# BA 932 Session 2 Traditional Models of Capital Markets Continued

Xu Jiang

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# What we have done

- Introduce (noisy) rational expectations models and how to solve such models
- Price is a (noisy) aggregator of information from each individual trader, results in price being an informative signal of asset payoffs
- How to solve such models: conjecture price as a linear function of (signals about) asset payoffs and asset supply -> such conjecture has to be rational -> the coefficients need to match in equilibrium that corresponds to each traders' optimal behavior

# This class

- Apply the noisy rational expectations models to study the implications of more public disclosure
- Introduce (noisy) rational expectations models when some traders have market power: Kyle model
- Discrete order version of Kyle model -> connects with Glosten-Milgrom model to explain bid-ask spreads

# Implications of more public disclosure - model setup

- Still two assets:
  - One risky asset with uncertain payoff $\widetilde{v} \sim N(0, \tau_v^{-1})$ and noisy supply $\widetilde{x} \sim N(0, \tau_x^{-1})$.
  - one risk-free asset with a return of 1 and unlimited supply
- Disclosure is modelled as a public signal $\widetilde{y} = \widetilde{v} + \widetilde{\eta}$ with $\widetilde{\eta} \sim N(0, \tau_\eta^{-1})$.

# Implications of more public disclosure - model setup

- Still a continuum $[0, 1]$ of traders with CARA utility $U(c) = -e^{-\gamma C}$.
- A proportion $\mu \in [0, 1]$ is informed and each informed trader $i$ gets

$$\widetilde{s}_i = \widetilde{v} + \widetilde{\varepsilon}_i \text{ with } \widetilde{\varepsilon}_i \text{ i.i.d. } \sim N(0, \tau_\varepsilon^{-1}).$$

- In Grossman and Stiglitz $\mu$ is endogenously determined. For now, treat $\mu$ as exogenous.

# Implications of more public disclosure - solving for the equilibrium

- Again conjecture a linear price function $\widetilde{P} = p_y\widetilde{y} + p_v\widetilde{v} + p_x\widetilde{x}$ (note that we do not have the prior term as the prior mean is zero and any weight times zero equals zero), where $p_y$, $p_v$, and $p_x$ are constants to be endogenously determined.
- Again $\widetilde{P}$ is a noisy signal of $\widetilde{v}$, and is informationally equivalent to $\frac{\widetilde{P}-p_y\widetilde{y}}{p_v} = \widetilde{v} + \rho^{-1}\widetilde{x}$ where $\rho \equiv \frac{p_v}{p_x}$, which is another noisy signal about $\widetilde{v}$ with precision $\rho^2\tau_x$.

# Implications of more public disclosure - solving for the equilibrium

- Again, for each informed trader $i$,

$$D_{Ii} = \frac{E[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}] - \widetilde{p}}{\gamma var[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}]}.$$

- Standard Bayesian updating has

$$\begin{aligned} E[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}] &= \frac{\tau_\eta \widetilde{y} + \tau_\varepsilon \widetilde{s}_i + \rho^2 \tau_x \frac{\widetilde{P} - p_y \widetilde{y}}{p_v}}{\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v}, \\ var[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}] &= \frac{1}{\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v}. \end{aligned}$$

# Implications of more public disclosure - solving for the equilibrium

- Therefore

$$
\begin{aligned}
D_{Ii} &= \frac{\tau_\eta \widetilde{y} + \tau_\varepsilon \widetilde{s}_i + \rho^2 \tau_x \frac{\widetilde{P} - p_y \widetilde{y}}{p_v} - \widetilde{P}(\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v)}{\gamma} \\
&= \frac{(\tau_\eta - \frac{\rho^2 \tau_x p_y}{p_v})\widetilde{y} + \tau_\varepsilon \widetilde{s}_i - (\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v})\widetilde{p}}{\gamma}.
\end{aligned}
$$

# Implications of more public disclosure - solving for the equilibrium

- Again, for each informed trader $i$,

$$D_{Ii} = \frac{E[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}] - \widetilde{p}}{\gamma var[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}]}.$$

- Standard Bayesian updating has

$$\begin{aligned} E[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}] &= \frac{\tau_\eta \widetilde{y} + \tau_\varepsilon \widetilde{s}_i + \rho^2 \tau_x \frac{\widetilde{P} - p_y \widetilde{y}}{p_v}}{\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v}, \\ var[\widetilde{v}|\widetilde{y}, \widetilde{s}_i, \widetilde{p}] &= \frac{1}{\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v}. \end{aligned}$$

# Implications of more public disclosure - solving for the equilibrium

- Therefore

$$
\begin{aligned}
D_U &= \frac{\tau_\eta \tilde{y} + \rho^2 \tau_x \frac{\tilde{P} - p_y \tilde{y}}{p_v} - (\tau_\eta + \rho^2 \tau_x + \tau_v)\tilde{P}}{\gamma} \\
&= \frac{(\tau_\eta - \frac{\rho^2 \tau_x p_y}{p_v})\tilde{y} - (\tau_\eta + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v})\tilde{P}}{\gamma}.
\end{aligned}
$$

