SINGULARITIES AND MIXING IN FLUID MECHANICS

By

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Abstract

Among the most important and most difficult open problems in the field of analysis are questions about the behavior of solutions to differential equations modeling the dynamics of fluids. The main issues that one must overcome in addressing them are frequently the nonlinearity and nonlocality of these equations. In this thesis we study these and related models, focusing on the possibility of singularity formation for their solutions as well as on ways such singular behavior can be suppressed.

In the first chapter of this thesis, we discuss the small scale creation and possible singularity formation in PDEs of fluid mechanics, especially the Euler equations and the related models. Recently, Tom Hou and Guo Luo proposed a new scenario, so called the hyperbolic flow scenario, for the development of a finite time singularity in solutions to 3D incompressible Euler equation. We first give a clear and understandable picture of hyperbolic flow restricted in 1D. Then, based on the recent work by Alexander Kiselev and Vladimir Šverák, we look into the hyperbolic geometry in 2D. Finally, we go back to 3D problem, and analyze a simplified 1D model for the potential singularity of the 3D Euler equation by Tom Hou and Guo Luo.

In the second chapter of this thesis, we investigate the problem about how to suppress the blowup. At the end of the second chapter, we demonstrate that incompressible mixing flow can indeed arrest the finite time blow up phenomenon. We first concentrate on understanding the mechanisms involved in mixing, studying mixing properties of the flows with different structure, and finding most efficient mixing flows. We resolve the problem of finding the optimal lower bound of the “mixing norm” under an enstrophy

constraint on the velocity field. On the basis of this result, we evaluate the role of mixing in systems where chemotaxis is present. We prove the result that the presence of fluid flow can affect singularity formation by mixing the density thus making concentration harder to achieve. This is an example to show that the fluid advection can regularize singular nonlinear dynamics.

This thesis resulted in the publications [31, 32, 48, 57, 87].

To my parents and my beloved wife

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Contents

Chapter 1

Singularity Formation for Some Active Scalar Equations

1.1 Introduction

The following transport equation

$$\omega_t + u \cdot \nabla \omega = 0. \tag{1.1}$$

is a basic mathematical model in fluid dynamics. If u depends on ω, (1.1) is called an active scalar equation. The problem of deciding whether blowup can occur for smooth initial data becomes very hard if the dependence of ω is nonlocal in space.

The relationship expressing u in terms of ω is commonly called Biot-Savart law. We have the following examples in 2D:

$$u = \nabla^{\perp}(-\triangle)^{-1}\omega, \tag{1.2}$$

where $\nabla^{\perp} = (-\partial_y, \partial_x)$ is the perpendicular gradient. Equations (1.1) and (1.2) are the vorticity form of 2D Euler equation. When we take

$$u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\omega,$$

(1.1) becomes the surface quasi-geostrophic (SQG) equation, which has important applications in geophysics, or can be regarded as a toy model for the 3D Euler equations.

For more details we refer to [19].

The 3D incompressible Euler equation describes the motion of an ideal (inviscid and incompressible) fluid. Whether the solution of this equation exists globally has remained open for over 250 years and has a close connection to the Clay Millennium Prize Problem on the Navier-Stokes equations [1]. Recently, Tom Hou and Guo Luo proposed a new scenario for the development of a finite time singularity in solutions to 3D axisymmetric Euler equation at a boundary point in [44]. This part of this thesis involves development of the analytical approach to understand this new numerical simulation based on analyzing some active scalar equations.

1.1.1 1D model

In [44], so called hyperbolic flow scenario was proposed to obtain singular solutions for the 3D Euler equations. The hyperbolic flow scenario in two dimensions can be explained in the following way. Consider e.g. a flow in the upper half-plane $\{x_2 > 0\}$. The essential properties required are (see Figure 1 for an illustration):

- There is a stagnant point of the flow at one boundary point (e.g. the origin) for all times.
- Along the boundary, the flow is essentially directed towards that point for all times.

Such flows can be created by imposing symmetry and other conditions on the initial data. For incompressible flows the stagnant point is a hyperbolic point of the velocity field, hence the name.

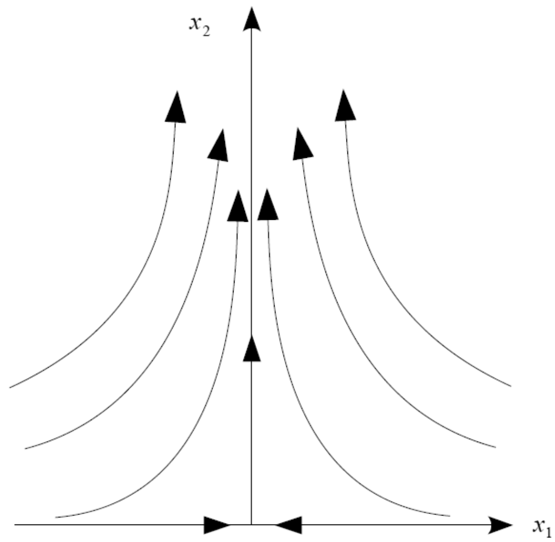


Figure 1 Illustration of hyperbolic flow scenario in two dimensions.

The scenario is a natural candidate for creating flows with strong gradient growth or finite-time blowup, since the fluid is compressed along the boundary. Due to non-linear and non-local interactions however, the flow remains hard to control, so a rigourous proof of blowup for the 3D Euler equations using hyperbolic flow remains a challenge.

One way to make progress in understanding and to gain insight into the hyperbolic blowup scenario is to study it in the context of one-dimensional model equations. This was begun in [12, 13], where one-dimensional models for the 2D-Boussinesq and 3D axisymmetric Euler equations were introduced and blowup was proven.

One-dimensional models capturing other aspects of fluid dynamical equations have a long-standing tradition, one of the earliest being the celebrated Constantin-Lax-Majda model [18]. We refer to the introduction of [80] for a more thorough review of known one-dimensional model equations, and to [12] for discussion of the aspects relating to the hyperbolic flow scenario. In section 1.2, we study a 1D model of this scenario.

1.1.2 2D Euler equation

The 1D models may help people to understand the hyperbolic scenario intuitively, nevertheless we still need to look into the incompressibility structure, which can hardly be modelled by 1D model. In section 1.3, we study the 2D Euler equation given by (1.1) and (1.2).

Global regularity of solutions to 2D Euler equation is well-known, it was first proved in [86] and [39], and see also [50], [68], [11] for more work. By the standard estimates (see [89]), to obtain the higher regularity of the solution corresponding to smooth initial data, we only need to bound the L^∞ norm of the gradient of vorticity. The best known upper bound for this quantity has double exponential growth in time. This result is well-known and has first appeared in [89].

The sharpness of the double exponential growth bound has remained open for a long time. In [90], Yudovich provided an example showing unbounded growth of the vorticity gradient at the boundary of the domain. Nadirashvili [73] constructed an example on the annulus where the gradient of vorticity grows linearly in time. In recent years, Denisov did a series of work on this problem. In [26], he constructed an example on the torus with superlinear growth in time. In [27], he proved that double exponential growth can be possible for any fixed but finite time. In [28], he constructed a patch solution to smoothly forced 2D Euler equation where the distance of two patches decreases double exponentially in time. We refer to [56] and [92] for more information on these questions.

In 2014, Kiselev and Šverák [56] constructed an example of initial data in the disk where such double exponential growth was observed. This means the double exponential growth in time is the best upper bound we can get for the gradient of vorticity, at least for solutions to 2D Euler equation in the disk. How generic is such growth is an interesting question. If this growth happens in many situations, it would highlight the challenges involved into numerical simulation of the solutions. In section 1.3, we make the first step towards better understanding of this question, by generalizing the construction of Kiselev and Šverák to the case of an arbitrary sufficiently regular domain with a symmetry axis. The main new ingredient of our proof is a more general version of the key lemma in [56]. The lemma captures hyperbolic structure of the velocity field near the point on the boundary where the fast growth happens. The analysis in [56] is specific to the disk, and the Green's function estimates used to prove the lemma do not translate easily to the more general case. We develop a more flexible natural construction that allows to carry the estimates through.

1.1.3 A modified 1D model

The main purpose of section 1.4 is to generalize the results of [12]. In [12], the authors derive a 1D model to study regularity for axisymmetric 3D Euler with swirl, which is the following system:

$$\partial_t \left(\frac{\omega^\theta}{r}\right) + u^r \left(\frac{\omega^\theta}{r}\right)_r + u^z \left(\frac{\omega^\theta}{r}\right)_z = -\left(\frac{(ru^\theta)^2}{r^4}\right)_z, \tag{1.3}$$

$$\partial_t(ru^\theta) + u^r(ru^\theta)_r + u^z(ru^\theta)_z = 0, \tag{1.4}$$

where u^r and u^z can be calculate by the following equations:

$$u^r = \frac{\psi_z}{r}, \tag{1.5}$$

$$u^z = -\frac{\psi_r}{r}, \tag{1.6}$$

where ψ satisfies the following elliptic equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial z^2} = \omega. \tag{1.7}$$

One can write u^r and u^z in terms of ω by computing the Green's function of the above elliptic PDE, more details can be found on [67].

The 1D model was inspired by the numerics presented in [44], which strongly suggest singularity formation on the boundary of a cylindrical domain. The 1D model suggested by Hou and Luo [44] to study the dynamics near the boundary is following:

$$\omega_t + u\omega_x = \theta_x \tag{1.8}$$

$$\theta_t + u\theta_x = 0 \tag{1.9}$$

$$u_x = H\omega \tag{1.10}$$

where H is the Hilbert transform and the space domain is taken to be $\mathbb{R}$ or $\mathbb{S}^1$(periodic). Equivalently, we can write u as

$$u(x,t) = k * \omega(x,t) \quad \text{where} \quad k(x) = \frac{1}{\pi}\log|x|. \tag{1.11}$$

In [12], blow up is shown for (1.8)-(1.10) for a large class of smooth initial data.

There are numerous related 1D models that have been used to study fluid equations. One of the earliest of these models was the one proposed by Constantin-Lax-Majda [18], which have later inspired other models [25] [21]. We refer the reader to [12] for a survey of results on this subject.

