general $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n-1$.

(4) Let X be an abelian variety. $\omega_X \cong \mathcal{O}_X$ since T_X is trivial. Thus, $P_m(X) = 1$ for $m \geq 0$ and $H^i(X, \mathcal{O}_X) \cong \wedge^i H^1(X, \mathcal{O}_X) \neq 0$.

Definition 3.2 (Calabi-Yau's)**.** We will call a weak Calabi-Yau variety a smooth projective variety X with $\omega_X \cong \mathcal{O}_X$. if in addition

$$H^i(X, \mathcal{O}_X) = 0, \forall 0 < i < \dim X$$

we will say that X is Calabi-Yau.
A $K3$ surface is a Calabi-Yau variety X of dimension 2. In other words. $\omega_X = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Example. A hypersurface of degree $d = n+1$ in $\mathbb{P}^n$ is Calabi-Yau and an abelian variety is weak Calabi-Yau. A quartic surface in $\mathbb{P}^3$ is a $K3$ surface.

Let X be a smooth projective variety, and let L be a line bundle on X. For each $m \geq 0$ such that $h^0(X, L^{\otimes m}) \neq 0$, the linear system $|L^{\otimes m}|$ induces a rational map from X to a projective space, and more precisely a morphism

$$\phi_m : X - B_m \to \mathbb{P}^{N_m}, N_m = h^0(X, L^{\otimes m}) - 1,$$

where B_m is the base locus of $L^{\otimes m}$. We write $\phi_m(X)$ as the closure of the image of ϕ_m in $\mathbb{P}^{N_m}$.

Definition 3.3. The Iitaka dimension of L is $\kappa(X, L) = \max_{m \geq 1} \dim \phi_m(X)$ if $\phi_m(X) \neq \emptyset$ for some m. We set $\kappa(X, L) = -\infty$ otherwise. Thus, $\kappa(X, L) \in \{-\infty, 0, 1, \ldots, \dim X\}$. A line bundle is call big if $\kappa(X, L) = \dim X$. The Kodaira dimension of X is $\kappa(X) := \kappa(X, \omega_X)$. X is call general type if $\kappa(X) = \dim X$, i.e. if ω_X is big.

Proposition 3.4. *Let $\kappa = \kappa(X, L)$, then there exist constants $a, b \geq 0$ such that $a \cdot m^\kappa \leq h^0(X, L^{\otimes m}) \leq b \cdot m^\kappa$ for sufficiently large m with $h^0(X, L^{\otimes m}) \neq 0$.*

Example. $\kappa(\mathbb{P}^n) = -\infty$. Use above proposition, we can easily tell the Kodaira dimension of curves, hypersurfaces and Calabi-Yau varieties.

Proposition 3.5. *If X, Y are smooth varieties, then $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$.*

4. Murre-Lectures on Algebraic Cycles and Chow Groups

We assume our varieties are over $\mathbb{C}$. Let X be a smooth irreducible projective variety and we can consider X as a complex compact connected manifold. As a complex analytic manifold, we have Hodge decomposition, $H^{2p}(X, \mathbb{C}) = \bigoplus_{r+s=2p} H^{r,s}(X)$. We define $\mathrm{Hdg}^p(X) := H^{2p}(X, \mathbb{Z}) \cap j^{-1}(H^{p,p}(X))$ where $j : H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C})$ is the natural map. We want to construct a cycle map $\gamma_{\mathbb{Z}} : Z^p(X) \to \mathrm{Hdg}^p(X)$ which induces a cycle map $\gamma_{\mathbb{Z}} : CH^p(X) \to \mathrm{Hdg}^p(X)$

Proposition 4.1 (Construction of $\gamma_{\mathbb{Z}}$)**.** *Let $Z_q \subset X_d$ be a closed subvariety. $p = d - q$. We have the following long exact sequence*

$$\cdots \to H^{2p-1}(U;\mathbb{Z}) \to H^{2p}(X,U;\mathbb{Z}) \xrightarrow{\rho} H^{2p}(X;\mathbb{Z}) \to H^{2p}(U;\mathbb{Z}) \to \cdots$$

Note that we also have an isomorphism by Thom $T : H^{2p}(X,U;\mathbb{Z}) \xrightarrow{\cong} H^0(Z;\mathbb{Z}) = \mathbb{Z}$. Now we take $\gamma_{\mathbb{Z}}(Z) = \rho \circ T^{-1}(1)$.

