1. Bloch-Lectures on Algebraic Cycles

**Remark.** For a given divisor $D$, $l(D)$ is the dimension of space of functions $f$ such that $(f) + D$ is effective.

**Remark.** For Riemann surface $X$, given a cycle $D = \sum n_ip_i$, when does $D = (f)$. First we observe that a necessary condition is $D = \sum n_i(p_i - p_0)$. A necessary and sufficient condition is that

$$\sum n_i \int_{p_0}^{p_i} \omega = \int_{\gamma} \omega$$

for some $\gamma \in H_1(X, \mathbb{Z})$ and all $\omega \in \Gamma(X, \Omega_X^1)$ where $\omega$ can be written as $f(z)dz$ locally. We may also consider $\int_{\gamma} \in \Gamma(X, \Omega_X^1)^*$. Abel-Jacobi theorem says that we have map

$$A_0(X) \to \Gamma(X, \Omega_X^1)^*/H_1(X, \mathbb{Z}) := J(X)$$

is an isomorphism where $A_0(X) := \frac{\text{divisors of degree 0}}{\text{divisors of functions}}$ (I have no idea about the surjectivity of this map).

**Definition 1.1** (Chow group).

$$CH^r(X) = z^r(X)/z^r_{\text{rat}}(X)$$

$$A^r(X) = z^r_{\text{alg}}(X)/z^r_{\text{rat}}(X)$$

**Proposition 1.2.** $CH_0(X)/A_0(X) \cong \mathbb{Z}$ given by the degree map. $CH^1(X) \cong \text{Pic}(X)$ and if $X$ has a rational point then $A^1(X) \cong k$-points of the Picard variety of $X$.

We will consider $A_0(X)$ of algebraic surface $X$ with $P_g \neq 0$. Let $X$ be a projective variety of dimension 2 over $\mathbb{C}$.

**Definition 1.3.** $q = \dim \Gamma(X, \Omega_X^1)$ and $P_g = \dim \Gamma(X, \Omega_X^2)$. Albanese of $X$, $\text{Alb}(X) := \Gamma(X, \Omega_X^1)^*/H_1(X, \mathbb{Z})$ which is a complex torus. Fixing a point $p_0$, we have a map $\phi : X \to \text{Alb}(X)$ given by $\phi(p) = \int_{p_0}^p$

**Proposition 1.4.** With above hypothesis, we have

1. $\phi : \text{Sym}^n(X) \to \text{Alb}(X)$ is surjective when $n \gg 0$.
2. Albanese variety satisfies the universal property.
3. $\sigma_0(X) \to \text{Alb}(X)$ factors through $CH_0(X) \to \text{Alb}(X)$. The induced map $A_0(X) \to \text{Alb}(X)$ is surjective and independent of choice of $p_0$.

**Lemma 1.5.** Let $Y$ be a smooth projective variety, and then $T(Y) = \ker(A_0(Y) \to \text{Alb}(Y))$ is a divisible group.

**Example.** Let $E$, $F$ be elliptic curves. Fix a point $\eta \in E$ of order 2. Let $X = (F \times E)/\{1, \sigma\}$ where $\sigma$ is an involution given by $\sigma(e, f) = (-f, e + \eta)$. Let $E' = E/\{1, \eta\}$. We have a natural projection $\rho : X \to E'$ with fiber $F$. We have $\Gamma(X, \Omega_X^1) = \Gamma(F \times E, \Omega_{F \times E}^1)^{\{1, \sigma\}} \cong [\Gamma(F, \Omega_F^1) \oplus \Gamma(E, \Omega_E^1)]^{\{1, \sigma\}} = \mathbb{C}$. Therefore $\text{Alb}(X)$ is a Riemann surface. We conclude that $\text{Alb}(X) \cong E'$. Now we want to show $A_0(X) \cong E'$ and $T(X) = 0$. Note that $4T(X) = 0$ and $T(X)$ is divisible group and thus $T(X) = 0$. This surface is called hyperelliptic with $P_g = 0$.

**Theorem 1.6** (Mumford, Bloch). When $P_g > 0$, $T(X)$ is enormous and $A_0(X)$ is not finite dimensional. We say $A_0(X)$ is finite dimensional if there exists a complete smooth curve $C$ mapping to $X$ such that $J(C) \to A_0(X)$ is surjective.
Definition 1.7. We define Neron-Severi group of projective variety $X$ by $\text{NS}(S) = \text{im}(c_1 : H^1(X, O_X^*) \to H^2(X, \mathbb{Z}2\pi i))$. An alternative definition is $\text{NS}(X) = CH^1(X)/A^1(X) = \text{Pic}(X)/\text{Pic}^0(X)$.