In section 1.4, for one of our results, we generalize their results to the model with the following choice of Biot-Savart law:

$$u(x,t) = k * \omega(x,t) \quad \text{where} \quad k(x) = \frac{1}{\pi}\log\frac{|x|}{\sqrt{x^2+a^2}}. \tag{1.12}$$

Using the reduction scheme from [12] to reduce 3D Euler to a 1D system, we will see that (1.12) is perhaps a more natural choice of Biot Savart law in section 1.4.1. The choice of (1.11) considerably simplifies estimates needed to prove finite time blow-up. In the section **??**, we prove blow up of the system (1.8) and (1.9) with law (1.12). The main task will be to prove estimates that allow the framework for the blow-up with (1.11) to remain.

For our second and closely related result, we prove the solutions to (1.8), (1.9) with even more generalized kernels can blow-up as well. We will modify (1.11) by adding a smooth function, which preserve the symmetries of (1.8), (1.9). The details will be in section **??**. To prove blow-up, in vague terms, we show that to "leading order" that the dynamics which lead to (1.8)-(1.10) to form singularity persist even with a more general law. Below, it will be apparent that our first result is not a direct corollary of our second result as we will see that the class of initial data used in the first result will be larger.

One can think of our results as strengthening the case for studying this family of equations. By proving singularity formation for more general Biot-Savart laws, one can view the blow up of (1.8)-(1.10) as not being solely the result of algebraic cancellations and manipulations due to the particular choice of Biot-Savart law, but as a more general phenomenon.

1.2 Hyperbolic fluid flow: 1D model equations

The results in this section come from a joint work with Tam Do, Vu Hoang and Maria Radosz [31].

1.2.1 Model discussion and the main results

In this section, we will study 1D models of (1.1) on $\mathbb{R}$ with the following two choices of u:

$$u_x = H\omega, \tag{1.13}$$

$$u = (-\Delta)^{-\frac{\alpha}{2}}\omega = -c_\alpha \int_{\mathbb{R}} |y-x|^{-(1-\alpha)}\omega(y,t)\, dy. \tag{1.14}$$

The choice (1.13) leads to a 1D analogue of the 2D Euler equation. One arrives at this model simply by restricting the dynamics to the boundary. In section 1.2.2 we give a brief heuristic argument which works by assuming that ω is concentrated in a small boundary layer.

We note that the model (1.13) was mentioned in [12], where it was stated that (1.13) has properties analogous to the 2D Euler equation, without giving details. In particular, in [12] a 1D model of the 2D Boussinesq equations (an extended version of (1.13)) was introduced and studied. One of our goals here is to validate the 1D model introduced in [12] in a setting where comparison with 2D results are available. The fact shown below, that the solutions to the model problem (1.13) behave similarly to the full 2D Euler case, provides support to the usefulness of the extended version of this model in [12] for getting insight into behavior of solutions to 2D Boussinesq system and 3D Euler equation.

The model defined by (1.14) is called α-patch model and appears in [35] (also with viscosity term). From the regularity standpoint, the α-patch model is between 1D Euler ($u_x = H\omega$) and the Cordoba-Cordoba-Fontelos model ($u = H\omega$) (see [21, 80]), which is an analogue of the SQG equation. These two models differ however from a geometric perspective, since the symmetry properties of the Biot-Savart laws are different. For the

CCF model, the velocity field is odd for even ω, whereas (1.14) is odd for odd ω. It is important to choose data with the right symmetry to make u odd, and thus to create a stagnant point of the flow at the origin for all times.

We note that local existence and blowup results for (1.14) were given in [35], where also dissipation is allowed. There the authors rely on a suitable Lyapunov function to show blowup, whereas we emphasize the more geometric aspects in this section. That is, we will be studying the analogue of the hyperbolic flow scenario for the above 1D models and show that this leads to natural and intuitive constructions of solutions with strong gradient growth and finite-time blowup.

Another blowup result related to hyperbolic flow was recently proven by A. Kiselev, L. Ryzhik, Y. Yao and A. Zlatoš [55] and concerns a α-patch model in 2D for small $\alpha > 0$.

In this section, we will prove the following theorems.

Theorem 1.2.1. *The solution ω to (1.1) and (1.13) satisfies the following inequality:*

$$\|\omega_x\|_{L^\infty} \leqslant C(1 + \|(\omega_0)_x\|) \exp(e^{Ct}),$$

for some universal C, provided that the initial data ω_0 is smooth. And there is ω_0 such that the equality holds.

Theorem 1.2.2. *There exists smooth initial data ω_0, such that the solution to (1.1) and (1.14) blows up in finite time.*

1.2.2 Euler 1D model

1.2.2.1 Heuristic derivation.

Recall the 2D Euler equations in vorticity form

$$\omega_t + u \cdot \nabla \omega = 0$$

where $u = \nabla^{\perp}(-\Delta)^{-1}\omega$.

We first indicate a simple heuristic motivation for the choice (1.13) (see also [12]). Consider the 2D Euler equation in a half-space $\{x_2 \geqslant 0\}$ and denote $\overline{x} = (x_1, -x_2)$. The x_1-component of the velocity (up to a normalization constant) for compactly supported vorticity ω is given by

$$u_1(x,t) = -\int_{\mathbb{R}^2} \frac{(y_2 - x_2)}{|y-x|^2}\omega(y,t)\ dy \tag{1.15}$$

where ω has been extended to $\{x_2 \leqslant 0\}$ by odd reflection ($\omega(\overline{x},t) = -\omega(x,t)$).

Suppose now that ω is concentrated in a boundary layer of width $a > 0$ and that $\omega(x_1, x_2, t) = \omega(x_1, t)$ in this boundary layer. Then a calculation gives

$$u_1(x_1, 0, t) = -2\int_{\mathbb{R}} \log\left(\frac{(y_1 - x_1)^2 + a^2}{(y_1 - x_1)^2}\right)\omega(y_1, t)\ dy_1. \tag{1.16}$$

If we now retain only the singular part of the kernel $\log\left(\frac{z^2+a^2}{z^2}\right) \sim -2\log|z|$ and identify u with u_1, we get (dropping the constants)

$$u(x,t) = \int_{\mathbb{R}} \log|y - x|\omega(y,t)\ dy.$$

So a reasonable 1D model is

$$\omega_t + u\omega_x = 0, \quad u_x = H\omega, \quad \omega(x,0) = \omega_0(x) \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{1.17}$$

where H is the Hilbert transform, using the convention

$$H\omega(x,t) = P.V.\int \frac{\omega(y,t)}{x-y}\,dy.$$

For this model, we have the following local well-posedness property:

Proposition 1.2.3. *Given initial data* $\omega_0 \in H_0^m((0,1))$ *with* $m \geqslant 2$*, there exists* $T = T(\|\omega_0\|_{H_0^m}) > 0$ *such that the system has a unique classical solution* $\omega \in C([0,T]; H_0^m)$*.*

The proof is standard so we skip it here.

An alternative argument to motivate (1.13) is to observe that the gradient of the 2D Euler velocity is given by a zero-th order operator acting on ω. In one dimension, this leaves only the choice $u_x = cH\omega$ or $u_x = c\omega$, c being a nonzero constant. So we could also consider the model

$$\omega_t + u\omega_x = 0, \quad u_x = -\omega, \quad \omega(x,0) = \omega_0(x) \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{1.18}$$

(1.18) is however not a close analogue of 2D Euler (see remark 1.2.7).

1.2.2.2 Sharp a-priori bounds for gradient growth.

We will first prove the global regularity of the solution to equation (1.17) by showing that ω_x can grow at most with double exponential rate in time. Then we will give an example of a smooth solution to (1.17) where such growth of the gradient of ω is achieved, meaning the bound is sharp.

Due to the Biot-Savart law relating u and ω, the proof of an upper bound for $\|\omega_x(\cdot,t)\|_\infty$ is very similar to the proof for the full 2D Euler equations. For the reader's convenience, we give the proof. Recall first the definition of the Hölder norm

$$||\omega||_{C^\alpha} = \sup_{|x-y|\leqslant 1, x\neq y} \frac{|\omega(x)-\omega(y)|}{|x-y|^\alpha}$$

for compactly supported ω.

We will need an estimate on the Hilbert transform:

Lemma 1.2.4. *Let* $0 < \alpha < 1$. *Suppose* $\operatorname{supp}(\omega) \subset [-D(t), D(t)]$ *and assume without loss of generality that* $||\omega_0||_{L^\infty} = 1$. *Then*

$$\|u_x\|_\infty \leqslant C(\alpha)\left(1 + |\log(D(t))| + \log(1 + \|\omega\|_{C^\alpha}\right))$$

Proof. For any $\delta > 0$, we have

$$\left|\int_{[-D(t),D(t)]\setminus(x-\delta,x+\delta)} \frac{\omega(y)}{x-y}\, dy\right| \leqslant C\int_\delta^{D(t)} \frac{1}{y}\, dy \leqslant C(|\log\delta| + |\log(D(t))|).$$

Using the oddness of $\frac{1}{x}$, we have

$$\left|P.V.\int_{x-\delta}^{x+\delta} \frac{\omega(y)}{x-y}\, dy\right| = \left|\int_{x-\delta}^{x+\delta} \frac{1}{x-y}(\omega(y) - \omega(x))\, dy\right| \leqslant C(\alpha)\|\omega(x,t)\|_{C^\alpha}\delta^\alpha.$$

Choosing $\delta = \min\left\{1, (\frac{1}{\|\omega\|_{C^\alpha}})^{\frac{1}{\alpha}}\right\}$, we get the desired estimate of $\|u_x\|_\infty$. □

The following lemma gives an estimate on $D(t)$.

Lemma 1.2.5. *Suppose the support of* ω_0 *is in* $[-1, 1]$ *and* $||\omega_0||_{L^\infty} = 1$. *Then the support of* $\omega(x, t)$ *will be inside* $[-C\exp(Ce^{Ct}), C\exp(Ce^{Ct})]$, *for some universal constant* $C > 0$.