# Implications of more public disclosure - solving for the equilibrium

- Market clearing implies that

$$\int_0^\mu D_{Ii} di + (1-\mu) D_U = \widetilde{x},$$

which is equivalent to

$$\begin{aligned} & \frac{(\tau_\eta - \frac{\rho^2 \tau_x p_y}{p_v})\widetilde{y} + \tau_\varepsilon \widetilde{v} - (\tau_\eta + \tau_\varepsilon + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v})\widetilde{P}}{\gamma}\mu \\ & + \frac{(\tau_\eta - \frac{\rho^2 \tau_x p_y}{p_v})\widetilde{y} - (\tau_\eta + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v})\widetilde{P}}{\gamma}(1-\mu) \\ = \ & \widetilde{x}, \end{aligned}$$

so

$$\widetilde{P} = \frac{(\tau_\eta - \frac{\rho^2 \tau_x p_y}{p_v})\widetilde{y} + \tau_\varepsilon \mu \widetilde{v} - \gamma \widetilde{x}}{\tau_\eta + \tau_\varepsilon \mu + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v}}.$$

# Implications of more public disclosure - solving for the equilibrium

- Matching coefficients results in

$$
\begin{aligned}
p_y &= \frac{\tau_\eta - \frac{\rho^2 \tau_x p_y}{p_v}}{\tau_\eta + \tau_\varepsilon \mu + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v}}, \\
p_v &= \frac{\tau_\varepsilon \mu}{\tau_\eta + \tau_\varepsilon \mu + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v}}, \\
p_x &= -\frac{\gamma}{\tau_\eta + \tau_\varepsilon \mu + \rho^2 \tau_x + \tau_v - \frac{\rho^2 \tau_x}{p_v}}.
\end{aligned}
$$

# Implications of more public disclosure - solving for the equilibrium

- Solving results in

$$
\begin{aligned}
p_y &= \frac{\gamma^2 \tau_\eta}{\mu^2 \tau_\varepsilon^2 \tau_x + \gamma^2 (\tau_\varepsilon \mu + \tau_v + \tau_\eta)}, \\
p_v &= \frac{\mu \tau_\varepsilon (\gamma^2 + \mu \tau_\varepsilon \tau_x)}{\mu^2 \tau_\varepsilon^2 \tau_x + \gamma^2 (\tau_\varepsilon \mu + \tau_v + \tau_\eta)}, \\
p_x &= -\frac{\gamma (\gamma^2 + \mu \tau_\varepsilon \tau_x)}{\mu^2 \tau_\varepsilon^2 \tau_x + \gamma^2 (\tau_\varepsilon \mu + \tau_v + \tau_\eta)}.
\end{aligned}
$$

# Implications of more public disclosure - comparative statics

- Price informativeness (or price efficiency): $\frac{1}{var[\widetilde{v}|\widetilde{p}]}$. We can show that

$$
\begin{aligned}
\frac{1}{var[\widetilde{v}|\widetilde{p}]} &= \frac{1}{var[\widetilde{v}] - \frac{cov^2(\widetilde{v},\widetilde{p})}{var(\widetilde{p})}} \\
&= \frac{1}{\frac{1}{\tau_v} - \frac{\frac{(p_y+p_v)^2}{\tau_v^2}}{\frac{(p_y+p_v)^2}{\tau_v} + \frac{p_y^2}{\tau_\eta} + \frac{p_x^2}{\tau_x}}} \\
&= \frac{\tau_v[\frac{(p_y+p_v)^2}{\tau_v} + \frac{p_y^2}{\tau_\eta} + \frac{p_x^2}{\tau_x}]}{\frac{p_y^2}{\tau_\eta} + \frac{p_x^2}{\tau_x}} \\
&= \tau_v + \frac{\tau_x[\mu\tau_\varepsilon(\gamma^2 + \mu\tau_\varepsilon\tau_x) + \gamma^2\tau_\eta]^2}{\gamma^2\tau_v[(\gamma^2 + \mu\tau_\varepsilon\tau_x)^2 + \gamma^2\tau_x\tau_\eta}.
\end{aligned}
$$

# Implications of more public disclosure - comparative statics

- It can be shown that the second term is increasing in $\tau_\eta$ (see the Mathematica file for the detailed expression of the derivative). So more precise disclosure increases price informativeness. Intuition: price reflects the informativeness of public disclosure.

# Implications of more public disclosure - comparative statics

- Illiquidity, captured by $|p_x| = -\, p_x$ as $p_x$ is a negative number. The larger $|p_x|$ is, the more illiquid the stock is (as forced trading by noise traders will move price more).
- $\frac{\partial |p_x|}{\partial \tau_\eta} = -\frac{\gamma^3(\gamma^2+\mu\tau_\varepsilon\tau_x)}{[\mu^2\tau_\varepsilon^2\tau_x+\gamma^2(\tau_\varepsilon\mu+\tau_v+\tau_\eta)]^2} < 0$. Therefore, more precise public disclosure increases liquidity. Intuition: 1) more precise public disclosure makes price more informative of $\widetilde{v}$ and thus respond less strongly to noisy traders' order flow; 2) more precise public disclosure reduces the information asymmetry between informed and uninformed, resulting in uninformed more willing to trade and thus price response less strongly to noise traders' order flow.
- Earnings response coefficient (ERC), captured by $p_y$ as $p_y$ is a positive number. The larger $p_y$ is, the larger ERC is.
- $\frac{\partial p_y}{\partial \tau_\eta} = \frac{\gamma^4(\mu\tau_\varepsilon+\tau_v)+\gamma^2\mu^2\tau_\varepsilon^2\tau_x}{[\mu^2\tau_\varepsilon^2\tau_x+\gamma^2(\tau_\varepsilon\mu+\tau_v+\tau_\eta)]^2} > 0$. Therefore, more precise public disclosure increases ERC. Intuition: price responds more to signals that are more informative.