Proposition 1.8. $X$ is a smooth variety over $k$ and $Y$ is a variety over $k$. Then, writing $K = k(Y)$,

$$CH^n(X_K) \cong \lim_{U \subseteq Y \text{ open}} CH^n(X \times_k U).$$

Let $X$ be a smooth projective surface over an algebraically closed field $k$. Let $A_0(X) \subset CH_0(X)$ be the subgroup of cycles of degree 0. We say $A_0(X)$ is finite dimensional if there exists a complete smooth curve $C$ mapping to $X$ such that the map $J(C) \to A_0(X)$ is surjective.

Lemma 1.9. Let $k \subset K \subset K'$ be extensions of fields. Then the kernel of $CH^2(X_K) \to CH^2(X_{K'})$ is torsion.

Proof. Need clarification. \hfill $\square$

Proposition 1.10. Let $\Omega \supset k$ be a universal domain. Assume $A_0(X_\Omega)$ is finite dimensional. Then there exist one-dimensional subschemes $C', C'' \subset X$ and a 2-cycle Gamma supported on $C' \times X) \cup (X \times C'')$ such that some non-zero multiple of the diagonal $\Delta$ on $X \times_k X$ is rationally equivalent to $\Gamma$. ($\Omega$ is the algebraic closure of $k(x_1, \ldots, x_n, \ldots)$.)

Proof. \hfill $\square$

2. Abelian Varieties

Remark. Give a curve $C$ over field $k$, Pic$^0(C)$ is one to one corresponding to $k$-rational points of $J(C)$.

Lemma 2.1 (Rigidity Lemma). Let $X, Y$ and $Z$ be algebraic variety and suppose that $X$ is complete. If $f : X \times Y \to Z$ is a morphism such that for some $y \in Y(k)$, $X \times y$ is mapped to a point $z \in Z(k)$ then $f$ is completely determined by $Y$.

Proof. An important ingredient of the proof is realizing that $X \to V$ is constant if $X$ is complete and $V$ is affine. \hfill $\square$

Corollary 2.2. When we fix identity $e \in X(k)$, there is at most one structure of an abelian variety on $X$. All abelian varieties are commutative.

Now we want to show that Abelian varieties are projective.

Theorem 2.3. Let $X$ and $Y$ be varieties. Suppose $X$ is complete. Let $L$ and $M$ be two line bundles on $X \times Y$. If for all closed points $y \in Y$ we have $L_y \cong M_y$ there exists a line bundle $N$ on $Y$ such that $L \cong M \otimes p^* N$ where $p$ is the projection $X \times Y \to Y$.

Proof. $L_y \otimes M_y^{-1}$ is trivial and $X_y$ is complete, so we have $H^0(X_y, L_y \otimes M_y^{-1}) \cong k(y)$. By Chapter III Theorem 12.11 on Hartshorne, $p_*(L \otimes M^{-1})$ is a line bundle. We can show that $p^*p_*(L \otimes M^{-1}) \cong L \otimes M^{-1}$. \hfill $\square$

Lemma 2.4. Let $X, Y$ be varieties, with $X$ complete. For a line bundle $L$ on $X \times Y$, the set $\{y \in Y \mid L_y$ is trivial$\}$ is closed in $Y$. 


Proof. We have
\[
\{ y \in Y | L_y \text{ is trivial} \} = \{ y \in Y | h^0(L_y) > 0 \} \cap \{ y \in Y | h^0(L_y^{-1}) > 0 \}
\]
since \( X \) is complete. By semi-continuous property, we know the two sets on right hand side are closed in \( Y \).
\[
\square
\]

**Theorem 2.5** (Important!). Let \( X \) and \( Y \) be complete varieties and let \( Z \) be a connected, locally noetherian scheme. Let \( x \in X(k), y \in Y(k) \) and let \( z \) be a point of \( Z \). If \( L \) is a line bundle on \( X \times Y \times Z \) whose restriction to \( \{ x \} \times Y \times Z \), to \( X \times \{ y \} \times Z \) and to \( X \times Y \times \{ z \} \) is trivial then \( L \) is trivial.

Several corollaries follow from this theorem. We write \( I = \{ i_1, \ldots, i_r \} \subset [n] \) and we define
\[
p_I : X^n \to X
\]
sending \((x_1, x_2, \ldots, x_n) \mapsto x_{i_1} + \cdots + x_{i_r} \).

**Corollary 2.6** (Theorem of Cube). \( \Theta := \bigotimes p_i^* L \otimes (1)^{1+\# I} \) on \( X^3 \) is trivial.

**Corollary 2.7** (Theorem of the Square). Let \( X \) be an abelian variety and let \( L \) be a line bundle on \( X \). Then for all \( x, y \in X(k), \)
\[
t_{x+y}^* L \otimes L \cong t_x^* L \otimes t_y^* L
\]
More generally, let \( T \) be a \( k \)-scheme and write \( L_T \) for the pull-back of \( L \) to \( X_T \). Then
\[
t_{x+y}^* L_T \otimes L_T \cong t_x^* L_T \otimes t_y^* L_T \otimes p_1^*(x+y)^L \otimes x^* L^{-1} \otimes y^* L^{-1}
\]
for all \( x, y \in X(T) \).