We want to show that $\gamma_{\mathbb{Z}}(Z) \in j^{-1}(H^{p,p}(X))$. In fact, we can find a "de Rham representative" such that

$$\int_X j \circ \gamma_{\mathbb{Z}}(Z) \wedge \beta = \int_Z i^*\beta$$

where $i : Z \hookrightarrow X$. The last integration is nonzero if and only if β is of type $(q,q) = (d-p, d-p)$ since Z is a q-dimensional complex manifold. Thus, $j \circ \gamma_{\mathbb{Z}}(Z) \in H^{p,p}(X)$. To be more precise, we write $\gamma^p_{\mathbb{Z}} : Z^p(X) \to \mathrm{Hdg}^p(X)$ and we have the following famous theorem:

Theorem 4.2 (Lefschetz (1,1)-theorem)**.** *Let X be a smooth, irreducible, projective variety over $\mathbb{C}$. Then $\gamma^1_{\mathbb{Z}}$ is surjective, i.e. every Hodge class of type $(1,1)$ is algebraic.*

Proof. Using the GAGA theorems, we work everything in the analytic category. We have an exact sequence

$$H^1(X,\mathcal{O}^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \xrightarrow{\beta} H^2(X,\mathcal{O}) \cong H^{0,2}(X)$$

We claim that $c_1 = \gamma^1_{\mathbb{Z}}$ and $\beta = \pi^{0,2} \circ j$. Given $\gamma \in H^2(X,\mathbb{Z}) \cap j^{-1}(H^{1,1}(X)) = \mathrm{Hdg}^1(X)$, we have $\beta(\gamma) = 0$ and there exist $[D] \in \mathrm{Div}(X)$ such that $\gamma^1_{\mathbb{Z}}([D]) = \gamma$. $\square$

Now we start discussing intermediate Jacobian. Let X be a smooth, irreducible, porjective variety over $\mathbb{C}$. We have a Hodge filtration

$$F^jH^i(X,\mathbb{C}) = \bigoplus_{r \geq j} H^{r,i-r} = H^{i,0} + H^{i-1,1} + \cdots H^{j,i-j}$$

Definition 4.3. The p-th intermediate Jacobian of X is

$$J^p(X) = H^{2p-1}(X,\mathbb{C})/(F^pH^{2p-1}(X,\mathbb{C}) + H^{2p-1}(X,\mathbb{Z}))$$

So writing $V = H^{p-1,p} + \cdots + H^{0,2p-1}$ we have that $J^p(X) = V/H^{2p-1}(X,\mathbb{Z})$.

Lemma 4.4. *$J^p(X)$ is a complex torus of dimension half the $(2p-1)$-th Betti number of X:*

$$\dim J^p(X) = \frac{1}{2} b_{2p-1}(X)$$

Proof. Since $H^{r,s}(X) = \overline{H^{s,r}(X)}$, we have Betti number is even and let $b_{2p-1} = 2m$. Now we need to show image of $H^{2p-1}(X,\mathbb{Z})$ is a lattice in V. Pick $\alpha_1, \ldots, \alpha_{2m}$ as the $\mathbb{R}$-basis for $H^{2p-1}(X,\mathbb{R})$, we want to show that this basis is linear independent in V. Suppose $\omega = \sum r_i\alpha_i \in F^pH^{2p-1}(X,\mathbb{C}) = \overline{V}$ with $r_i \in \mathbb{R}$. We have $\omega = \overline{w}$ and therefore $\omega \in V \cap \overline{V} = 0$. Therefore $\omega = 0$ and $r_i = 0$ for all i. $\square$

A complex torus $T = V/L$ to be an abelian variety it is necessary and sufficient that there exists a so-called Riemann form. This is a $\mathbb{R}$-bilinear form $E : V \times V \to \mathbb{R}$ satisfying

(1) $E(iv, iw) = E(v,w)$.
(2) $E(v,w) \in \mathbb{Z}$ whenever $v, w \in L$.
(3) $E(v, iw)$ symmetric and positive definite.

When $p = 1, p = d$, $J^p(X)$ is an abelian variety. When $p = 1$, $J^1(X) = H^1(X, \mathbb{C})/H^{1,0} + H^1(X, \mathbb{Z})$ which is the Picard variety. $p = d$, $J^d(X) = H^{2d-1}(X, \mathbb{C})/H^{d,d-1} + H^{2d-1}(X, \mathbb{Z})$ which is the Albanese variety.