# Implications of more public disclosure - comparative statics

- Cost of capital, defined as $E[\widetilde{v} - \widetilde{P}]$: think of this as (future) investors buy firm shares at price $\widetilde{P}$ and gets a terminal payoff of $\widetilde{v}$.
- (Perhaps) counterintuitively, in this model $E[\widetilde{v} - \widetilde{P}] = 0$, which is independent of public disclosure quality.
- This result is driven by assuming $\widetilde{x}$, the noisy supply, having a mean zero. Any positive (negative) mean will result in more precise public disclosure decreasing (increasing) cost of capital.
- Intuition: mean zero supply means that investors on average do not bear any risk of holding the shares so cost of capital is equal to expected terminal payoff on average. Positive average supply means that investors on average bear a positive risk of holding the shares so the cost of capital has to be smaller than the expected terminal payoff to account for the risk premium. More precise public disclosure reduces the residual variance of $\widetilde{v}$ thus reduces the risk premium so decreases the cost of capital.
- See Goldstein and Yang (2017) for more detailed discussion.

# Implications of more public disclosure - comparative statics

- Ex-post return volatility (in the sense of volatility measured in the period after disclosure), defined as

$$
\begin{aligned}
& var(\widetilde{v} - \widetilde{P}) \\
= \; & var[(1-p_v)\widetilde{v} - p_y\widetilde{y} - p_x\widetilde{x}] \\
= \; & \frac{(1-p_v)^2}{\tau_v} + p_y^2\left(\frac{1}{\tau_v} + \frac{1}{\tau_\eta}\right) + \frac{p_x^2}{\tau_x} - 2(1-p_v)p_y\frac{1}{\tau_v} \\
= \; & \frac{(1-p_v-p_y)^2}{\tau_v} + \frac{p_y^2}{\tau_\eta} + \frac{p_x^2}{\tau_x}.
\end{aligned}
$$

- It can be shown that $\frac{\partial var(\widetilde{v}-\widetilde{P})}{\partial \tau_\eta} < 0$ (see the Mathematica file for the detailed expression). Therefore, higher disclosure quality reduces ex-post return volatility. Intuition: more precise public disclosure increases price efficiency and thus brings $\widetilde{P}$ closer to $\widetilde{v}$.

# Implications of more public disclosure - incorporating information acquisition

- In the example above, $\mu$ (the proportion of informed traders) is exogenously given. However, given that information acquisition is costly, $\mu$ should be endogenously determined by public disclosure quality.
- See Goldstein and Yang (2017) for more detailed discussion: more precise public disclosure crowding out private information acquisition (i.e., $\mu$ decreases in $\tau_\eta$) is a quite robust result.
- This implies that some of the capital market consequences of more public disclosure may be overturned, as less private information acquisition will (ceterus paribus) reduce price informativeness and move price away from $\tilde{v}$.
- Other than liquidity, every other capital market consequence result can be overturned, as crowding out private information acquisition reduces information asymmetry among the informed and uninformed and further increases liquidity.

# Kyle model - continuous order flow

- In the noisy rational expectations model, traders take price as given when choosing their demand function, i.e., $\frac{\partial \widetilde{P}}{\partial D_i} = 0$.
- This assumption may be reasonable for some markets, e.g., blue chip stocks (think of Apple) with high trading volume so each trader has negligible impact on prices, even for big hedge funds.
- It may not be reasonable for small and thinly traded stocks with low trading volume so some traders can have a large impact on prices.
- Again, the core issue is inference: I do not observe information from the traders but I observe their actions (buy or sell). What information should I infer?
- If I know you know more than me about a firm and I observe that you are selling stocks, what does that tell me? How should I respond? Knowing this, how should you respond?
- Kyle (1985) provides an elegant model to precisely capture those phenomena.

# Continuous Kyle model - model setup

- One risky asset with terminal payoff $\widetilde{v} \sim N(p_0, \Sigma_0)$.
- There is an informed trader who observes $\widetilde{v}$ perfectly and there is a market maker who observes total order flow. Both parties are risk-neutral.
- Suppose there is no other party, then there will be no trade.
- Intuition: if the informed trader sells (buys), the market maker knows that the informed trader knows $\widetilde{v}$ is low (high) and is not willing to buy (sell). If the informed trader makes money, then the market maker loses money and so the market maker will not want to trade.

# Continuous Kyle model - model setup

- How to make trade happen: assume noise traders that submit random orders (as in Grossman and Stiglitz) and let the noise traders lose money to the informed trader, the market maker breaks even. Noise traders lose money but they have liquidity needs so do not care.
- Why submit random orders? Again to prevent prices from fully revealing the informed trader's info. (the Grossman-Stiglitz paradox)
- Assume noise traders submit order $\widetilde{u} \sim N(0, \sigma_u^2)$ and independent of $\widetilde{v}$. The informed trader submit order $x(\widetilde{v})$.
- The market maker observes total order flow $x(\widetilde{v}) + \widetilde{u}$ and sets up price $\widetilde{P} = E[\widetilde{v}|x(\widetilde{v}) + \widetilde{u}]$. Again price will be partially revealing (but not fully revealing) of $\widetilde{v}$.

# Continuous Kyle model - solving the model

- We again face uneven distribution of information.
- Again think of Texas Holdem: the market maker does not observe the informed trader's information $\widetilde{v}$ but will conjecture the informed trader's strategy $x(\widetilde{v})$ and take this into account the strategy in choosing his or her own strategy (i.e., setting prices)
- The conjecture has to be rational: it has to be consistent with the informed trader's optimal behavior.
- Given normal distribution, linear conjecture seems a good start.

# Continuous Kyle model - solving the model

- Conjecture that

$$x(\widetilde{v}) = \beta(\widetilde{v} - p_0),$$

and

$$\widetilde{P} = E[\widetilde{v}|x(\widetilde{v}) + \widetilde{u}] = p_0 + \lambda(x(\widetilde{v}) + \widetilde{u}),$$

where $\beta > 0$, $\gamma > 0$ are constants to be determined.