**Corollary 2.8.** We get a homomorphism \( \phi_L : X(k) \to \text{Pic}(X) \) given by \( x \mapsto [t_x^* L \otimes L^{-1}] \).

**Proof.** This follows from previous corollary.
\[
\square
\]
We define \([n] : X \to X \) by multiplication by \( n \). We get the following corollary,

**Corollary 2.9.**
\[
n^* L \cong n^{(n+1)/2} \otimes (1)^* L^{n(n-1)/2}
\]

**Proof.** Applying Theorem of the square, we get \((n+1)^* L \cong n^* L \otimes (n-1)^* L \otimes L^{-1}\) and apply induction.
\[
\square
\]

**Remark.** We note that for any line bundle \( M = (M \otimes (1)^* M) \otimes (M \otimes (1)^* M^{-1}) = L_+ \otimes L_- \), and \( L_+ \) is symmetric and \( L_- \) is antisymmetric, i.e. \((1)^* L_+ \cong L_+^n, (1)^* L_- \cong L_-^{-1} \). By previous corollary, we get \( n^* L_+ \cong n^2 L_+ \) and \( n^* L_- \cong n^2 L_- \).

**Definition 2.10** (Mumford line bundle). Let \( L \) be a line bundle on an abelian variety \( X \). On \( X \times X \) we define \( \Lambda(L) \) by
\[
\Lambda(L) := m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}
\]
The restriction of \( \Lambda(L) \) to \( \{ x \} \times X \) and \( X \times \{ x \} \) are both \( t_x^* L \otimes L^{-1} \). In particular, \( \Lambda(L) \) is trivial on \( \{ 0 \} \times X \) and on \( X \times \{ 0 \} \). We define \( K(L) \subset X \) as the maximal closed subscheme such that \( \Lambda|_{X \times K(L)} \) is trivial over \( K(L) \), i.e. \( \Lambda|_{X \times K(L)} \cong p^* M \) for some line bundle on \( K(L) \).

**Lemma 2.11.** We have \( M = O_{K(L)} \) and \( \Lambda|_{X \times K(L)} \cong O_{X \times K(L)} \).

**Proposition 2.12.** The scheme \( K(L) \) is a subgroup scheme of \( X \).
Lemma 2.13. If $L$ is ample then $K(L)$ is finite group scheme.

Proof. $K(L)^{0}_{red}$ has a trivial ample line bundle. Then $\dim(Y) = 0$. Thus $K(L)$ is finite. □

Proposition 2.14 (Very important!). $X$ is an abelian variety over algebraically closed field $k$. Let $f : X \to Y$ be a morphism of $k$-varieties. For $x \in X$, let $C_x$ denote the connected component of the fiber over $f(x)$ such that $x \in C_x$, and write $F_x$ for the reduced scheme underlying $C_x$. Then $F_0$ is an abelian subvariety of $X$ and $F_x = t_x(F_0) = x + F_0$ for all $x \in X(k)$.

Now I plan to talk about the Picard schemes and dual varieties. I start with an analytic approach. I suppose that $X$ is a smooth projective variety over $\mathbb{C}$. We have the first Chern map $c_1 : \text{Pic}(X) \to H^2(X, \mathbb{Z})$. Note that $\text{im}(c_1) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C}) := NS(X)$. Now we consider the short exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\times 2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 1$$

which induces an exact of cohomologies

$$0 \to H^1(X, \mathcal{O})/H^1(X, \mathbb{Z}) \to \text{Pic}(X) \to NS(X) \to 0$$

We call $\text{Pic}^0(X) := H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$ which picks out the topologically trivial holomorphic line bundles on $X$. As a complex manifold, we have $\dim(\text{Pic}^0(X)) = \frac{1}{2} b_1$ and $\dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega_X)$.

Now we define the relative Picard functor.

Definition 2.15. We define contravariant Picard functor as $\underline{\text{Pic}}_X : \text{Sch}_{/S} \to \text{Ab}$ by

$$\underline{\text{Pic}}_X(Y) = \{\text{line bundles on } X \times Y \text{ trivialized on } * \times Y\}$$

Theorem 2.16 (Grothendieck). $\underline{\text{Pic}}_X$ is representable by the countable union of quasi-projective schemes.

I also denote the scheme representing the Picard functor also as $\text{Pic}_X$. I want to show this is a group scheme which means I want to impose group structure on $\text{Pic}_X$. Denote $h_{\text{Pic}_X} := \text{hom}(-, \text{Pic}_X)$. We have multiplication structure $m : h_{\text{Pic}_X \times \text{Pic}_X} = h_{\text{Pic}_X} \times h_{\text{Pic}_X} \to h_{\text{Pic}_X}$ induced by the multiplication of $\underline{\text{Pic}}_X$ since it is group valued. By Yoneda lemma, this multiplication structure is induced by a unique morphism $m : \text{Pic}_X \times \text{Pic}_X \to \text{Pic}_X$. We denote $\text{Pic}_X^0$ as the component of containing 0.