Proof. Suppose $\operatorname{supp}\omega = [-D(t), D(t)]$. Then for any point x inside of this interval, we have

$$|u(x)| \leqslant \int_{-D(t)}^{D(t)} |\log|x-y||\, dy \leqslant C\int_0^{2D(t)} |\log|s||\, ds \leqslant CD(t)(|\log(D(t))| + 1).$$

By following the trajectory of the particle at $D(t)$,

$$D'(t) \leqslant CD(t)(|\log(D(t))| + 1).$$

A simple argument using differential inequalities shows that $D(t)$ is always less than $z(t)$, where $z(t)$ is the solution of

$$z'(t) = Cz(t)(\log z(t) + 1), \quad z(0) = \min\{D(0), 2\}.$$

This yields the double-exponential upper bound on $D(t)$. □

The following Theorem gives the double exponential upper bound for ω_x.

Theorem 1.2.6. *There is universal constant C such that if ω_0 is smooth, compactly supported with* $\operatorname{supp}\omega_0 \subset [-1, 1]$ *and* $\|\omega\|_{L^\infty} = 1$,

$$\log(1 + \|\omega_x\|_{L^\infty}) \leqslant C \log(1 + \|(\omega_0)_x\|_{L^\infty})e^{Ct} \quad (t \geqslant 0). \tag{1.19}$$

Proof. We follow the proof in [56]. Let us denote the flow map corresponding to the evolution by $\Phi_t(x)$. Then

$$\frac{\partial}{\partial t}\Phi_t(x) = u(\Phi_t(x), t), \quad \Phi_0(x) = x,$$

and

$$\left| \frac{\partial_t |\Phi_t(x) - \Phi_t(y)|}{|\Phi_t(x) - \Phi_t(y)|} \right| \leqslant ||u_x||_{L^\infty}.$$

After integration, and by Lemma 1.2.4 and Lemma 1.2.5, this gives

$$f(t)^{-1} \leqslant \frac{|\Phi_t(x) - \Phi_t(y)|}{|x - y|} \leqslant f(t),$$

where

$$f(t) = \exp\left(C \int_0^t (1 + \exp(Cs) + \log(1 + ||\omega_x||_{L^\infty}))\, ds \right).$$

This bound also holds for Φ_t^{-1}. On the other hand,

$$\|\omega_x\|_{L^\infty} = \sup_{x \neq y} \frac{|\omega_0(\Phi_t^{-1}(x)) - \omega_0(\Phi_t^{-1}(y))|}{|x - y|} \leqslant \|(\omega_0)_x\| \sup_{x \neq y} \frac{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|}{|x - y|}.$$

Which means we have

$$(1 + \|\omega_x\|_{L^\infty}) \leqslant (1 + \|(\omega_0)_x\|_{L^\infty}) \exp\left(C \int_0^t 1 + \exp(Cs) + \log(1 + \|\omega_x\|_{L^\infty})\ ds \right),$$

or

$$\log(1 + \|\omega_x\|_{L^\infty}) \leqslant \log(1 + \|(\omega_0)_x\|_{L^\infty}) + C\exp(Ct) + C\int_0^t (1 + \log(1 + \|\omega_x\|_{L^\infty}))\ ds.$$

So $y(t) := \log(1 + \|\omega_x\|_{L^\infty})$ satisfies the integral inequality

$$y(t) \leqslant y(0) + Ce^{Ct} + \int_0^t (1 + y(s))\ ds$$

and by the integral form Gronwall's inequality and some elementary manipulations, we arrive at the bound $y(t) \leqslant C_1 y(0) e^{C_2 t}$. This yields the desired bound on $\|\omega_x\|_\infty$. □

Remark 1.2.7. *If we choose our Biot-Savart law to be $u_x = -\omega$, then from a modification of the above proof we get an exponential upper bound for $\|\omega_x\|_{L^\infty}$. This is different from the 2D Euler equation, which suggests that* (1.17) *is a better analogue of the 2D Euler equation than* (1.18)*. Moreover the equation* (1.18) *also has different symmetry properties.*

Next we construct initial data ω_0 such that $\|\omega_x(\cdot, t)\|_{L^\infty}$ grows with double-exponential rate, proving the sharpness of the a-priori bound (1.19). The hyperbolic flow scenario is created in the following way: First, we require that the initial data ω_0 is odd with respect to the origin, and has compact support. By Proposition 1.2.3, the oddness is easily seen to be preserved by the evolution. Consequently, the velocity field (which is also an odd function) can be written as

$$u(x,t) = -x \int_0^\infty K\left(\frac{x}{y}\right) \frac{\omega(y,t)}{y}\ dy \quad (x > 0), \tag{1.20}$$

where

$$K(s) := \frac{1}{s} \log \left| \frac{s+1}{s-1} \right| . \tag{1.21}$$

Note that the origin is a stagnant point of the flow for all times. By taking ω_0 to be positive on the right, the direction of the flow is towards the origin. More precisely, ω_0 is defined as follows (see Figure 2):

- Let ω_0 be supported on $[-1, 1]$, smooth and odd. Choose numbers $0 < x_1(0) < 2x_2(0) < 1$ such that $Mx_1(0) \leqslant x_2(0)$ where M will be determined later. Require that ω_0 is increasing on $[0, x_1(0)]$, decreasing on $[x_2(0), 1]$ and identically 1 on $[x_1(0), x_2(0)]$.

Using the earlier notation Φ_t for the flow map associated to (1.17), let

$$x_1(t) := \Phi_t(x_1(0))$$

$$x_2(t) := \Phi_t(x_2(0))$$

It is easy to see that the general structure of ω_0 will be preserved by the flow: For fixed t, $\omega(x, t)$ will be increasing on $[0, x_1(t)]$, decreasing on $[x_2(t), 1]$ and identically 1 on $[x_1(t), x_2(t)]$. In fact, since $u(x, t) \leqslant 0$ for $x \geqslant 0$, $x_1(t)$ and $x_2(t)$ will be moving towards the origin in time. We will show that the quantity $\frac{x_2(t)}{x_1(t)}$ increases double exponentially in time. This is sufficient to conclude the desired growth of $\|\omega_x(\cdot, t)\|_{L^\infty}$.

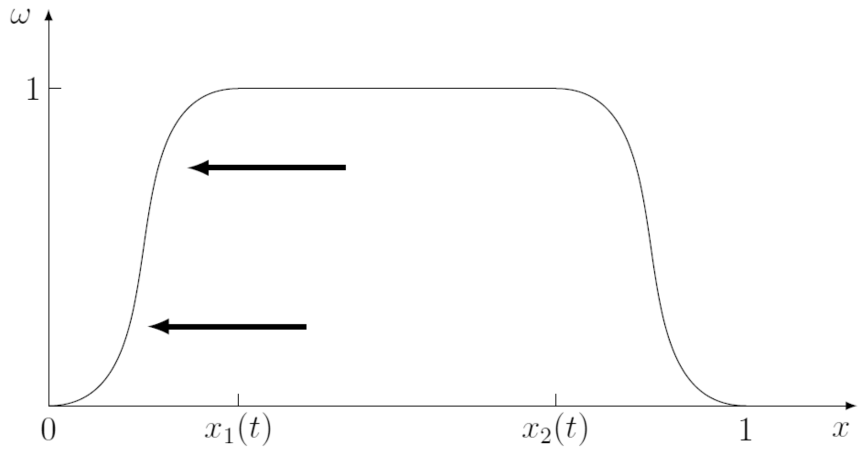


Figure 2 Structure of $\omega(x,t)$.

Theorem 1.2.8. *Assume our initial data is defined as above, then*

$$\log\frac{x_2(t)}{x_1(t)} \geqslant \log\frac{x_2(0)}{x_1(0)}\exp(Ct) \quad (t>0),$$

for some positive constant C. As a consequence,

$$\log\|\omega_x(\cdot,t)\|_{L^\infty} \geqslant C_1\exp(C_2 t) \quad (t>0)$$

for some $C_1, C_2 > 0$.

Theorem 2.4 quickly follows from the following lemma:

Lemma 1.2.9. *Suppose $1 \geqslant x_2 \geqslant 8x_1$. There are universal constants C_0 and C_1 so that*

$$\frac{d}{dt}\left(\frac{x_2}{x_1}\right) \geqslant C_1\frac{x_2}{x_1}\left(\log\left(\frac{x_2}{x_1}\right) - C_0\right).$$

Proof. First observe

$$\begin{aligned}\frac{d}{dt}\left(\frac{x_2}{x_1}\right) &= \frac{x_2'x_1 - x_1'x_2}{x_1^2} = \frac{u(x_2)x_1 - u(x_1)x_2}{x_1^2} = \frac{x_2}{x_1}\left(\frac{u(x_2)}{x_2} - \frac{u(x_1)}{x_1}\right)\\ &= \frac{x_2}{x_1}\int_0^1\left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right]\frac{\omega(y)}{y}\,dy.\end{aligned}$$

We decompose the integral into 4 pieces which we will estimate separately:

$$
\begin{aligned}
&\int_0^1 \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy \\
=&\int_0^{2x_1} + \int_{2x_1}^{\frac{1}{2}x_2} + \int_{\frac{1}{2}x_2}^{2x_2} + \int_{2x_2}^{1} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy \\
=&I + II + III + IV.
\end{aligned}
$$

For I, we use $0 \leqslant \omega(y) \leqslant 1$ and $2x_1 \leqslant x_2 \leqslant 1$:

$$
\begin{aligned}
0 \leqslant I \leqslant &\int_0^{2x_1} \frac{1}{x_1} \log \frac{(x_1+y)}{|x_1-y|}\, dy + \int_0^{2x_1} \frac{1}{x_2} \log \frac{(x_2+y)}{|x_2-y|}\, dy \\
=& \frac{1}{x_1} 3x_1 \log 3 + \frac{1}{x_2}\left[2x_1 \log \frac{1+\frac{2x_1}{x_2}}{1-\frac{2x_1}{x_2}} + x_2 \log\left(1 - \frac{2x_1}{x_2}\right) + x_2 \log\left(1 + \frac{2x_1}{x_2}\right)\right] \\
\leqslant& 3\log 3 + 2\log 2.
\end{aligned}
$$

Using the fact that $K(s)$ is increasing in $[0,1)$ and decreasing in $(1,\infty]$ and that $\omega(y) = 1$ for $y \in (2x_1, \frac{1}{2}x_2)$ we get

$$
\begin{aligned}
II =& \int_{2x_1}^{\frac{1}{2}x_2} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy \geqslant \int_{2x_1}^{\frac{1}{2}x_2} (2 - \frac{1}{2}\log(3)) \frac{1}{y}\, dy \\
=& (2 - \frac{1}{2}\log(3)) \log\left(\frac{x_2}{x_1}\right) - C.
\end{aligned}
$$