- Intuitively,
  - from the informed trader's perspective: $x(\widetilde{v}) > 0$ if and only $\widetilde{v} > p_0$, buy if and only if my information is better than the prior mean;
  - from the market maker's perspective: the larger $x(\widetilde{v}) + \widetilde{u}$ is, the more likely it comes from the informed trader buying, the higher the price should be. When $x(\widetilde{v}) + \widetilde{u} = 0$, then it is equally likely that the informed trader is buying vs. selling so $\widetilde{P} = E[\widetilde{v}] = v_0$.

# Continuous Kyle model - solving the model

- Need two equations of $\beta$ and $v$.
- First one comes from the rationality of the informed trader: based on the conjecture $\widetilde{P}$, the informed trader chooses $x(\widetilde{v})$ to maximize expected profit, i.e.,

$$\begin{aligned} & E[(\widetilde{v} - \widetilde{P})x(\widetilde{v})|\widetilde{v}] \\ = \;& E[(\widetilde{v} - p_0 - \lambda x(\widetilde{v}) - \lambda \widetilde{u})x(\widetilde{v})|\widetilde{v}] \\ = \;& (\widetilde{v} - p_0)x(\widetilde{v}) - \lambda x^2(\widetilde{v}). \end{aligned}$$

- Note the difference from Grossman-Stiglitz: $\widetilde{P}$ depends on $x(\widetilde{v})$ so when choosing demand cannot take price as given has to take into account the effect of demand on price.
- First order condition results in $x(\widetilde{v}) = \frac{\widetilde{v} - p_0}{2\lambda}$, so matching coefficient results in

$$\beta = \frac{1}{2\lambda}.$$

# Continuous Kyle model - solving the model

- Second one comes from the rationality of the market maker: based on the conjecture $x(\widetilde{v})$, $\widetilde{v}|x(\widetilde{v}) + \widetilde{u}$ is a normally distributed random variable and Bayesian updating results in

$$
\begin{aligned}
& E[\widetilde{v}|x(\widetilde{v}) + \widetilde{u}] \\
= \; & E[\widetilde{v}] + \frac{cov(\widetilde{v}, x(\widetilde{v}) + \widetilde{u})}{var(x(\widetilde{v}) + \widetilde{u})}(x(\widetilde{v}) + \widetilde{u} - E[x(\widetilde{v}) + \widetilde{u}]) \\
= \; & p_0 + \frac{cov(\widetilde{v}, x(\widetilde{v}))}{var(x(\widetilde{v}) + \widetilde{u})}(x(\widetilde{v}) + \widetilde{u} - E[x(\widetilde{v})]) \\
= \; & p_0 + \frac{cov(\widetilde{v}, \beta(\widetilde{v} - p_0))}{var(\beta(\widetilde{v} - p_0) + \widetilde{u})}(x(\widetilde{v}) + \widetilde{u} - E[x(\widetilde{v})]) \\
= \; & p_0 + \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}(x(\widetilde{v}) + \widetilde{u}).
\end{aligned}
$$

- Matching coefficient results in

$$\lambda = \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}.$$

# Continuous Kyle model - solving the model

- We thus have
$$\beta = \frac{1}{2\lambda}$$
and
$$\lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2}.$$
- Notice that the first equation means the higher $\lambda$, the smaller the $\beta$: when price responds more to order flow (means price increase more when you buy the same amount), the informed trader trades less intensively on price information.
- Solving (with Mathematica or not) results in
$$\beta = (\frac{\sigma_u^2}{\Sigma_0})^{\frac{1}{2}},$$
and
$$\lambda = \frac{1}{2}(\frac{\Sigma_0}{\sigma_u^2})^{\frac{1}{2}}.$$

# Continuous Kyle model - implications

- Note that $\beta$ (and thus $\lambda$) increases in $\sigma_u^2$ and decreases in $\Sigma_0$
  - Intuitively, the informed trader trades more aggressively when there is more noisy trading (as a camouflage) and less aggressively when there is more information asymmetry
  - $\lambda$ is the famous "Kyle's lambda", which is a measure of the illiquidity of the market; the higher $\lambda$ is, the more sensitive price is to order flow, so the same amount of order flow results in larger change in prices.
  - If public disclosure is modelled as an increase in prior precision (i.e., a decrease in $\Sigma_0$) as in Verrecchia (1982 JAR), then we have the intuitive result that more precise public disclosure increases liquidity.
  - You will verify this in the homework when more precise public information is modelled as an increase in the precision of an ex-post signal.

# Continuous Kyle model - implications

- In this setting, price informativeness can be again captured by (the inverse of)

$$
\begin{aligned}
var(\widetilde{v}|\widetilde{P}) &= var(\widetilde{v}) - \frac{cov^2(\widetilde{v},\widetilde{P})}{var(\widetilde{P})} \\
&= \Sigma_0 - \frac{\lambda^2\beta^2\Sigma_0^2}{\lambda^2(\beta^2\Sigma_0+\sigma_u^2)} \\
&= \Sigma_0 - \Sigma_0\beta\frac{\beta\Sigma_0}{\beta^2\Sigma_0+\sigma_u^2)} \\
&= \Sigma_0 - \Sigma_0\beta\lambda = \frac{1}{2}\Sigma_0.
\end{aligned}
$$

In other words, price resolve 50% of the prior uncertainty and so incorporate 50% of the informed trader's information. Price informativeness is independent of $\sigma_u^2$ (as more noisy trading results in insiders trade more intensively), which is different from Grossman-Stiglitz.