Definition 2.17. An alternative definition for Picard functor is $\underline{\text{Pic}}_X$ is the sheafification of the functor $Y \mapsto \text{Pic}(X \times_k Y)/\text{Pic}(Y)$

Our goal is to understand the tangent space of $\text{Pic}_X^0$ at $0 \in \text{Pic}_X^0$.

Proposition 2.18. $T_0\text{Pic}_X \cong H^1(X, \mathcal{O}_X)$

3. Kodaira Dimension

We denote $\omega_X$ as the canonical bundle of a projective variety.

Definition 3.1. The plurigenera of $X$ is $P_m(X) = \dim_k H^0(X, \omega_X^{\otimes m})$.

Example. (1) $X = \mathbb{P}^n$, $\omega_X = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, and so $P_m(X) = 0$ for all $m \geq 0$. 

(2) $X = C$, a smooth curve of genus $g$. We have $P_1(X) = g$. If $g = 1$ then $\omega_C \cong \mathcal{O}_C$ and in particular $P_m(C) = 1$ for all $m$. If $g \geq 2$, $\deg \omega_C^m = m(2g - 2) > 2g - 1$ for $m \geq 2$. We have $l(K - mK) = 0$ where $K$ is the canonical divisor. Thus, $P_m(C) = (2m - 1)(g - 1)$ for $m \geq 2$.

(3) $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree $d$. We have $\omega_X \cong \mathcal{O}_X(d - n - 1)$. If $d \leq n$, then $P_m(X) = 0$ for all $m \geq 0$. If $d = n + 1$, $\omega_X \cong \mathcal{O}_X$ and $P_m(X) = 1$ for all $m \geq 0$. If $d \geq n + 2$, $\omega_X$ is very ample line bundle and we have $P_m(X) = \chi(X, \omega_X^m) = \frac{d(d-n-1)}{(n-1)!} m^{n-1} + O(m^{n-2})$. Note that in general $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n - 1$.

(4) Let $X$ be an abelian variety. $\omega_X \cong \mathcal{O}_X$ since $T_X$ is trivial. Thus, $P_m(X) = 1$ for $m \geq 0$ and $H^1(X, \mathcal{O}_X) \cong \Lambda^i H^1(X, \mathcal{O}_X) \neq 0$.

**Definition 3.2** (Calabi-Yau’s). We will call a weak Calabi-Yau variety a smooth projective variety $X$ with $\omega_X \cong \mathcal{O}_X$, if in addition

$$H^i(X, \mathcal{O}_X) = 0, \forall 0 < i < \dim X$$

we will say that $X$ is Calabi-Yau.

A $K3$ surface is a Calabi-Yau variety $X$ of dimension 2. In other words, $\omega_X = \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

**Example.** A hypersurface of degree $d = n + 1$ in $\mathbb{P}^n$ is Calabi-Yau and an abelian variety is weak Calabi-Yau. A quartic surface in $\mathbb{P}^3$ is a $K3$ surface.

Let $X$ be a smooth projective variety, and let $L$ be a line bundle on $X$. For each $m \geq 0$ such that $h^0(X, L^\otimes m) \neq 0$, the linear system $|L^\otimes m|$ induces a rational map from $X$ to a projective space, and more precisely a morphism

$$\phi_m : X - B_m \to \mathbb{P}^{N_m}, N_m = h^0(X, L^\otimes m) - 1,$$

where $B_m$ is the base locus of $L^\otimes m$. We write $\phi_m(X)$ as the closure of the image of $\phi_m$ in $\mathbb{P}^{N_m}$.

**Definition 3.3.** The Iitaka dimension of $L$ is $\kappa(X, L) = \max_{m \geq 1} \dim \phi_m(X)$ if $\phi_m(X) \neq \emptyset$ for some $m$. We set $\kappa(X, L) = -\infty$ otherwise. Thus, $\kappa(X, L) \in \{-\infty, 0, 1, \ldots, \dim X\}$. A line bundle is call big if $\kappa(X, L) = \dim X$. The Kodaira dimension of $X$ is $\kappa(X) := \kappa(X, \omega_X)$. $X$ is call general type if $\kappa(X) = \dim X$, i.e. if $\omega_X$ is big.

**Proposition 3.4.** Let $\kappa = \kappa(X, L)$, then there exist constants $a, b \geq 0$ such that $a \cdot m^\kappa \leq h^0(X, L^\otimes m) \leq b \cdot m^\kappa$ for sufficiently large $m$ with $h^0(X, L^\otimes m) \neq 0$.

**Example.** $\kappa(\mathbb{P}^n) = -\infty$. Use above proposition, we can easily tell the Kodaira dimension of curves, hypersurfaces and Calabi-Yau varieties.