Using the positivity of K,

$$
\begin{aligned}
III =& \int_{\frac{1}{2}x_2}^{2x_2} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy \geqslant -\int_{\frac{1}{2}x_2}^{2x_2} K\left(\frac{x_2}{y}\right) \omega(y) \frac{1}{y}\, dy \\
\geqslant& -\int_{\frac{1}{2}}^{2} \frac{1}{s^2} \log \frac{|s+1|}{|s-1|}\, ds \geqslant -C.
\end{aligned}
$$

We estimate IV in the following way, using that $\omega(y) \leqslant 1$ and $\frac{x_1}{y} \leqslant \frac{x_2}{y} \leqslant 1$ for $2x_2 \leqslant$

$y \leqslant 1$:

$$\begin{aligned}|IV| &= \left|\int_{2x_2}^{1} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy\right| \leqslant \int_{2x_2}^{1} \left[K\left(\frac{x_2}{y}\right) - K\left(\frac{x_1}{y}\right)\right] \frac{1}{y}\, dy \\ &\leqslant \int_{2x_2}^{1} \frac{1}{x_2} \log\frac{y+x_2}{y-x_2}\, dy - \int_{2x_2}^{1} \frac{1}{x_1} \log\frac{y+x_1}{y-x_1} \\ &= (i) - (ii).\end{aligned}$$

We can compute (i) directly and get

$$(i) = \frac{1}{x_2}\log\frac{1+x_2}{1-x_2} + \log(1+x_2)(1-x_2) - 2\log(x_2) - 3\log(3).$$

Similarly, for (ii), we have

$$(ii) = \frac{1}{x_1}\log\frac{1+x_1}{1-x_1} + \log(x_1+1)(1-x_1) - 2\frac{x_2}{x_1}\log\frac{2x_2+x_1}{2x_2-x_1} - \log(2x_2+x_1)(2x_2-x_1).$$

Then using that $x_1 < x_2$,

$$|IV| \leqslant C - 2\log(x_2) + \log(4x_2^2 - x_1^2) = C - \log\left(4 - \left(\frac{x_1}{x_2}\right)^2\right) \leqslant C.$$

□

The proof of Theorem 1.2.8 is now completed as follows: choose $M > 8$ so large such that $\frac{1}{2}\log(M) - C_0 \geqslant 0$. We have thus $\frac{1}{2}\log\left(\frac{x_2^0}{x_1^0}\right) - C_0 \geqslant 0$. From Lemma 1.2.9 it follows that $\frac{x_2(t)}{x_1(t)}$ is growing in time and that we have

$$\frac{d}{dt}\left(\frac{x_2}{x_1}\right) \geqslant \frac{C_1}{2}\frac{x_2}{x_1}\log\left(\frac{x_2}{x_1}\right),$$

or $\frac{d}{dt}\log\left(\frac{x_2}{x_1}\right) \geqslant \frac{C_1}{2}\log\left(\frac{x_2}{x_1}\right)$ for all times. This clearly implies that $\frac{x_2}{x_1}$ grows double-exponentially.

Remark 1.2.10. *In [56], the Biot-Savart law is decomposed into a main contribution and an error term. In our case* (1.20)*, the main contribution would be*

$$-x \int_x^\infty \frac{\omega(y)}{y}\, dy. \tag{1.22}$$

If we replace (1.20) *by* (1.22)*, then double-exponential growth of* $\frac{x_2}{x_1}$ *can be proven by a straightforward argument. In this case, the computation for the estimate in Lemma 1.2.9 becomes much easier.*

1.2.3 α-patch 1D model

In this section, we consider the 1D model equation

$$\omega_t + u\omega_x = 0 \tag{1.23}$$

with a different Biot-Savart law

$$u(x,t) = (-\Delta)^{-\alpha/2}\omega(x,t) = -c_\alpha \int_{\mathbb{R}} |y-x|^{-(1-\alpha)}\omega(y,t)\, dy, \qquad \alpha \in (0,1) \tag{1.24}$$

For convenience, we will assume the constant c_α associated with the fractional Laplacian is 1, and we write $\gamma = 1-\alpha$.

This problem has been studied in [35], where local existence and uniqueness results for smooth initial data are proven. From these, we can show that this equation preserves oddness and $u(0,t) = 0$ holds with odd initial datum. For odd data, we can write

$$u(x,t) = -\int_0^\infty k(x,y)\omega(y,t)\, dy \tag{1.25}$$

where $k(x,y) = |y-x|^{-\gamma} - |y+x|^{-\gamma}$. Note that $k(x,y) \geqslant 0$ for $x \neq y \in (0,\infty)$.

Following similar ideas as for 1D Euler, we specifiy our initial data ω_0 as follows:

- Pick $0 < x_1(0), x_2(0)$ with $Mx_1(0) < x_2(0)$. Let ω_0 be smooth, odd, $\omega_0(x) \geqslant 0$ for $x > 0$ and have its support in $[-2x_2(0), 2x_2(0)]$. $M > 1$ is to be chosen below. Let ω_0 moreover be bounded by 1, smoothly increasing in the interval $[0, x_1(0)]$ and $\omega_0 = 1$ between $x_1(0)$ and $x_2(0)$.

As long as the solution remains smooth, the general structure of the solution does not change. Let $x_1(t), x_2(t)$ be again the position of the particles starting at $x_1(0), x_2(0)$.

Theorem 1.2.11. *There exists a choice of $x_1(0), x_2(0), M$ and a time $T > 0$ such that the smooth solution of* (1.23) *for the above initial data cannot be continued beyond T. Provided the solution remains smooth on the time interval $[0, T)$, the particle starting at $x_1(0)$ reaches the origin at time $t = T$, i.e.*

$$\lim_{t \to T} x_1(t) = 0. \tag{1.26}$$

In this sense, the solution forms a "shock".

Remark 1.2.12. *In [35], the existence of blowup solutions to* (1.23) *is shown using energy methods. The advantage is that they are able to include a dissipation term. The drawback of energy methods in general, however, is that the blowup mechanism is obscured. Our proof uses the dynamics of the solution and gives a more intuitive picture of the blowup, and is easily generalized to other even kernels having the same singular behavior.*

In the rest of this section, we will prove Theorem 1.2.11. First of all, we track the movement of the particle starting at $x_1(0)$, which is the following lemma.

Lemma 1.2.13. *There exists a universal constant $M > 2$ so that if $Mx_1(t) \leqslant x_2(t)$,* *the velocity at $x_1(t)$ will satisfy*

$$u(x_1(t), t) \leqslant -Cx_1(t)^{1-\gamma}, \tag{1.27}$$

for some universal constant C.

Proof. Let $u_1 = u(x_1(t), t)$. Since $k, \omega \geqslant 0$ on $(0, \infty)$

$$\begin{aligned}
-u_1 &\geqslant \int_{2x_1}^{x_2} k(x_1, y)\, dy \\
&= c_\gamma \left[-(x_2 + x_1)^{1-\gamma} + (x_2 - x_1)^{1-\gamma} + (3x_1)^{1-\gamma} - x_1^{1-\gamma}\right] \\
&= c_\gamma \left[(3^{1-\gamma} - 1)x_1^{1-\gamma} + (x_2 - x_1)^{1-\gamma} - (x_2 + x_1)^{1-\gamma}\right] \\
&= c_\gamma x_1^{1-\gamma} \left[(3^{1-\gamma} - 1) + \frac{1}{x_1^{1-\gamma}} \left((x_2 - x_1)^{1-\gamma} - (x_2 + x_1)^{1-\gamma}\right)\right]
\end{aligned}$$

for some constant $c_\gamma > 0$. Note that $(3^{1-\gamma} - 1) > 0$. We can write

$$\begin{aligned}
\frac{1}{x_1^{1-\gamma}}(x_2 - x_1)^{1-\gamma} - (x_2 + x_1)^{1-\gamma} &= \frac{x_2^{1-\gamma}}{x_1^{1-\gamma}} \left[\left(1 - \frac{x_1}{x_2}\right)^{1-\gamma} - \left(1 + \frac{x_1}{x_2}\right)^{1-\gamma}\right] \\
&=: \frac{x_2^{1-\gamma}}{x_1^{1-\gamma}} f(x_1/x_2).
\end{aligned}$$

There exists a constant $C > 0$ with $|f(x_1/x_2)| \leqslant C|x_1/x_2|$ for $|x_1/x_2| \leqslant 1/2$, and so

$$-u_1 \geqslant c_\gamma x_1^{1-\gamma} \left[(3^{1-\gamma} - 1) - CM^{-\gamma}\right]$$

if $Mx_1(t) \leqslant x_2(t)$. Now choose M large enough so that $CM^{-\gamma}$ is smaller than the number $\frac{1}{2}(3^{1-\gamma} - 1)$. □

This estimate of velocity field will lead to a blowup in finite time, provided we can show $Mx_1(t) \leqslant x_2(t)$. More precisely,

$$\frac{d}{dt}x_1(t) = u(x_1) \leqslant -Cx_1^{1-\gamma},$$

implying

$$x_1 \leqslant C(x_1(0)^\gamma - Ct)^{\frac{1}{\gamma}}.$$

This shows that no later than $T_0 := C^{-1}x_1(0)^\gamma$, the particle $x_1(t)$ will reach the origin, and the solution cannot be continued smoothly. Note that T_0 does not depend on $x_2(0)$.

It remains therefore to control the motion of $x_2(t)$, concluding the proof.

Lemma 1.2.14. *For $x_2(0)$ large enough, $Mx_1(t) < x_2(t)$ for $t \in [0, T_0)$.*

Proof. We write $u(x_2(t), t) = u_2$. Observe that the support of $\omega(\cdot, t)$ is always contained in $[-2x_2(0), 2x_2(0)]$ because of $u(x, t) \leqslant 0$ for $x > 0$.