Theorem 4.5. *There exists a homomorphism $AJ^p : Z^p_{hom} \to J^p(X)$ which factors through CH^p_{hom}. AJ is called the Abel-Jacobi map.*

Proof. First note that V is dual to $F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})$ and $J^p(X) = V/H^{2p-1}(X, \mathbb{Z})$. Therefore, an element $v \in V$ is a functional on $F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})$. Let $Z \in Z^p_{hom}(X)$, and there exists a topological $(2d-2p+1)$-chain Γ such that $Z = \partial\Gamma$. Now Γ is a functional on $F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})$ because $\omega \in F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C})$ is represented by a closed smooth differential form φ of degree $2d - 2p + 1$. The functional is given by $\omega \mapsto \int_\Gamma \varphi$. Thus the choice of Γ determines an element of V; and thus also of the intermediate jacobian $J^p(X) = V/H^{2p-1}(X, \mathbb{Z})$. Note that both choice of φ and Γ are well-defined. $\square$

Our next question is what is $\mathrm{im}(AJ^p(Z_{alg}))$. Given $T \in Z^p(Y_e \times X_d)$, we have $T : J^r(Y) \to J^{p+r-e}(X)$. So in particular if $Y = C$ and $T \in Z^p(C \times X)$, we get a homomorphism $T : J(C) \to J(X)$. The tangent space at the origin of $J^p(X)$ is V. We can see that $H^{0,1}(C)$ as the tangent space of $J(C)$ is mapped into subsspace of $H^{p,p-1}(X) \subset V$. Let $J^p(X)_{alg} \subset J^p(X)$ be the largest subtorus of $J^p(X)$ for which the tangent space is contained in $H^{p,p-1}(X)$. Note that $J^p(X)_{alg}$ is an abelian variety.

Lemma 4.6. $AJ^p(Z^p_{alg}(X)) \subset J^p(X)_{alg}$

Now we start discussing the difference of algebraic versus homological equivalence and this leads to Griffiths groups. We define that $Gr^i(X) := Z^i_{hom}(X)/Z^i_{alg}(X)$. We first note that $\mathrm{Gr}^1(X) = 0$ and $\mathrm{Gr}^d(X_d) = 0$. Griffiths showed that

Theorem 4.7 (Griffiths 1969)**.** *There exist smooth irreducible, projective varieties of dimension 3 such that $Gr(X) \otimes \mathbb{Q} \neq 0$.*

Theorem 4.8. *Let $V_{d+1} \subset \mathbb{P}^N$ be a smooth, irreducible variety and $W = V \cap H$ a smooth hyperplane section. Then*

$$H^j(V, \mathbb{Z}) \to H^j(W, \mathbb{Z})$$

is an isomorphism for $j < d = \dim W$ and injective for $j = d$. This is also true for homology map induced by inclusion. i.e. the induced map is isomorphic for $j < d$ and surjective for $j = d$.

Special case: Let $V = \mathbb{P}^{d+1}$ and $W \subset \mathbb{P}^{d+1}$ hypersurface. We get $H^j(W, \mathbb{Z}) = 0$ when $j < d$

Theorem 4.9 (Hard Lefschetz theorem)**.** *Let $V_{d+1} \subset \mathbb{P}^N$ and $W = V \cap H$ be a smooth hyperplane section. Let $h = \gamma_{\mathbb{Z},V}(W) \in H^2(V, \mathbb{Z})$. Then there is a Lefschetz operator*

$$L_V : H^j(V, \mathbb{Z}) \to H^{j+2}(V, \mathbb{Z})$$

$$\alpha \mapsto h \cup \alpha$$

By repeating we get ($n = d + 1 = \dim V$)

$$L^r : H^{n-r}(V, \mathbb{Z}) \to H^{n+r}(V, \mathbb{Z}), (0 \leq r \leq n).$$

L^r induces an isomorphism for all $r \leq n$ in coefficient $\mathbb{Q}$. i.e.