**Proposition 3.5.** If $X, Y$ are smooth varieties, then $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$.

4. MURRE-LECTURES ON ALGEBRAIC CYCLES AND CHOW GROUPS

We assume our varieties are over $\mathbb{C}$. Let $X$ be a smooth irreducible projective variety and we can consider $X$ as a complex compact connected manifold. As a complex analytic manifold, we have Hodge decomposition, $H^{2p}(X, \mathbb{C}) = \bigoplus_{r+s = 2p} H^{r,s}(X)$. We define $\text{Hdg}^p(X) := H^{2p}(X, \mathbb{Z}) \cap j^{-1}(H^{p,p}(X))$ where $j : H^{2p}(X, \mathbb{Z}) \to H^{2p}(X, \mathbb{C})$ is the natural map. We want to construct a cycle map $\gamma_Z : Z^p(X) \to \text{Hdg}^p(X)$ which induces a cycle map $\gamma_Z : CH^p(X) \to \text{Hdg}^p(X)$.
Proposition 4.1 (Construction of $\gamma_Z$). Let $Z \subset X$ be a closed subvariety, $p = d - q$. We have the following long exact sequence
\[
\cdots \rightarrow H^{2p-1}(U; \mathbb{Z}) \rightarrow H^{2p}(X; U; \mathbb{Z}) \xrightarrow{\omega} H^{2p}(X; \mathbb{Z}) \rightarrow H^{2p}(U; \mathbb{Z}) \rightarrow \cdots
\]
Note that we also have an isomorphism by Thom $T : H^{2p}(X; U; \mathbb{Z}) \xrightarrow{\cong} H^0(Z; \mathbb{Z}) = \mathbb{Z}$. Now we take $\gamma_Z(Z) = \rho \circ T^{-1}(1)$.

We want to show that $\gamma_Z(Z) \in j^{-1}(H^{p-p}(X))$. In fact, we can find a "de Rham representative" such that
\[
\int_X j \circ \gamma_Z(Z) \wedge \beta = \int_Z i^* \beta
\]
where $i : Z \hookrightarrow X$. The last integration is nonzero if and only if $\beta$ is of type $(q, q) = (d - p, d - p)$ since $Z$ is a $q$-dimensional complex manifold. Thus, $j \circ \gamma_Z(Z) \in H^{p-p}(X)$. To be more precise, we write $\gamma_Z^p : Z^p(X) \rightarrow \text{Hdg}^p(X)$ and we have the following famous theorem:

Theorem 4.2 (Lefschetz (1,1)-theorem). Let $X$ be a smooth, irreducible, projective variety over $\mathbb{C}$. Then $\gamma_Z^1$ is surjective, i.e. every Hodge class of type $(1, 1)$ is algebraic.

Proof. Using the GAGA theorems, we work everything in the analytic category. We have an exact sequence
\[
H^1(X, \mathcal{O}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{\beta} H^2(X, \mathcal{O}) \cong H^{0,2}(X)
\]
We claim that $c_1 = \gamma_Z^1$ and $\beta = \pi^{0,2} \circ j$. Given $\gamma \in H^2(X, \mathbb{Z}) \cap j^{-1}(H^{1,1}(X)) = \text{Hdg}^1(X)$, we have $\beta(\gamma) = 0$ and there exist $[D] \in \text{Div}(X)$ such that $\gamma_Z^1([D]) = \gamma$.

Now we start discussing intermediate Jacobian. Let $X$ be a smooth, irreducible, projective variety over $\mathbb{C}$. We have a Hodge filtration
\[
F^jH^i(X, \mathbb{C}) = \bigoplus_{r \geq j} H^{r,i-r} = H^{i,0} + H^{i-1,1} + \cdots H^{j,i-j}
\]
Definition 4.3. The $p$-th intermediate Jacobian of $X$ is
\[
J^p(X) = H^{2p-1}(X, \mathbb{C})/(F^pH^{2p-1}(X, \mathbb{C}) + H^{2p-1}(X, \mathbb{Z}))
\]
So writing $V = H^{p-1, p} + \cdots + H^{0, 2p-1}$ we have that $J^p(X) = V/H^{2p-1}(X, \mathbb{Z})$.

Lemma 4.4. $J^p(X)$ is a complex torus of dimension half the $(2p - 1)$-th Betti number of $X$:
\[
\dim J^p(X) = \frac{1}{2} b_{2p-1}(X)
\]
Proof. Since $H^{r,s}(X) = \overline{H^{s,r}(X)}$, we have Betti number is even and let $b_{2p-1} = 2m$. Now we need to show image of $H^{2p-1}(X, \mathbb{Z})$ is a lattice in $V$. Pick $\alpha_1, \ldots, \alpha_{2m}$ as the $\mathbb{R}$-basis for $H^{2p-1}(X, \mathbb{R})$, we want to show that this basis is linear independent in $V$. Suppose $\omega = \sum r_i\alpha_i \in F^pH^{2p-1}(X, \mathbb{C}) = V$ with $r_i \in \mathbb{R}$. We have $\omega = \overline{\omega}$ and therefore $\omega \in V \cap \overline{V} = 0$. Therefore $\omega = 0$ and $r_i = 0$ for all $i$.