Next we find an upper bound on u_2:

$$|u_2(t)| \leqslant \int_{-2x_2(0)}^{2x_2(0)} |y - x|^{-\gamma} \leqslant Cx_2(0)^{1-\gamma}. \tag{1.28}$$

Hence,

$$x_2(t) \geqslant x_2(0) - \int_0^{T_0} |u_2(s)|\, ds \geqslant x_2(0)(1 - Cx_2(0)^{-\gamma}T_0). \tag{1.29}$$

Now choose $x_2(0)$ so large that $Mx_1(0) < x_2(0)(1 - Cx_2(0)^{-\gamma}T_0)$. But then

$$Mx_1(t) \leqslant Mx_1(0) < x_2(0)(1 - Cx_2(0)^{-\gamma}T_0) \leqslant x_2(t),$$

giving the statement of the Lemma. □

1.3 Hyperbolic fluid flow: fast growth in 2D Euler equation

The results in this section come from [87].

1.3.1 Main result

Recall the two dimensional Euler equation in vorticity form for an incompressible fluid is given by

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \quad \omega(x,0) = \omega_0(x). \tag{1.30}$$

Here $\omega = \text{curl } u$ is the vorticity of the flow, and the velocity u can be determined from ω by the Biot-Savart law. Here we consider the flow in a smooth bounded domain Ω, and we assume u satisfies a no flow boundary condition, namely $u \cdot n = 0$ on $\partial\Omega$, here n is the outer normal vector of $\partial\Omega$. This implies that

$$u = \nabla^{\perp} \int_{\Omega} G_{\Omega}(x,y)\omega(y)dy.$$

Where $G_{\Omega}(x,y)$ is the Green's function of the Dirichlet problem in Ω and $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$ is the perpendicular gradient.

In this section, we will prove the following double exponential growth bound for more general domains instead of disk:

Theorem 1.3.1. *Let Ω be a C^3 bounded open domain in $\mathbb{R}^2$, tangent at the origin to the x_1-axis and symmetric about x_2-axis. Consider the 2D Euler equation on Ω. There exists a smooth initial data ω_0 with $\|\nabla\omega_0\|_{L^\infty} > \|\omega_0\|_{L^\infty}$ such that the corresponding solution $\omega(x,t)$ satisfies*

$$\frac{\|\nabla\omega(\cdot,t)\|_{L^\infty}}{\|\omega_0\|_{L^\infty}} \geqslant \left(\frac{\|\nabla\omega_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\right)^{c(\Omega)\exp(c(\Omega)\|\omega_0\|_{L^\infty} t)} \tag{1.31}$$

for some $c(\Omega) > 0$ that only depends on Ω and for all $t \geqslant 0$.

Based on the ideas of [56], we will focus on the appropriate representation for the Biot-Savart law for the fluid velocity u. The key will be obtaining the representation which will show that $u_1 \sim x_1 \log(x_1)$ on appropriate (time dependent) length scales. After that, based on the construction of [56], we will get the desired growth for the gradient of vorticity in domain Ω.

1.3.2 The key lemma

To construct a flow with fast growth in the gradient, we need a technical lemma for the expansion of velocity field.

We use the same notation as in [56], which means ω is the vorticity field in Ω, odd in the first variable. Note that this property is conserved by Euler evolution in a symmetric domain. Let u be the corresponding velocity field. $D^+ = \{x \in \Omega : x_1 > 0\}, \tilde{x} = (-x_1, x_2)$ and $Q(x_1, x_2)$ is a region that is the intersection of D^+ and the quadrant $\{y : x_1 \leqslant y_1 < \infty,\ x_2 \leqslant y_2 < \infty\}$.

Lemma 1.3.2. *[Key Lemma.] Suppose Ω is a C^3 bounded open domain in $\mathbb{R}^2$, symmetric about x_2-axis, and tangent to x_1-axis at the origin. Take any γ, $\frac{\pi}{2} > \gamma > 0$. Denote D_1^γ the intersection of D^+ with a sector $\frac{\pi}{2} - \gamma \geqslant \phi \geqslant -\frac{\pi}{2}$, where ϕ is the usual angular variable. Then there exists $\delta > 0$ such that for all $x \in D_1^\gamma$ such that $|x| \leqslant \delta$ we have*

$$u_1(x,t) = -\frac{4}{\pi} x_1 \int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4} \omega(y,t) dy + x_1 B_1(x,t), \tag{1.32}$$

where $|B_1(x,t)| \leqslant C(\gamma,\Omega) \|\omega_0\|_{L^\infty}$.

Similarly, if we denote D_2^γ the intersection of D^+ with a sector $\frac{\pi}{2} \geqslant \phi \geqslant \gamma$, then for all $x \in D_2^\gamma$ such that $|x| \leqslant \delta$, we have

$$u_2(x,t) = \frac{4}{\pi} x_2 \int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4} \omega(y,t) dy + x_2 B_2(x,t). \tag{1.33}$$

where $|B_2(x,t)| \leqslant C(\gamma,\Omega) \|\omega_0\|_{L^\infty}$.

In the argument below, δ and constant C may change from line to line, but since we change it only for finite many times, at the end we choose the smallest δ and the biggest C among all of them instead.

We want to define y^* to be the mirror image point of y, about the boundary. Namely, we want y^* to be written as $y^* = 2e(y) - y$, where $e(y)$ is a point on $\partial\Omega$ so that $y - e(y)$ is orthogonal to the tangent line at $e(y)$. More intuitively, $e(y)$ is the closest point to y on $\partial\Omega$. However, it is not clear that this $e(y)$ is well-defined. Given y, it may be possible to find more than one $e(y)$ on the whole $\partial\Omega$. However, we will show $y^*(y)$ is locally well-defined close to the origin by lemma 1.3.3 below.

Since $\partial\Omega$ is tangent to the x_1 axis at the origin, we can choose the parameterization near the origin of $\partial\Omega$ such that we have $\partial\Omega = (s, f(s))$ for some function $f \in C^3$ and sufficiently small s.

Lemma 1.3.3. *There exists $r = r(\Omega) > 0$ only depending on Ω, so that for any $y \in B_r(0) \cap \Omega$, there is a unique s in $(-2r, 2r)$ such that the following equation holds:*

$$(s - y_1) + f'(s)(f(s) - y_2) = 0. \tag{1.34}$$

Moreover, this s, as a function of y, is C^2.

Remark 1.3.4. *If we call $e(y) = (s(y), f(s(y)))$, then (1.34) means $y - e(y)$ is orthogonal to the tangent line of $\partial\Omega$ at $e(y)$. Note that if $e(y)$ is one of the points on $\partial\Omega$ closest to y then (1.34) holds.*

Proof of lemma 1.3.3. We call the left hand side of (1.34) the function $F(s, y)$. We take derivative of $F(s, y)$ in s and get

$$1 + f'(s)^2 + f''(s)(f(s) - y_2). \tag{1.35}$$

So $D_sF(0,(0,0)) = 1$ and $F(0,(0,0)) = 0$. Thus by implicit function theorem we know the solution to (1.34) exists and is unique in a neighborhood of $\{0\} \times \{(0,0)\}$. By choosing r small enough, $s(y)$ is uniquely defined. Moreover, as in the very beginning we choose $\partial\Omega$ to be C^3, which means f is C^3. Hence, $F(s,y)$ is a C^2 function as it only contains function f' and f, which means $s(y)$ is C^2 in y by the implicit function theorem. □

By lemma 1.3.3, we can give the following definition:

Definition 1.3.5. *Given r small enough, for any $|y| \leqslant r$, we define $e(y)$ to be the only point in $B_{2r}(0) \cap \partial\Omega$ so that $e(y) - y$ is orthogonal to the tangent line of $\partial\Omega$ at $e(y)$. And we define $y^*(y) = 2e(y) - y$. We denote $y^*(y) = (y_1^*(y), y_2^*(y)) = (y_1^*, y_2^*)$.*

Then, we have the following lemma.

Lemma 1.3.6. *Take r small enough as in lemma 1.3.3. If $|s_0| \leqslant 2r$ and $y \in B_r(0) \cap \Omega$, we have*

$$\begin{aligned} y_1^* - s_0 &= \frac{1 - f'(s_0)^2}{1 + f'(s_0)^2}(y_1 - s_0) + \frac{2f'(s_0)}{1 + f'(s_0)^2}(y_2 - f(s_0)) + O(|y - (s_0, f(s_0))|^2), \\ y_2^* - f(s_0) &= \frac{2f'(s_0)}{1 + f'(s_0)^2}(y_1 - s_0) + \frac{f'(s_0)^2 - 1}{1 + f'(s_0)^2}(y_2 - f(s_0)) + O(|y - (s_0, f(s_0))|^2). \end{aligned} \tag{1.36}$$

Here the constant in capital O notation only depends on the domain Ω. In addition the map y^* is invertible in y in $B_r(0) \cap \Omega$.

Proof of lemma 1.3.6. This is an elementary calculation by Taylor's expansion formula. Like in the proof of lemma 1.3.3, we call the left hand side of (1.34) $F(s,y)$. We take the partial derivative of $F(s,y)$ with $s = s(y)$ in y_1 and denote $\partial_1 = \partial_{y_1}$, $\partial_2 = \partial_{y_2}$. We get

$$\partial_1 s - 1 + f'(s)^2\partial_1 s + f''(s)\partial_1 s(f(s) - y_2) = 0.$$

Plug in $y_1 = s_0$,$y_2 = f(s_0)$, noticing that we have $s(s_0, f(s_0)) = s_0$, we get

$$\partial_1 s|_{y=(s_0,f(s_0))} = \frac{1}{1 + f'(s_0)^2}.$$

Which means by the definition of y^* we get

$$\partial_1 y_1^*|_{y=(s_0,f(s_0))} = 2\partial_1 s|_{y=(s_0,f(s_0))} - 1 = \frac{1 - f'(s_0)^2}{1 + f'(s_0)^2}$$

And similarly by taking the partial derivative in y_2 in $F(s, y)$ we get

$$\partial_2 s|_{y=(s_0,f(s_0))} = \frac{f'(s_0)}{1 + f'(s_0)^2}.$$

So

$$\partial_2 y_1^*|_{y=(s_0,f(s_0))} = 2\partial_2 s|_{y=(s_0,f(s_0))} = \frac{2f'(s_0)}{1 + f'(s_0)^2}$$

By chain rule, we get

$$\partial_1 f(s)|_{y=(s_0,f(s_0))} = \frac{f'(s_0)}{1 + f'(s_0)^2}, \quad \partial_2 f(s)|_{y=(s_0,f(s_0))} = \frac{f'(s_0)^2}{1 + f'(s_0)^2}.$$

Which means we can get

$$\partial_1 y_2^*|_{y=(s_0,f(s_0))} = \frac{2f'(s_0)}{1 + f'(s_0)^2}, \quad \partial_2 y_2^*|_{y=(s_0,f(s_0))} = \frac{f'(s_0)^2 - 1}{1 + f'(s_0)^2}.$$

Thus by Taylor's expansion we get (1.36). And we can also see the invertibility of y^* for r small enough, this is simply by the inverse function theorem because $|\det(\nabla y^*)|_{y\in\partial\Omega}| = 1$. □

To understand y^* better, we need another lemma. The following lemma shows that although $\partial\Omega$ could be crazy, the intuition that y^* must be outside of Ω is always true.