$$L^r : H^{n-r}(V, \mathbb{Q}) \xrightarrow{\cong} H^{n+r}(V, \mathbb{Q})$$

Definition 4.10 (Lefschetz pencil)**.** Let $V_n \subset \mathbb{P}^n$ be a smooth, irreducible. Let $H_\lambda = \lambda_0 H_0 + \lambda_1 H_1$ where $\lambda = [\lambda_0 : \lambda_1] \in \mathbb{P}^1$ and H_0, H_1 are two hpyerplanes. We get a pencil of hyperplane sections $\{W_\lambda = V \cap H_\lambda\}$ on V. Lefschetz pencil satisfies two following properties:

(1) There is a finite set S of points $t \in \mathbb{P}^1$ such that W_t is smooth outside S. Put $U := \mathbb{P}^1 - S$.
(2) For $s \in S$, W_s has only one singular point x and it is an "ordinary double point".

For $t \in U$, we have the Lefschetz theorem

$$\iota_t^* : H^j(V, \mathbb{Z}) \to H^j(W_t, \mathbb{Z})$$

which is an isomorphism for $j < \dim W_t$ and injective for $j = \dim W_t$. Now we can talk about the monodromy of Lefschetz pencils. Fix $t_0 \in U$ and write $W = W_{t_0}$. Then $\pi_1(U)$ operates on $H^j(W, \mathbb{Q})$, but due to the Lefschetz theorem it acts trivially if $j \neq d = \dim W$. Consider the action

$$\rho : \pi_1(U) \to \mathrm{Aut}(H^d(W_d, \mathbb{Q})).$$

$\Gamma = \mathrm{im}(\rho)$ is called the monodromy group.

Let $\iota : W \hookrightarrow V$ and $\iota_* : H^j(W, \mathbb{Q}) \to H^{j+2}(W, \mathbb{Q})$. By Lefschetz theorem $H^j(W, \mathbb{Q})_{van} := \ker \iota_* = 0$ when $j \neq d$. And we note that $L_W = \iota^* \circ \iota_*$.

5. Voisin-The Griffiths Group of a General Calabi-Yau Threefold is not Finitely Generated

This section is the collection of materials that I learned when I read Voisin's paper. I will follow the notation in this paper which is different from the previous section.

As stated in previous section, Abel-Jacobi map induces a morphism

$$\Phi_X^k : \mathrm{Gr}^k(X) \to J^{2k-1}(X)_{tr}$$

where $J^{2k-1}(X)_{tr}$ is the quotient of $J^{2k-1}(X)$ by its maximal subtorus having its tangent space contained in $H^{k-1,k}(X)$.

We will focus on when $n = 3, k = 2$. Let X be a projective Calabi-Yau threefold and it is well know that $\dim H^1(T_X) = \dim H^{1,2}(X)$.

Theorem 5.1. *let X be a Calabi-Yau threefold. If $\dim H^1(T_X) \neq 0$, the general deformation X_t of X satisfies that the Abel-Jacobi map*

$$\Phi_{X_t} : Z^2(X_t) \to J^2(X_t)$$

of X_t is nontrivial, even module torsion.

We are going to prove the following theorem

Theorem 5.2. *Let X be a Calabi-Yau threefold. If $h^1(T_X) \neq 0$, the general deformation X_t of X has the property that the Abel-Jacobi map*

$$\Phi_{X_t} : Z^2(X_t) \to J(X_t)$$

is such that $im\Phi_{X_t} \otimes \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space. In particular, $Gr(X_t) \otimes \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space.

For the discussion of basic deformation theory and Koddaira-Spencer map, please refer to next subsection.

5.1. **Kodaira-Spencer map.** We define a deformation functor

$$F : \textbf{Categroy of local Artin } \mathbb{C}\textbf{-algebras} \to \textbf{Sets}$$

where local Artin $\mathbb{C}$-algebras are rings of the form $\mathbb{C}[x_1,\ldots,x_n]/I$ with one maximal ideal and $\dim_{\mathbb{C}} \mathbb{C}[x_1,\ldots,x_n]/I$ is finite. For example $\mathbb{C}[x]/(x^4), \mathbb{C}[x,y]/(x^3,y^4)$. GIven a smooth projective variety W_0 over $\mathbb{C}$, deformation problem is to solve the following commutative diagram

$$\begin{tikzcd}
W_0 \arrow[r] \arrow[d] & W_n \arrow[r] \arrow[d] & W \arrow[d] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathbb{C}[x]/x^n) \arrow[r] & \mathrm{Spec}(\mathbb{C}[x])
\end{tikzcd}$$

with vertical maps being flat. For each Artin $\mathbb{C}$-algebra $\mathcal{A} \leftarrow \mathbb{C}$, we consider the isomorphism classes of commutative diagrams

$$\begin{tikzcd}
W_0 \arrow[r] \arrow[d, "F_0"] & W \arrow[d, "F_A"] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r] & \mathrm{Spec}(\mathcal{A})
\end{tikzcd}$$