A complex torus $T = V/L$ to be an abelian variety it is necessary and sufficient that there exists a so-called Riemann form. This is a $\mathbb{R}$-bilinear form $E : V \times V \rightarrow \mathbb{R}$ satisfying
\begin{enumerate}
  \item $E(iv, iw) = E(v, w)$.
  \item $E(v, w) \in \mathbb{Z}$ whenever $v, w \in L$.
  \item $E(v, iw)$ symmetric and positive definite.
\end{enumerate}
When \( p = 1, p = d \), \( J^p(X) \) is an abelian variety. When \( p = 1 \), \( J^1(X) = H^1(X, \mathbb{C})/H^{1,0} + H^1(X, \mathbb{Z}) \) which is the Picard variety. \( p = d \), \( J^d(X) = H^{2d-1}(X, \mathbb{C})/H^{d,d-1} + H^{2d-1}(X, \mathbb{Z}) \) which is the Albanese variety.

**Theorem 4.5.** There exists a homomorphism \( AJ^p : \mathcal{P}^p_{\text{hom}} \to J^p(X) \) which factors through \( CH^p_{\text{hom}} \). \( AJ \) is called the Abel-Jacobi map.

**Proof.** First note that \( V \) is dual to \( F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C}) \) and \( J^p(X) = V/H^{2p-1}(X, \mathbb{Z}) \). Therefore, an element \( v \in V \) is a functional on \( F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C}) \). Let \( Z \in Z^p_{\text{hom}}(X) \), and there exists a topological \((2d-2p+1)\)-chain \( \Gamma \) such that \( Z = \partial \Gamma \). Now \( \Gamma \) is a functional on \( F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C}) \) because \( \omega \in F^{d-p+1}H^{2d-2p+1}(X, \mathbb{C}) \) is represented by a closed smooth differential form \( \varphi \) of degree \( 2d-2p+1 \). The functional is given by \( \omega \mapsto \int_{\Gamma} \varphi \). Thus the choice of \( \Gamma \) determines an element of \( V \); and thus also of the intermediate jacobian \( J^p(X) = V/H^{2p-1}(X, \mathbb{Z}) \). Note that both choice of \( \varphi \) and \( \Gamma \) are well-defined. \( \square \)

Our next question is what is \( \text{im}(AJ^p(Z_{\text{alg}})) \). Given \( T \in \mathcal{P}^p(Y \times X_d) \), we have \( T : J^r(Y) \to J^{p+r-c}(X, \mathbb{C}) \). So in particular if \( Y = C \) and \( T \in \mathcal{P}^p(C \times X) \), we get a homomorphism \( T : J(C) \to J(X) \). The tangent space at the origin of \( J^p(X) \) is \( V \). We can see that \( H^0(C) \) as the tangent space of \( J(C) \) is mapped into subspace of \( H^{p,p-1}(X) \subset V \). Let \( J^p(X)_{\text{alg}} \subset J^p(X) \) be the largest subtorus of \( J^p(X) \) for which the tangent space is contained in \( H^{p,p-1}(X) \). Note that \( J^p(X)_{\text{alg}} \) is an abelian variety.

**Lemma 4.6.** \( AJ^p(Z^p_{\text{alg}}(X)) \subset J^p(X)_{\text{alg}} \)

Now we start discussing the difference of algebraic versus homological equivalence and this leads to Griffiths groups. We define that \( Gr^1(X) := Z^1_{\text{hom}}(X)/Z^1_{\text{alg}}(X) \). We first note that \( Gr^1(X) = 0 \) and \( Gr^d(X_d) = 0 \). Griffiths showed that

**Theorem 4.7** (Griffiths 1969). There exist smooth irreducible, projective varieties of dimension 3 such that \( Gr(X) \otimes \mathbb{Q} \neq 0 \).

**Theorem 4.8.** Let \( V_{d+1} \subset \mathbb{P}^N \) be a smooth, irreducible variety and \( W = V \cap H \) a smooth hyperplane section. Then

\[
H^j(V, \mathbb{Z}) \to H^j(W, \mathbb{Z})
\]

is an isomorphism for \( j < d = \dim W \) and injective for \( j = d \). This is also true for homology map induced by inclusion. i.e. the induced map is isomorphic for \( j < d \) and surjective for \( j = d \).