Lemma 1.3.7. *There exists r so that for any $y \in B_r(0) \cap \Omega$, $y^*(y) \notin \Omega$.*

Proof of lemma 1.3.7. First, since y is close to the origin, the slope of inner normal line at $e(y)$ of $\partial\Omega$ is close to $+\infty$. Recall by our definition the inner normal line has the same direction as $y - e(y)$, which means the second component of $e(y)$ is less than y_2. By the definition of y^*, $y_2^* < y_2$.

Now we argue by contradiction. Suppose for every r_0 we can find y so that $y^*(y)$ is inside Ω. By the expansion of y^* near zero, we know that $|y^*| \approx |y|$. Here the notation " $\approx$ " means there are constants C_1, C_2 only depending on Ω, such that $C_1|y| \leqslant |y^*| \leqslant C_2|y|$. So, if r_0 is small enough, say, less than $\frac{r}{C_2}$, where r is the same r in lemma 1.3.3, then $y^*(y)$ is also in the domain of map y^* by lemma 1.3.3. By definition, $e(y) - y^*(y)$ is orthogonal to the tangent line at $e(y)$. However, by lemma 1.3.3, such a boundary point $e(y)$ is unique, which means $e(y) = e(y^*(y))$. So we know $y^*(y^*(y)) = y$. But then $y_2 = y^*(y^*(y))_2 < y_2^* < y_2$, which is a contradiction. □

We will use y^* as a sort of conjugate point for y in the context of the Dirichlet reflection principle for the representation of the Green's function. We note that for the case of the disk in [56], by the well known explicit formula for the Green's function, the natural choice of y^* is given by circular inversion of y. For more general Ω the choice of y^* is less obvious. We will see that our definition of y^* will work well for the estimates that we have in mind.

Without loss of generality, we assume $\Omega \subset [-2, 2] \times [-2, 2]$. Then we have the following proposition.

Proposition 1.3.8. *Suppose Ω is a C^3 bounded open domain in $\mathbb{R}^2$, symmetric about x_2-axis, and tangent to x_1-axis at the origin. Then there exists $r = r(\Omega) > 0$ so that for*

$x, y \in B_r(0)$, the Green function of Ω can be written as:

$$G_\Omega(x,y) = \frac{1}{2\pi}(\log|x-y| - \log|x-y^*|) + B(x,y). \tag{1.37}$$

Here $B(x,y)$ satisfies for any $\omega \in L^\infty(\Omega)$, $\int_{B_r(0)\cap\Omega} B(x,y)\omega(y)dy \in C^{2,\alpha}(B_r(0)\cap\Omega)$, for any $0 < \alpha < 1$. More precisely, we have

$$\|\partial_{x_i}\partial_{x_j}\int_{B_r(0)\cap\Omega} B(x,y)\omega(y)dy\|_{L^\infty(B_r(0)\cap\Omega)} \leqslant C(\Omega)\|\omega\|_{L^\infty} \quad i,j = 1,2$$

To prove this proposition, we need a technical lemma.

Lemma 1.3.9. *Let $x = (s, f(s))$. Let $K(z_1, z_2)$ be a integral kernel such that it is C^1 on the set $\{z_1 \in \Omega, z_2 \in \Omega : z_1 \neq z_2\}$. Suppose we have $K(s, f(s), y) = O(\frac{1}{|x-y|})$ and $D_s K(s, f(s), y) = O(\frac{1}{|x-y|^2})$. Then $\int_\Omega K(s, f(s), y)\omega(y)dy$ has modulus of continuity $s\log(s)$, with the constant equal to $C(\Omega)||\omega||_{L^\infty}$. Here $C(\Omega)$ is a constant that only depends on Ω.*

Proof of lemma 1.3.9. Without loss of generality, let $s_1 < s_2$. Suppose $|s_1 - s_2| = \zeta$. Let $(\frac{(s_1+s_2)}{2}, f(\frac{(s_1+s_2)}{2}))$ be Z. By the smoothness of f, there is a constant $C_1 = C_1(\Omega)$, so that for any s between s_1 and s_2, $(s, f(s)) \in B_{C_1\zeta}(Z)$. Then, for any $\tau > C_1\zeta$, we have

$$\begin{aligned}
&\int_\Omega (K(s_1, f(s_1), y) - K(s_2, f(s_2), y))\omega dy \\
&= \int_{\Omega\cap B_\tau(Z)} (K(s_1, f(s_1), y) - K(s_2, f(s_2), y))\omega(y)dy + \\
&\int_{\Omega\cap B_\tau^c(Z)} (K(s_1, f(s_1), y) - K(s_2, f(s_2), y))\omega(y)dy \\
&\leqslant C||\omega||_{L^\infty} \int_{\Omega\cap B_\tau(Z)} (\frac{1}{|y-(s_1, f(s_1))|} + \frac{1}{|y-(s_2, f(s_2))|})dy + \\
&C||\omega||_{L^\infty} \int_{\Omega\cap B_\tau^c(Z)} \int_{s_1}^{s_2} |D_s K(t, f(t), y)|dtdy
\end{aligned} \tag{1.38}$$

For $y \in B_\tau(Z)$, by the smoothness of f,

$$|y-(s_i,f(s_i))| \leqslant |y-Z| + |Z-(s_i,f(s_i))| \leqslant \tau + C_1\zeta,$$

for $i=1,2$. In addition, for any $s_1 \leqslant s \leqslant s_2$ and $y \in B_\tau^c(Z)$, we have

$$|y-(s,f(s))| \geqslant |y-Z| - |Z-(s,f(s))| \geqslant \tau - C_1\zeta.$$

Hence, the right hand side of (1.38) is no more than

$$\begin{aligned} &C||\omega||_{L^\infty}\int_{(\Omega-Z)\cap B_{\tau+C_1\zeta}(O)} \frac{1}{|y|}dy + C||\omega||_{L^\infty}\int_{(\Omega-Z)\cap B^c_{\tau-C_1\zeta}(O)} \frac{|s_1-s_2|}{|y|^2}dy \leqslant \\ &C||\omega||_{L^\infty}(\int_0^{\tau+C_1\zeta}\frac{1}{r}\cdot rdr + \int_{\tau-C_1\zeta}^2 \frac{\zeta}{r^2}\cdot rdr) \qquad (1.39) \\ &= C||\omega||_{L^\infty}(\tau + \zeta\log(\tau-C_1\zeta)+C_1\zeta). \end{aligned}$$

Here $\Omega - Z$ means the translation of Ω by Z. So if we choose $\tau = 4C_1\zeta$, we get the desired modulus of continuity. □

Remark 1.3.10. *In this lemma, it's easy to see that if $K(s,f(s),y)$ is not differentiable but $|K(s_1,f(s_1),y)-K(s_2,f(s_2),y)| = |s_1-s_2|^\alpha O(\frac{1}{|(s_1,f(s_1))-y|^2}+\frac{1}{|(s_2,f(s_2)-y)|^2})$, we can still get the similar result. More precisely, $\int_\Omega K(s,f(s),y)\omega(y)dy$ has modulus of continuity $x^\alpha\log(x)$. This can be used to extend the results of this section to less regular domains with $C^{2,\alpha}$ boundary.*

Now we prove proposition 1.3.8.

Proof of proposition 1.3.8. The idea is to use the properties of elliptic equations.

First, remember

$$B(x,y) = G_\Omega(x,y) - \frac{1}{2\pi}(\log|x-y| - \log|x-y^*|),$$

where $G_\Omega(x,y)$ is the Green function of domain Ω, y^* is a function of y defined by definition 1.3.5. As a well-known result, Green function is a smooth function for $x \neq y$. Here we would like to show the subtraction of $\frac{1}{2\pi}\log|x-y|$, which is the Green function of $\mathbb{R}^2$, can eliminate the singularity of $G_\Omega(x,y)$ with the help of the term $\log|x-y^*|$.

As a result of lemma 1.3.7, we know that for all y, $y^* \notin \Omega$. Therefore for any fixed $y \in \Omega$, $\log|x-y^*|$ is smooth and harmonic in x. This means $B(x,y)$ is harmonic as x varies in Ω, and satisfies the boundary condition $B(x,y)|_{x\in\partial\Omega} = \frac{1}{2\pi}\log(\frac{|x-y^*|}{|x-y|})$. Hence $\int_{\Omega\cap B_r(0)} B(x,y)\omega(y)dy$ is also harmonic and satisfies

$$\int_{\Omega\cap B_r(0)} B(x,y)\omega(y)dy|_{x\in\partial\Omega\cap B_r(0)} = \int_{\Omega\cap B_r(0)} \frac{1}{2\pi}\log(\frac{|x-y^*|}{|x-y|})\omega(y)dy.$$

Since the boundary of the domain Ω is C^3, by the well-known results on elliptic regularity (see, e.g., Lemma 6.18 of [37]), we know that in order to show that $\int_{\Omega\cap B_r(0)} B(x,y)\omega(y)dy$ is $C^{2,\alpha}$ near the origin, we only need to show that this harmonic function is $C^{2,\alpha}$ on the boundary near the origin. Recall the notation $x = (s, f(s))$. We only need

$$\iota(s) = \int_{\Omega\cap B_r(0)} \frac{1}{2\pi}\log(\frac{|(s,f(s))-y^*|}{|(s,f(s))-y|})\omega(y)dy \tag{1.40}$$

to be $C^{2,\alpha}$ in s for s small. Here remember y^* is only a function in y, so $\iota(s)$ is a well defined function in s. The proof for regularity of ι is simply by calculation. Here we will only use the expansion of $x-y^*$ and the corresponding cancellation of $y-y^*$. As $\iota(s)$ can be seen as the integral of a difference of the same function in different points y and y^*, we will essentially need to calculate the finite differences of some complicated functions.