Tangent space to deformation functor= First order deformations of W_0 is given by assigning $\mathcal{A} = \mathbb{C}[\epsilon]/\epsilon^2$ and this is isomorphic to $H^1(W_0, T_{W_0})$. By Serre's duality, we have $H^i(W_0, \mathscr{F}) = H^{\dim(W_0)-i}(W_0, \mathscr{F}^* \otimes \omega)^*$. In the case when W_0 is Calabi-Yau which means canonical bundle is trivial, we have $\dim H^1(W_0, T_{W_0}) = \dim H^2(W_0, \Omega^1) = h^{1,2}$. Now we consider the following commutative diagram of deformation

$$\begin{tikzcd}
W_0 \arrow[r] \arrow[d] & W \arrow[d] \\
0 \arrow[r] & B
\end{tikzcd}$$

where 0 is the base point of the deformation. We have short exact sequences on W_0

$$0 \to \mathscr{I}_{W_0} \to \mathcal{O}_W \to \mathcal{O}_{W_0} \to 0$$

and

$$0 \to \mathscr{I}_{W_0}/\mathscr{I}_{W_0}^2 \to \Omega_W|_{W_0} \to \Omega_{W_0} \to 0$$

Note that $\mathscr{I}_{W_0}/\mathscr{I}_{W_0}^2 = \mathcal{O}_{W_0} \otimes T_0^*B$ since conormal sheaf of fiber is trivial. We now dualize the above sequence and we get

$$0 \to T_{W_0} \to T_W|_{W_0} \to \mathcal{O}_{W_0} \otimes T_0 B \to 0$$

and it induces a map $KS : H^0(W_0, \mathcal{O}_{W_0} \otimes T_0 B) \to H^1(W_0, T_{W_0})$. We have $T_0 B = \mathscr{N}_{0/B} \cong H^0(W_0, \mathscr{N}_{W_0/W}) = H^0(W_0, \mathcal{O}_{W_0} \otimes T_0 B)$ and denote $KS : T_0 B \to H^1(W_0, T_{W_0})$.

6. Some Random Math

This section is contributed to various topics in mathematics.

6.1. **Hilbert scheme.** Both Grassmannian and the Fano scheme are special cases of Hilbert scheme. First we introduce universal property of Grassmannian. We define $G = G(k+1, V) = \mathbb{G}(k, \mathbb{P}V)$ and $\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}$ as the universal family of k-planes in $\mathbb{P}V$ in the following sense: For any scheme B, we will say that a subscheme $\mathcal{L} \subset B \times \mathbb{P}V$ is a flat family of k-planes in $\mathbb{P}V$ if the restriction $\pi : \mathcal{L} \to B$ of the projection $\pi_1 : B \times \mathbb{P}V \to B$ is flat, and the fibers over closed points of B are linearly embedded k-planes in $\mathbb{P}V$. We then have:

Proposition 6.1. *If $\pi : \mathcal{L} \subset B \times \mathbb{P}V \to B$ is a flat family of k-planes in $\mathbb{P}V$, then there is a unique map $\alpha : B \to G$ such that $\mathcal{L}$ is equal to the pullback of the family Φ via α:*

$$\begin{tikzcd}
\mathcal{L} \arrow[r, equal] \arrow[dr, "\pi"'] & B \times_G \Phi \arrow[r] \arrow[d] & \Phi \arrow[d, "\pi"] \\
& B \arrow[r, "\alpha"] & G
\end{tikzcd}$$

6.2. **Elliptic surfaces.** Let $k = \overline{k}$ and C a smooth projective curve over k.

Definition 6.2. An elliptic surface S over C is a smooth porjective surface S with an elliptic fibration over C, i.e. a surjective morphism

$$f : S \to C,$$

such that

(1) generic fibers are elliptic curves;
(2) no fiber contains an exceptional curve of the first kind.

The second condition ensures that S is a smooth minimal model.

Example.