# Continuous Kyle model - implications

- The informed trader's expected profit is

$$\begin{aligned} & E[(\widetilde{v}-\widetilde{P})x(\widetilde{v})] \\ = \;& E[E[(\widetilde{v}-\widetilde{P})x(\widetilde{v})|\widetilde{v}]] \\ = \;& = E[(\widetilde{v}-p_0)x(\widetilde{v})-\lambda x^2(\widetilde{v})] \\ = \;& E[\beta(\widetilde{v}-p_0)^2-\lambda\beta^2(\widetilde{v}-p_0)^2] \\ = \;& \frac{\beta}{2}E[(\widetilde{v}-p_0)^2]=\frac{\beta}{2}\Sigma_0=\frac{1}{2}(\sigma_u^2\Sigma_0)^{\frac{1}{2}}. \end{aligned}$$

- Therefore, the informed trader's expected profit is increasing in $\sigma_u^2$ and $\Sigma_0$. This is intuitive, as the more volatile the noisy trading is, the better the camouflage for the informed trader; the higher the information asymmetry, the better off the informed trader is.
- Note that when $\Sigma_0$ increases, liquidity decreases but the informed trader's expected profit increases whereas when $\sigma_u^2$ increases both increase so liquidity and informed trader's profit do not necessarily move in the same direction.

# Kyle model - discrete order flow

- The continuous order flow Kyle model may become intractable when adding more ingredients (e.g., add manager learning from prices, i.e., the feedback effect)
- The essential feature in the Kyle model is that there is information asymmetry between the informed trader and the uninformed (i.e., market maker). The uninformed takes that into account, resulting in price moving (partially) against the informed trader's trade (increase when the informed trader wants to buy and decrease when the informed trader wants to sell).
- Discrete order flow can generate similar features and the bid-ask spread naturally arises.

# Discrete Kyle - model setup

- Assume that the terminal payoff of the risky asset, $\widetilde{v}$, is binary, $\widetilde{v} \in \{0, 1\}$ with prior probability $\Pr(\widetilde{v} = 0) = \frac{1}{2}$.
- Assume that the noisy trader order flow $\widetilde{u} \in \{-1, 1\}$ with distribution of $\Pr(\widetilde{u} = -1) = \frac{1}{2}$.
- Assume that the informed trader observes $\widetilde{v}$ and can buy or sell at most one share (e.g., because of cost considerations), i.e., $x(\widetilde{v}) \in \{-1, 0, 1\}$.
- The market maker still observes total order flow $S = x(\widetilde{v}) + \widetilde{u} \in \{-2, -1, 0, 1, 2\}$ and chooses price $\widetilde{P} = E[\widetilde{v}|S]$.

# Discrete Kyle - solving the model

- Again, market maker conjectures the informed trader's strategy.
- From the continuous model, we know that the informed trader will buy if $\widetilde{v} > E[\widetilde{v}]$ and sell if $\widetilde{v} < E[\widetilde{v}]$.
- Because $E[\widetilde{v}] = \frac{1}{2}$, the conjecture is that $x(1) = 1$ and $x(0) = -1$ (need to eventually verify this is the case).
- This implies that $S \in \{-2, 0, 2\}$. The maker maker needs to set $P(S)$ for each possible value of $S$, taking into account the informed trader's strategy.

# Discrete Kyle - solving the model

- When $S = 2$, this implies that $x = 1$ so the informed trader must be buying. Therefore $P(+2) = 1$.
- When $S = -2$, this implies that $x = -1$ so the informed trader must be buying. Therefore $P(-2) = 0$.
- When $S = 0$, need to use Bayes' rule to calculate $\Pr(\widetilde{v} = 1 | S = 0)$.

# Discrete Kyle - solving the model

- $$\begin{aligned}\Pr(\widetilde{v} &= 1|S=0) \\ &= \frac{\Pr(S=0|\widetilde{v}=1)\Pr(\widetilde{v}=1)}{\Pr(S=0|\widetilde{v}=1)\Pr(\widetilde{v}=1)+\Pr(S=0|\widetilde{v}=0)\Pr(\widetilde{v}=0)} \\ &= \frac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}} = \frac{1}{2} = \Pr(\widetilde{v}=1).\end{aligned}$$
- $S=0$ is completely uninformative as the posterior is the same as the prior.
- Intuitively, $S=0$ can be caused by the informed buying but noisy trader selling or the informed selling but noisy trader buying. Each event has equal likelihood, resulting in $S=0$ is completely uninformative regarding whether the informed is buying or selling.
- Therefore $P(S=0)=\frac{1}{2}$.

# Discrete Kyle - solving the model

- Still need to verify that $x(1) = 1$ and $x(0) = -1$ is optimal.
- When the informed trader observing $\widetilde{v} = 1$,
  - buying results in an expected payoff of

$$\frac{1}{2}[1 - P(+2)] + \frac{1}{2}[1 - P(0)] = \frac{1}{4}.$$

  - Not trading results in an expected payoff of 0.
  - Selling results in an expected payoff of

$$\frac{1}{2}[P(0) - 1] + \frac{1}{2}[P(-2) - 1] = -\frac{3}{4}.$$

  - Therefore buying is optimal.
- Can similarly verify that $x(0) = -1$ is optimal so characterization of the equilibrium is complete.