First we find the second derivative in s of $\log(\frac{|(s,f(s))-y^*|}{|(s,f(s))-y|})$. We call it $K(s,f(s),y)$.

More precisely, we have

$$
\begin{aligned}
-K(s,\mu,y) = &\frac{1+f''(s)(\mu-y_2)+f'(s)^2}{|x-y|^2} - 2\frac{((s-y_1)+f'(s)(\mu-y_2))^2}{|x-y|^4}\\
&-\frac{1+f''(s)(\mu-y_2^*)+f'(s)^2}{|x-y^*|^2} + 2\frac{((s-y_1^*)+f'(s)(\mu-y_2^*))^2}{|x-y^*|^4}.
\end{aligned}
\tag{1.41}
$$

Then, observe that by lemma 1.3.6 and simple computation, for y close to $(s, f(s))$ we have $f(s)-y_2^* = f(s)-y_2+O(|f(s)-x_2|)+O(|s-x_1|)+O(|x-y|^2) = f(s)-y_2+O(|x-y|)$, which means $y_2 - y_2^* = O(|x-y|)$ for y close to x, and $|x-y^*|^2 = |x-y|^2 + O(|x-y|^3)$, for x, y close to the origin and y close to x. So $K(s, f(s), y)$ can be written as

$$
\begin{aligned}
&\frac{f''(s)(f(s)-y_2+O(|x-y|))+O(|x-y|))}{|x-y|^2+O(|x-y|^3)}-\\
&2\frac{A(s,y)(s-y_1+s-y_1^*+f'(s)(f(s)-y_2+f(s)-y_2^*))}{|x-y|^4+O(|x-y|^5)}+\\
&2\frac{((s-y_1)+f'(s)(f(s)-y_2))^2 O(|x-y|)}{|x-y|^4+O(|x-y|^5)}.
\end{aligned}
\tag{1.42}
$$

Where $A(s,y) = (s-y_1)-(s-y_1^*)+f'(s)((f(s)-y_2)-(f(s)-y_2^*)))$. By lemma 1.3.6,

$$
\begin{aligned}
A(s,y) = (s-y_1) &+ \frac{1-f'(s)^2}{1+f'(s)^2}(y_1-s) + \frac{2f'(s)}{1+f'(s)^2}(y_2-f(s))+\\
&f'(s)\left(f(s)-y_2+\frac{2f'(s)}{1+f'(s)^2}(y_1-s)+\frac{f'(s)^2-1}{1+f'(s)^2}(y_2-f(s))\right)+\\
&O(|x-y|^2)\\
&=\frac{-2f'(s)^2}{1+f'(s)^2}(y_1-s)+\frac{2f'(s)}{1+f'(s)^2}(y_2-f(s))+\\
&f'(s)\left(\frac{2f'(s)}{1+f'(s)^2}(y_1-s)+\frac{-2}{1+f'(s)^2}(y_2-f(s))\right)+O(|x-y|^2)\\
&=O(|x-y|^2).
\end{aligned}
\tag{1.43}
$$

Again by lemma 1.3.6 we have $s-y_1^* = O(|x-y|)$, $f(s)-y_2^* = O(|x-y|)$. And also we have $s-y_1 = O(|x-y|)$ and $f(s)-y_2 = O(|x-y|)$. Plug in all of these into (1.42) we get $K(s, f(s), y) = O(\frac{1}{|x-y|})$.

Then, we take the derivative of $K(s, f(s), y)$ in terms of s again. We get

$$\begin{aligned}
D_sK(s,f(s),y) = &\frac{f'''(s)(f(s)-y_2)+3f''(s)f'(s)}{|x-y|^2} - \\
&\frac{(1+f''(s)(f(s)-y_2)+f'(s)^2)((s-y_1)+f'(s)(f(s)-y_2))}{|x-y|^4} \\
&+4\left(\frac{1+f''(s)(f(s)-y_2)+f'(s)^2}{|x-y|^2} - 2\frac{((s-y_1)+f'(s)(f(s)-y_2))^2}{|x-y|^4}\right) \\
&\cdot \frac{((s-y_1)+f'(s)(f(s)-y_2))}{|x-y|^2} \\
&- \frac{f'''(s)(f(s)-y_2^*)+3f''(s)f'(s)}{|x-y^*|^2} \\
&+ \frac{(1+f''(s)(f(s)-y_2^*)+f'(s)^2)((s-y_1^*)+f'(s)(f(s)-y_2^*))}{|x-y^*|^4} \\
&-4\left(\frac{1+f''(s)(f(s)-y_2^*)+f'(s)^2}{|x-y^*|^2} - 2\frac{((s-y_1^*)+f'(s)(f(s)-y_2^*))^2}{|x-y^*|^4}\right) \\
&\cdot \frac{((s-y_1^*)+f'(s)(f(s)-y_2^*))}{|x-y^*|^2}
\end{aligned} \tag{1.44}$$

$$\begin{aligned}
&= O\left(\frac{1}{|x-y|^2}\right) \\
&- 5(1+f'(s)^2)\left(\frac{(s-y_1+f'(s)(f(s)-y_2))}{|x-y|^4} - \frac{(s-y_1^*+f'(s)(f(s)-y_2^*))}{|x-y^*|^4}\right) \\
&- 8\left(\frac{((s-y_1)+f'(s)(f(s)-y_2))^3}{|x-y|^6} - \frac{((s-y_1^*)+f'(s)(f(s)-y_2^*))^3}{|x-y^*|^6}\right) \\
&= O\left(\frac{1}{|x-y|^2}\right) - 5(1+f'(s))\frac{A(s,y)}{|x-y|^4} - 8\frac{A(s,y)O(|x-y|^2)}{|x-y|^6} \\
&= O\left(\frac{1}{|x-y|^2}\right).
\end{aligned} \tag{1.45}$$

To complete the proof of this proposition, we only need to use lemma 1.3.9. □

Remark 1.3.11. *Notice that this proposition is true for all small enough r. Later in the proof of the key lemma this r may change from line to line, and finally we will choose the smallest r which is still only depend on Ω.*

Remark 1.3.12. *If f is not in C^3 but in $C^{2,\beta}$ for some $0 < \beta < 1$, by a longer but similar computation we can find*

$$|K(s_1, f(s_1), y) - K(s_2, f(s_2), y)| = |s_1 - s_2|^{\alpha} O(\frac{1}{|(s_1, f(s_1)) - y|^2} + \frac{1}{|(s_2, f(s_2)) - y|^2}).$$

Which means even if we have $C^{2,\beta}$ domain, we can still get some regularity of $K(s, f(s), y)$. By the remark 1.3.10, we still have $\int_{B_r(0)\cap\Omega} B(x, y)\omega(y)dy$ is $C^{2,\alpha}$ for any $0 < \alpha < \beta$.

The proposition can now be applied to prove the key lemma for the domain Ω.

Proof of the key lemma. By proposition 1.3.8, we know we can write the Green function of Ω as follows:

$$2\pi G_{\Omega}(x, y) = \eta_{B_r(0)}(y)(\log|x - y| - \log|x - y^*|) + C(x, y). \tag{1.46}$$

Here $\eta_{B_r}(y)$ is the smooth cut-off function. $C(x, y)$ is a function so that $\int_{\Omega} C(x, y)\omega(y)dy$ is $C^{2,\alpha}(B_{\delta}(0) \cap \Omega)$, for any small $\delta \leqslant \frac{r}{2}$, and $\omega(y)$ is a bounded function in $\overline{\Omega}$. Here y^* is the same as in proposition 1.3.8. Hence, by using the Taylor's expansion and x_2 can be controlled by x_1 in the sector D_1^{γ}, with $\frac{|x|}{x_1} \leqslant C(\gamma)$, the first order term of $\partial_{x_2} \int_{\Omega} C(x, y)\omega(y)dy$ can be written as $x_1 J_1(x, t) + M_1(\omega)$, for $J_1(x, t) \leqslant C(\gamma)||\omega_0||_{L^{\infty}}$ and $M_1(\omega) = \partial_{x_2} \int_{\Omega} C(x, y)\omega(y)dy|_{x=(0,0)}$.

We first prove (1.32). For (1.33), it is similar. By the expansion of y^* near the origin we know $|y^*| \approx |y|$ for any $y \in B_r(0)$. Without loss of generality, we assume $|y^*| \geqslant C_1|y|$ for some $C_1 \leqslant 1$. Fix a small $\gamma > 0$, fix $x \in D_1^{\gamma}$, $|x| \leqslant \delta$. Here δ is a small number that we will choose later. Now we would like to choose a number which is comparable to x_1 while it can control both x_1 and x_2. We define $a = \frac{100}{C_1}(1 + \cot(\gamma))x_1$. Let's first assume δ is small enough so that $a < \min\{0.01, \frac{r}{2}\}$ whenever $|x| \leqslant \delta$. Now the contribution to

u_1 from integration over $B_a(0)$ does not exceed

$$|2\pi \int_{\Omega\cap B_a(0)} \partial_{x_2} G_\Omega(x,y)\omega(y)dy| \leqslant C\|\omega_0\|_{L^\infty} \int_{D^+\cap B_a(0)} \left(\frac{1}{|x-y|}+1\right) dy \\ \leqslant Ca\|\omega_0\|_{L^\infty} \leqslant C(\gamma)x_1\|\omega_0\|_{L^\infty}. \tag{1.47}$$