Special case: Let \( V = \mathbb{P}^{d+1} \) and \( W \subset \mathbb{P}^{d+1} \) hypersurface. We get \( H^j(W, \mathbb{Z}) = 0 \) when \( j < d \)

**Theorem 4.9** (Hard Lefschetz theorem). Let \( V_{d+1} \subset \mathbb{P}^N \) and \( W = V \cap H \) be a smooth hyperplane section. Let \( h = \gamma_{\mathbb{Z},V}(W) \in H^2(V, \mathbb{Z}) \). Then there is a Lefschetz operator

\[
L_V : H^j(V, \mathbb{Z}) \to H^{j+2}(V, \mathbb{Z})
\]

\[
\alpha \mapsto h \cup \alpha
\]

By repeating we get \( (n = d + 1 = \dim V) \)

\[
L^r : H^{n-r}(V, \mathbb{Z}) \to H^{n+r}(V, \mathbb{Z}), (0 \leq r \leq n).
\]

\( L^r \) induces an isomorphism for all \( r \leq n \) in coefficient \( \mathbb{Q} \) i.e.

\[
L^r : H^{n-r}(V, \mathbb{Q}) \cong H^{n+r}(V, \mathbb{Q})
\]
**Definition 4.10** (Lefschetz pencil). Let $V_n \subset \mathbb{P}^n$ be a smooth, irreducible. Let $H_\lambda = \lambda_0 H_0 + \lambda_1 H_1$ where $\lambda = [\lambda_0 : \lambda_1] \in \mathbb{P}^1$ and $H_0, H_1$ are two hyperplanes. We get a pencil of hyperplane sections $\{W_\lambda = V \cap H_\lambda\}$ on $V$. Lefschetz pencil satisfies two following properties:

1. There is a finite set $S$ of points $t \in \mathbb{P}^1$ such that $W_t$ is smooth outside $S$. Put $U := \mathbb{P}^1 - S$.
2. For $s \in S$, $W_s$ has only one singular point $x$ and it is an "ordinary double point".

For $t \in U$, we have the Lefschetz theorem $\iota_t^*: H^j(V, \mathbb{Z}) \to H^j(W_t, \mathbb{Z})$ which is an isomorphism for $j < \dim W_t$ and injective for $j = \dim W_t$. Now we can talk about the monodromy of Lefschetz pencils. Fix $t_0 \in U$ and write $W = W_{t_0}$. Then $\pi_1(U)$ operates on $H^j(W, \mathbb{Q})$, but due to the Lefschetz theorem it acts trivially if $j \neq d = \dim W$.

Consider the action $\rho: \pi_1(U) \to \text{Aut}(H^d(W, \mathbb{Q}))$. $\Gamma = \text{im}(\rho)$ is called the monodromy group.

Let $\iota: W \hookrightarrow V$ and $\iota_*: H^j(W, \mathbb{Q}) \to H^{j+2}(W, \mathbb{Q})$. By Lefschetz theorem $H^j(W, \mathbb{Q})_{\text{van}} := \ker \iota_* = 0$ when $j \neq d$. And we note that $L_W = \iota^* \circ \iota_*$. 

5. **Voisin-The Griffiths Group of a General Calabi-Yau Threefold is not Finitely Generated**

This section is the collection of materials that I learned when I read Voisin’s paper. I will follow the notation in this paper which is different from the previous section.

As stated in previous section, Abel-Jacobi map induces a morphism

$$\Phi^k_X : \text{Gr}^k(X) \to J^{2k-1}(X)_{\text{tr}}$$

where $J^{2k-1}(X)_{\text{tr}}$ is the quotient of $J^{2k-1}(X)$ by its maximal subtorus having its tangent space contained in $H^{k-1,k}(X)$.

We will focus on when $n = 3, k = 2$. Let $X$ be a projective Calabi-Yau threefold and it is well known that $\dim H^1(T_X) = \dim H^{1,2}(X)$.

**Theorem 5.1.** let $X$ be a Calabi-Yau threefold. If $\dim H^1(T_X) \neq 0$, the general deformation $X_t$ of $X$ satisfies that the Abel-Jacobi map

$$\Phi_{X_t}: Z^2(X_t) \to J^2(X_t)$$

of $X_t$ is nontrivial, even module torsion.

We are going to prove the following theorem

**Theorem 5.2.** Let $X$ be a Calabi-Yau threefold. If $h^1(T_X) \neq 0$, the general deformation $X_t$ of $X$ has the property that the Abel-Jacobi map

$$\Phi_{X_t}: Z^2(X_t) \to J(X_t)$$

is such that $\text{im} \Phi_{X_t} \otimes \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space. In particular, $\text{Gr}(X_t) \otimes \mathbb{Q}$ is an infinite dimensional $\mathbb{Q}$-vector space.