# Discrete Kyle - implications

- Issue with this model:
    - While the informed trader does trade in the intuitive manner and price is not fully revealing, it does not generate a bid-ask spread.
    - The reason is that prices are either fully revealing or not revealing (of the informed trader's info) at all.
- How to fix this and (endogenously) generate spread:
    - Increase the volatility of noisy trades (so price may be partial revealing).
    - Introduce the possibility that the informed trader may not be present (Glosten-Milgrom approach).

# Discrete Kyle variant 1- model setup

- Consider now two noisy traders, $\widetilde{u}_1$, $\widetilde{u}_2 \in \{-1, 0, 1\}$, $\widetilde{u}_1$ and $\widetilde{u}_2$ independent of each other, with distribution of

$$\Pr(\widetilde{u}_i = -1) = \Pr(\widetilde{u}_i = 0) = \Pr(\widetilde{u}_i = +1) = \frac{1}{3}.$$

.

- This implies that total noisy trades $\widetilde{u} = \widetilde{u}_1 + \widetilde{u}_2 \in \{-2, -1, 0, +1, +2\}$ with

$$\begin{aligned}
\Pr(\widetilde{u} &= +2) = \Pr(\widetilde{u} = -2) = \frac{1}{9}, \\
\Pr(\widetilde{u} &= +1) = \Pr(\widetilde{u} = -1) = \frac{2}{9}, \\
\Pr(\widetilde{u} &= 0) = \frac{1}{3}.
\end{aligned}$$

- Again conjecture that $x(1) = 1$ and $x(0) = -1$ (can be similarly verified this is indeed the optimal strategy). This implies that total order flow $S \in \{+3, +2, +1, 0 - 1, -2, -3\}$.

# Discrete Kyle variant 1- solving the model

- When $S = +3$ or $+2$, this implies that $x = 1$ so the informed trader must be buying. Therefore $P(+3) = P(+2) = 1$.
- When $S = -3$ or $-2$, this implies that $x = -1$ so the informed trader must be selling. Therefore $P(-3) = P(-2) = 0$.
- The case when $S = \pm 1$ and 0 is more complicated. Still need to use Bayes' rule to calculate

$$\begin{aligned}
\Pr(\widetilde{v} &= 1|S) \\
&= \frac{\Pr(S|\widetilde{v}=1)\Pr(\widetilde{v}=1)}{\Pr(S|\widetilde{v}=1)\Pr(\widetilde{v}=1)+\Pr(S|\widetilde{v}=0)\Pr(\widetilde{v}=0)} \\
&= \frac{\Pr(S|\widetilde{v}=1)}{\Pr(S|\widetilde{v}=1)+\Pr(S|\widetilde{v}=0)}.
\end{aligned}$$

# Discrete Kyle variant 1- solving the model

- When $S = +1$,
  - $\Pr(S = +1|\widetilde{v} = 1) = \frac{1}{3}$, as when $\widetilde{v} = 1$, $x = 1$ so $S = +1$ is only possible if $\widetilde{u} = 0$, which happens with probability $\frac{1}{3}$.
  - $\Pr(S = +1|\widetilde{v} = 0) = \frac{1}{9}$, as when $\widetilde{v} = 0$, $x = -1$ so $S = +1$ is only possible if $\widetilde{u} = +2$, which happens with probability $\frac{1}{9}$.
  - Therefore $\Pr(\widetilde{v} = 1|S = +1) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{9}} = \frac{3}{4}$.
- When $S = -1$,
  - $\Pr(S = -1|\widetilde{v} = 1) = \frac{2}{9}$, as when $\widetilde{v} = 1$, $x = 1$ so $S = -1$ is only possible if $\widetilde{u} = -2$, which happens with probability $\frac{1}{9}$.
  - $\Pr(S = -1|\widetilde{v} = 0) = \frac{1}{3}$, as when $\widetilde{v} = 0$, $x = -1$ so $S = -1$ is only possible if $\widetilde{u} = 0$, which happens with probability $\frac{1}{3}$.
  - Therefore $\Pr(\widetilde{v} = 1|S = -1) = \frac{\frac{1}{9}}{\frac{1}{9}+\frac{1}{3}} = \frac{1}{4}$.

# Discrete Kyle variant 1- solving the model

- When $S = 0$,
  - $\Pr(S = 0|\widetilde{v} = 1) = \frac{2}{9}$, as when $\widetilde{v} = 1$, $x = 1$ so $S = 0$ is only possible if $\widetilde{u} = -1$, which happens with probability $\frac{2}{9}$.
  - $\Pr(S = 0|\widetilde{v} = 0) = \frac{2}{9}$, as when $\widetilde{v} = 0$, $x = -1$ so $S = 0$ is only possible if $\widetilde{u} = +1$, which happens with probability $\frac{2}{9}$.
  - Therefore $\Pr(\widetilde{v} = 1|S = 0) = \frac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{9}} = \frac{1}{2}$.
- We therefore have $P(+1) = \frac{3}{4}$, $P(0) = \frac{1}{2}$, and $P(-1) = \frac{1}{4}$.
- The bid-ask spread is therefore $P(+1) - P(-1) = \frac{1}{2}$.
- Intuition: bid-ask spread is generated by information asymmetry as the market maker is less informed about $\widetilde{v}$. A buy (sell) order implies that it is more likely that it comes from a positively (negatively) informed investor, resulting in a higher (lower) ask (bid).

# Discrete Kyle variant 1- effect of disclosure modelled as an ex post signal

- If bid-ask spread is generated by information asymmetry, then more disclosure, by reducing information asymmetry, should reduce bid-ask spread.
- Introduce disclosure: suppose there is a public signal $\widetilde{y} \in \{h, l\}$ such that $\Pr(\widetilde{y} = h|\widetilde{v} = 1) = \Pr(\widetilde{y} = l|\widetilde{v} = 0) = q \in [\frac{1}{2}, 1)$.
- Then price will be a function of both $S$ and $y$, i.e., $P(y, S)$.
- Again, conjecture that $x(1) = +1$ and $x(0) = -1$ (can be similarly verified those are indeed equilibrium strategies) independent of $y$.
- So $S = +3, +2$ and $S = -3, -2$ are still fully revealing.