For the discussion of basic deformation theory and Kodaira-Spencer map, please refer to next subsection.
5.1. **Kodaira-Spencer map.** We define a deformation functor

\[ F : \text{Category of local Artin } \mathbb{C}\text{-algebras} \rightarrow \text{Sets} \]

where local Artin \( \mathbb{C} \)-algebras are rings of the form \( \mathbb{C}[x_1, \ldots, x_n]/I \) with one maximal ideal and \( \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/I \) is finite. For example \( \mathbb{C}[x]/(x^4), \mathbb{C}[x, y]/(x^3, y^4) \). Given a smooth projective variety \( W_0 \) over \( \mathbb{C} \), deformation problem is to solve the following commutative diagram

\[
\begin{array}{ccc}
W_0 & \longrightarrow & W \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(\mathbb{C}[x]/x^n) \\
\end{array}
\]

with vertical maps being flat. For each Artin \( \mathbb{C} \)-algebra \( A \leftarrow \mathbb{C} \), we consider the isomorphism classes of commutative diagrams

\[
\begin{array}{ccc}
W_0 & \longrightarrow & W \\
\downarrow_{F_0} & & \downarrow_{F_A} \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(A) \\
\end{array}
\]

Tangent space to deformation functor= First order deformations of \( W_0 \) is given by assigning \( A = \mathbb{C}[\epsilon]/\epsilon^2 \) and this is isomorphic to \( H^1(W_0, T_{W_0}) \). By Serre’s duality, we have \( H^i(W_0, \mathcal{F}) = H^{\dim(W_0)-i}(W_0, \mathcal{F}^* \otimes \omega)^* \). In the case when \( W_0 \) is Calabi-Yau which means canonical bundle is trivial, we have \( \dim H^1(W_0, T_{W_0}) = \dim H^2(W_0, \Omega^1) = h^{1,2} \). Now we consider the following commutative diagram of deformation

\[
\begin{array}{ccc}
W_0 & \longrightarrow & W \\
\downarrow & & \downarrow \\
0 & \longrightarrow & B \\
\end{array}
\]

where 0 is the base point of the deformation. We have short exact sequences on \( W_0 \)

\[ 0 \rightarrow \mathcal{I}_{W_0} \rightarrow \mathcal{O}_W \rightarrow \mathcal{O}_{W_0} \rightarrow 0 \]

and

\[ 0 \rightarrow \mathcal{I}_{W_0}/\mathcal{I}_{W_0}^2 \rightarrow \Omega_W|_{W_0} \rightarrow \Omega_{W_0} \rightarrow 0 \]

Note that \( \mathcal{I}_{W_0}/\mathcal{I}_{W_0}^2 = \mathcal{O}_{W_0} \otimes T^*_0B \) since conormal sheaf of fiber is trivial. We now dualize the above sequence and we get

\[ 0 \rightarrow T_{W_0} \rightarrow T_W|_{W_0} \rightarrow \mathcal{O}_{W_0} \otimes T_0B \rightarrow 0 \]

and it induces a map \( KS : H^0(W_0, \mathcal{O}_{W_0} \otimes T_0B) \rightarrow H^1(W_0, T_{W_0}) \). We have \( T_0B = \mathcal{N}_{0/B} \cong H^0(W_0, \mathcal{N}_{W_0/W}) = H^0(W_0, \mathcal{O}_{W_0} \otimes T_0B) \) and denote \( KS : T_0B \rightarrow H^1(W_0, T_{W_0}) \).

6. **Some Random Math**

This section is contributed to various topics in mathematics.
6.1. **Hilbert scheme.** Both Grassmannian and the Fano scheme are special cases of Hilbert scheme. First we introduce universal property of Grassmannian. We define $G = G(k + 1, V) = G(k, \mathbb{P}V)$ and $\Phi = \{(\Lambda, p) \in G \times \mathbb{P}V \mid p \in \Lambda\}$ as the universal family of $k$-planes in $\mathbb{P}V$ in the following sense: For any scheme $B$, we will say that a subscheme $\mathcal{L} \subset B \times \mathbb{P}V$ is a flat family of $k$-planes in $\mathbb{P}V$ if the restriction $\pi : \mathcal{L} \to B$ of the projection $\pi_1 : B \times \mathbb{P}V \to B$ is flat, and the fibers over closed points of $B$ are linearly embedded $k$-planes in $\mathbb{P}V$. We then have:

**Proposition 6.1.** If $\pi : \mathcal{L} \subset B \times \mathbb{P}V \to B$ is a flat family of $k$-planes in $\mathbb{P}V$, then there is a unique map $\alpha : B \to G$ such that $\mathcal{L}$ is equal to the pullback of the family $\Phi$ via $\alpha$:

\[
\begin{array}{ccc}
\mathcal{L} & \xleftarrow{\pi} & B \times G \Phi \\
\downarrow & & \downarrow \pi \\
\mathcal{L} & \xrightarrow{\alpha} & \Phi \\
\end{array}
\]

6.2. **Elliptic surfaces.** Let $k = \overline{k}$ and $C$ a smooth projective curve over $k$.

**Definition 6.2.** An elliptic surface $S$ over $C$ is a smooth projective surface $S$ with an elliptic fibration over $C$, i.e. a surjective morphism

$$f : S \to C,$$

such that

1. generic fibers are elliptic curves;
2. no fiber contains an exceptional curve of the first kind.

The second condition ensures that $S$ is a smooth minimal model.

**Example.**