# Discrete Kyle variant 1- effect of disclosure modelled as an ex post signal

- Again, using Bayes' Rule,

$$
\begin{aligned}
\Pr(\widetilde{v} &= 1|y) = \frac{\Pr(y|\widetilde{v}=1)\Pr(\widetilde{v}=1)}{\Pr(y|\widetilde{v}=1)\Pr(\widetilde{v}=1)+\Pr(y|\widetilde{v}=0)\Pr(\widetilde{v}=0)} \\
&= \frac{\Pr(y|\widetilde{v}=1)}{\Pr(y|\widetilde{v}=1)+\Pr(y|\widetilde{v}=0)},
\end{aligned}
$$

$$
\begin{aligned}
\Pr(\widetilde{v} &= 1|S,y) \\
&= \frac{\Pr(S|\widetilde{v}=1,y)\Pr(\widetilde{v}=1|y)}{\Pr(S|\widetilde{v}=1,y)\Pr(\widetilde{v}=1|y)+\Pr(S|\widetilde{v}=0)\Pr(\widetilde{v}=0|y)} \\
&= \frac{\Pr(S|\widetilde{v}=1,y)\Pr(\widetilde{v}=1|y)}{\Pr(S|\widetilde{v}=1,y)\Pr(\widetilde{v}=1|y)+\Pr(S|\widetilde{v}=0,y)\Pr(\widetilde{v}=0|y)}.
\end{aligned}
$$

- Note that $\Pr(S|\widetilde{v},y)=\Pr(S|\widetilde{v})$ as both the informed and the noisy traders' demand are independent of $y$.

# Discrete Kyle variant 1 - effect of disclosure modelled as an ex post signal

- Note that $\Pr(\widetilde{v} = 1|h) = q$ and $\Pr(\widetilde{v} = 0|h) = 1 - q$.
- Therefore $P(+1, h) = \frac{\frac{1}{3}q}{\frac{1}{3}q + \frac{1}{9}(1-q)} = \frac{3q}{1+2q}$ and $P(-1, h) = \frac{\frac{1}{9}q}{\frac{1}{9}q + \frac{1}{3}(1-q)} = \frac{q}{3-2q}$.
- So the bid-ask spread

$$P(+1, h) - P(-1, h) = \frac{3q}{1+2q} - \frac{q}{3-2q} = \frac{8q(1-q)}{(1+2q)(3-2q)} \leq \frac{1}{2},$$

as the inequality is equivalent to

$$12(q - \frac{1}{2})^2 \geq 0.$$

- In addition, $\frac{\partial(P(+1,h) - P(-1,h))}{\partial q} = -\frac{24(2q-1)}{(1+2q)^2(3-2q)^2} < 0$. Therefore, higher disclosure quality reduces bid-ask spread when $y = h$.

# Discrete Kyle variant 1- effect of disclosure modelled as an ex post signal

- Note that $\Pr(\widetilde{v}=1|l)=1-q$ and $\Pr(\widetilde{v}=0|l)=q$.
- Therefore $P(+1,l)=\frac{\frac{1}{3}(1-q)}{\frac{1}{3}(1-q)+\frac{1}{9}q}=\frac{3(1-q)}{3-2q}$ and $P(-1,l)=\frac{\frac{1}{9}(1-q)}{\frac{1}{9}(1-q)+\frac{1}{3}q}=\frac{1-q}{1+2q}$.
- So the bid-ask spread

  $$P(+1,l)-P(-1,l)=\frac{3(1-q)}{3-2q}-\frac{1-q}{1+2q}=\frac{8q(1-q)}{(1+2q)(3-2q)}\le\frac{1}{2},$$

  as the inequality is equivalent to

  $$12(q-\frac{1}{2})^2\ge 0.$$

- In addition, $\frac{\partial(P(+1,h)-P(-1,h))}{\partial q}=-\frac{24(2q-1)}{(1+2q)^2(3-2q)^2}<0$. Therefore, higher disclosure quality reduces bid-ask spread when $y=l$.
- Therefore higher disclosure quality reduces bid-ask spread for any $y$.

# Discrete Kyle variant 2 - model setup

- Assume one noisy trader with the order flow $\widetilde{u} \in \{-1, 0, 1\}$ with distribution of $\Pr(\widetilde{u} = -1) = \Pr(\widetilde{u} = 0) = \Pr(\widetilde{u} = 1) = \frac{1}{3}$.
- Assume that the informed trader observes $\widetilde{v}$ and can buy or sell at most one share (e.g., because of cost considerations), i.e., $x(\widetilde{v}) \in \{-1, 0, 1\}$.
- However, assume that because of some exogenous reasons the informed trader only appears in the market with probability $p > 0$. With probability $1 - p$ there is no informed trader participation. In other words, $x(\widetilde{v}) = 0$ with probability $1 - p$.
- The market maker still observes total order flow $S = x(\widetilde{v}) + \widetilde{u} \in \{-2, -1, 0, 1, 2\}$ and chooses price $\widetilde{P} = E[\widetilde{v}|S]$.

# Discrete Kyle variant 2 - solving the model

- Again conjecture that the informed trader will choose $x(1) = +1$ and $x(0) = -1$, when they appears in the market.
- Therefore, $S = +2$ ($S = -2$) are fully revealing that the informed trader buys (sells). So $P(S = +2) = 1$ and $P(S = -2) = 0$.
- For all other possible values of $S$, still use Bayes' Rule:

$$
\begin{aligned}
\Pr(\widetilde{v} &= 1|S) \\
&= \frac{\Pr(S|\widetilde{v}=1)\Pr(\widetilde{v}=1)}{\Pr(S|\widetilde{v}=1)\Pr(\widetilde{v}=1)+\Pr(S|\widetilde{v}=0)\Pr(\widetilde{v}=0)} \\
&= \frac{\Pr(S|\widetilde{v}=1)}{\Pr(S|\widetilde{v}=1)+\Pr(S|\widetilde{v}=0)}.
\end{aligned}
$$

# Discrete Kyle variant 2 - solving the model

- When $S = +1$,
  - $\Pr(S = +1|\widetilde{v} = 1) = \frac{p}{3} + \frac{1-p}{3} = \frac{1}{3}$, as $S = +1$ is possible if 1) the informed trader is present, chooses $x = +1$, and the noisy trader submits $\widetilde{u} = 0$, which happens with probability $p \times \frac{1}{3}$; 2) the informed trader is not present so $x = 0$, and the noisy trader submits $\widetilde{u} = +1$, which happens with probability $(1 - p) \times \frac{1}{3}$.
  - $\Pr(S = +1|\widetilde{v} = 0) = \frac{1-p}{3}$, as $S = +1$ is only possible if the informed trader is not present so $x = 0$, and the noisy trader submits $\widetilde{u} = +1$, which happens with probability $(1 - p) \times \frac{1}{3}$.
  - Therefore $\Pr(\widetilde{v} = 1|S = +1) = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1-p}{3}} = \frac{1}{2-p}$.

# Discrete Kyle variant 2 - solving the model

- When $S = -1$,
  - $\Pr(S = -1|\widetilde{v} = 1) = \frac{1-p}{3}$, as $S = -1$ is only possible if the informed trader is not present so $x = 0$, and the noisy trader submits $\widetilde{u} = -1$, which happens with probability $(1-p) \times \frac{1}{3}$.
  - $\Pr(S = -1|\widetilde{v} = 0) = \frac{1}{3}$, as $S = -1$ is possible if 1) the informed trader is present, chooses $x = -1$, and the noisy trader submits $\widetilde{u} = 0$, which happens with probability $p \times \frac{1}{3}$; 2) the informed trader is not present so $x = 0$, and the noisy trader submits $\widetilde{u} = -1$, which happens with probability $(1-p) \times \frac{1}{3}$.
  - Therefore $\Pr(\widetilde{v} = 1|S = -1) = \frac{\frac{1-p}{3}}{\frac{1-p}{3} + \frac{1}{3}} = \frac{1-p}{2-p}$.

# Discrete Kyle variant 2 - solving the model

- When $S = 0$,
  - $\Pr(S = 0|\widetilde{v} = 1) = \frac{p}{3} + \frac{1-p}{3} = \frac{1}{3}$, as $S = 0$ is possible if 1) the informed trader is present, chooses $x = +1$, and the noisy trader submits $\widetilde{u} = -1$, which happens with probability $p \times \frac{1}{3}$; 2) the informed trader is not present so $x = 0$, and the noisy trader submits $\widetilde{u} = 0$, which happens with probability $(1 - p) \times \frac{1}{3}$.
  - $\Pr(S = 0|\widetilde{v} = 0) = \frac{p}{3} + \frac{1-p}{3} = \frac{1}{3}$, as $S = 0$ is possible if 1) the informed trader is present, chooses $x = -1$, and the noisy trader submits $\widetilde{u} = +1$, which happens with probability $p \times \frac{1}{3}$; 2) the informed trader is not present so $x = 0$, and the noisy trader submits $\widetilde{u} = 0$, which happens with probability $(1 - p) \times \frac{1}{3}$.
  - Therefore $\Pr(\widetilde{v} = 1|S = 0) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}} = \frac{1}{2}$.

# Discrete Kyle variant 2 - solving the model

- We therefore have $P(+1) = \frac{1}{2-p}$, $P(0) = \frac{1}{2}$, and $P(-1) = \frac{1-p}{2-p}$.
- The bid-ask spread is therefore $P(+1) - P(-1) = \frac{p}{2-p} > 0$.
- The intuition for the bid-ask spread is the same as before: information asymmetry between the market maker and the informed trader resulting in price protect of the market maker.

# Discrete Kyle variant 2 - implications

- The bid-ask spread is increasing in $p$ (probability of informed trading) as higher $p$ implies more information asymmetry - a justification for PIN as an empirical proxy for (il)liquidity.
- Disclosure should reduce information asymmetry and thus the bid-ask spread. HW problem 3 asks you to figure this out.
- Glosten and Milgrom (1985) uses similar trick: rather than assuming that the informed trader enters the market with an exogenous probability but noisy trader is always present, they assume that the order flow is always 1 but can come from either the informed trader or noisy trader.
- Assuming informed and noisy trader enter the market with some exogenously specified stochastic processes, they can then examine how bid-ask spread evolves over time; also when information asymmetry is so severe, the market breaks down (bid-ask spread becomes sufficiently large that nobody is willing to trade) - this involves some tweak of liquidity trader's order flows.

# Next class

- Feedback effect - the firm can learn from the market prices to guide real decisions (currently the firm cannot)
- Algebra becomes more complicated - which is why the discrete Kyle models are widely used.
- Price efficiency and real efficiency are not necessarily the same.