For $y \in (D^+ \cap B_r(0)) \setminus B_a(0)$, we have $|y| \geqslant 100|x|$ and $|y^*| \geqslant 100|x|$. By symmetry, we can write the first term in (1.46) as $\eta_{D^+\cap B_r(0)}$ times the following terms:

$$\log|x-y| - \log|x-y^*| = \log\left(1 - \frac{2xy}{|y|^2} + \frac{|x|^2}{|y|^2}\right) - \log\left(1 - \frac{2xy^*}{|y^*|^2} + \frac{|x|^2}{|y^*|^2}\right) \\ - \log\left(1 - \frac{2\tilde{x}y}{|y|^2} + \frac{|x|^2}{|y|^2}\right) + \log\left(1 - \frac{2\tilde{x}y^*}{|y^*|^2} + \frac{|x|^2}{|y^*|^2}\right). \tag{1.48}$$

Here $\tilde{x} = (-x_1, x_2)$. For small t, we have

$$\log(1+t) = t - \frac{t^2}{2} + O(t^3).$$

Hence, (1.48) can be written as

$$-\frac{x_1y_1}{|y|^2} + \frac{x_1y_1^*}{|y^*|^2} - \frac{2x_1x_2y_1y_2}{|y|^4} + \frac{2x_1x_2y_1^*y_2^*}{|y^*|^4} + O(\frac{|x|^3}{|y|^3}).$$

In the last term, we used that $|y^*| \approx |y|$, this is true by taking $s_0 = 0$ in the expression of y^* in lemma 1.3.6 for $y \in B_r(0)$ and r small. Again by the expression near 0 of y^*, we have

$$\frac{y_1^*}{|y^*|^2} = \frac{y_1 + O(|y|^2)}{|y|^2 + O(|y|^3)} = \frac{y_1}{|y|^2 + O(|y|^3)} + b_1(y) = \frac{y_1}{|y|^2} + b_1(y).$$

Where $b_1(y)$ is a bounded function in y, and the bound is a universal constant. Similarly,

$$\frac{y_2^*}{|y^*|^2} = -\frac{y_2}{|y|^2} + b_2(y).$$

Here again $b_2(y)$ is also bounded by a universal constant. Therefore we get that the expression (1.48) can be written as

$$x_1b_1(y) - \frac{4x_1x_2y_1y_2}{|y|^4} + \frac{2x_1x_2y_1}{|y|^2}b_2(y) + \frac{2x_1x_2y_2}{|y|^2}b_1(y) + O(\frac{|x|^3}{|y|^3}). \tag{1.49}$$

Then we can differentiate the above expression with respect to x_2, since we know the explicit functions, and by direct computation we get

$$-\frac{4x_1y_1y_2}{|y|^4}+\frac{2x_1y_1}{|y|^2}b_2(y)+\frac{2x_1y_2}{|y|^2}b_1(y)+O(\frac{|x|^2}{|y|^3}).$$

Now

$$\int_{(D^+\cap B_r(0))\setminus B_a(0)}\frac{|x|^2}{|y|^3}dy\leqslant C|x|^2\int_a^1\frac{1}{s^2}ds\leqslant C\frac{|x|^2}{a}\leqslant C(\gamma)x_1.$$

Also,

$$\int_{(D^+\cap B_r(0))\setminus B_a(0)}\frac{y_i}{|y|^2}dy\leqslant C,$$

for $i=1,2$. Therefore, the last three terms of (1.49) only give regular contributions to u_1. Now we only need to show that adjusting the region $B_r(0)\setminus B_a(0)$ to $Q(x_1,x_2)$ will not change too much for the expression, namely,

$$\int_{(D^+\cap B_r(0))\setminus B_a(0)}\frac{y_1y_2}{|y|^4}\omega(y)dy=C(\Omega)b_3(x)\|\omega_0\|_{L^\infty}+\int_{Q(x_1,x_2)}\frac{y_1y_2}{|y|^4}\omega(y)dy.$$

Here again $b_3(x)$ is a bounded function whose bound is a universal constant. Indeed,

$$\int_{D^+\setminus B_r(0)}\frac{y_1y_2}{|y|^4}\omega(y)dy\leqslant C(r)\|\omega_0\|_{L^\infty}\leqslant C(\Omega)\|\omega_0\|_{L^\infty}.$$

And

$$\begin{aligned}\left|\int_{B_a\cap Q(x_1,x_2)}\frac{y_1y_2}{|y|^4}\omega(y)dy\right|&\leqslant C\|\omega_0\|_{L^\infty}\int_{B_a\cap Q(x_1,x_2)}\frac{y_1|y_2|}{|y|^4}dy\\&\leqslant C\|\omega_0\|_{L^\infty}2\int_{x_1}^{Cx_1}dy_1\int_0^{Cx_1}dy_2\frac{y_1y_2}{|y|^4}\leqslant C\|\omega_0\|_{L^\infty}.\end{aligned}\tag{1.50}$$

Finally, the set $D^+\setminus(B_a\cup Q(x_1,x_2))$ consists of two strips. The contribution of the strip along x_2 axis does not exceed the following quantity:

$$\begin{aligned}\left|\int_{D^+\setminus(B_a\cup Q(x_1,x_2))\cap\{y_1\leqslant x_1\}}\frac{y_1y_2}{|y|^4}\omega(y)dy\right|&\leqslant\|\omega_0\|_{L^\infty}\int_0^{x_1}dy_1\int_{x_1}^1dy_2\frac{y_1y_2}{|y|^4}\\&\leqslant\|\omega_0\|_{L^\infty}\int_0^{Cx_1}\frac{y_1}{C^2x_1^2+y_1^2}dy_1\leqslant C\|\omega_0\|_{L^\infty}.\end{aligned}$$

Similarly, the integral over the strip along x_1 axis can be bounded by

$$\begin{aligned}\left|\int_{D^+\backslash(B_a\cup Q(x_1,x_2))\cap\{y_2\leqslant x_2\}}\frac{y_1y_2}{|y|^4}\omega(y)dy\right| &\leqslant \left|\int_{-C(\Omega)|x_1|^2}^{|x_2|}dy_2\int_{Cx_1}^{1}dy_1\frac{y_1y_2}{|y|^4}\right|\|\omega_0\|_{L^\infty}\\ &\leqslant C\|\omega_0\|_{L^\infty}\int_0^{C(\gamma)x_1}dy_2\int_{Cx_1}^{1}dy_1\frac{y_1y_2}{|y|^4}\leqslant C(\gamma)\|\omega_0\|_{L^\infty}.\end{aligned}$$

Here the first step is due to the fact if we write $\partial\Omega=(s,f(s))$, then since $f'(0)=0$, near 0 we have $f(x_1)=Cx_1^2$. The second inequality is true since δ is small, $|x_1|^2\leqslant|x|^2\leqslant|x|\leqslant C(\gamma)x_1$. This completes the estimate of the first term of (1.46).

Finally notice that $u_1(0,0)=0$, so $M_1(\omega)$ will be canceled by the constant term of the first term. So we finish the proof of the key lemma. □

Remark 1.3.13. *By the remark 1.3.10 proposition 1.3.8, one can find that this key lemma is still true for $\partial\Omega$ to be $C^{2,\alpha}$, for any $\alpha>0$. Therefore one could have double exponential in time upper bound as well. On the other hand, it has been proved in [58] and [47] that if the boundary $\partial\Omega$ is only Lipschitz, one may expect finite time blowup or exponential in time upper bound for $\|\nabla\omega\|_{L^\infty}$. It is an interesting question whether we could get any similar estimate for $C^{1,\alpha}$ domain.*

Remark 1.3.14. *For this lemma, one can also try to prove it by taking the direct calculation. We provide this alternative approach in appendix A.*

1.3.3 The proof of the main theorem

Now based on the key lemma for Ω, we follow the idea of the proof in [56], we can prove Theorem 1.3.1.

Proof of Theorem 1.3.1. First of all, we set $1 \geqslant \omega_0 \geqslant 0$ with $\|\omega_0\|_{L^\infty} = 1$. Then we know $1 \geqslant \omega \geqslant 0$ as well. If $x_2 \leqslant 0$, observe that

$$\left| \int_{x_1}^{2} \int_{x_2}^{-x_2} \frac{y_1 y_2}{|y|^4} \omega(y) dy_2 dy_1 \right| \leqslant C \int_{x_1}^{2} \int_{0}^{-f(x_1)} \frac{y_1 y_2}{|y|^4} dy_1 dy_2 \leqslant C \log(1 + (\frac{f(x_1)}{x_1})^2) + C \leqslant C. \tag{1.51}$$

So if we take smooth ω_0 equal to one everywhere in D^+ except on a thin strip of width δ near the axis $x_1 = 0$, where $0 \leqslant \omega_0 \leqslant 1$, we will have

$$\int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4} \omega(y) dy_1 dy_2 \geqslant C_1 \int_{2\delta}^{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\omega(r\cos\phi, r\sin\phi)}{r} d\phi dr - C,$$

here in the second inequality we set $y_1 = r\cos\phi, y_2 = r\sin\phi$. Since $\omega < 1$ in D^+ only in an area not exceeding 2δ, for δ small enough, the right hand side will be at least

$$\frac{C_1}{2} \int_{\delta}^{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{r} dr \geqslant c \log(\delta^{-1}), \tag{1.52}$$

for some $c > 0$.

For $0 < x_1' < x_1'' < 1$ we denote

$$R(x_1', x_1'') = \{(x_1, x_2) \in D^+, x_1' < x_1 < x_1'', x_2 < x_1\}. \tag{1.53}$$

For $0 < x_1 < 1$ we define

$$u_1^l(x_1, t) = \min_{(x_1,x_2) \in D^+, x_2 < x_1} u_1(x_1, x_2, t) \tag{1.54}$$

and

$$u_1^u(x_1, t) = \max_{(x_1,x_2) \in D^+, x_2 < x_1} u_1(x_1, x_2, t). \tag{1.55}$$

By the smoothness of u, it is easy to see that these functions are locally Lipschitz in x_1, with the Lipschitz constant bounded in finite time. As a result, we can define $a(t)$ and

$b(t)$ by

$$\dot{a} = u_1^u(a,t), \quad a(0) = \epsilon^{10},$$
$$\dot{b} = u_1^l(b,t), \quad b(0) = \epsilon. \tag{1.56}$$

Where $0 < \epsilon < \delta$ is small and to be determined later. Let $R_t = R(a(t), b(t))$. Notice by definition R_t can only be guaranteed to be non-empty for small enough t, but we will see that in fact R_t is not empty for all $t > 0$.

We assume $\omega_0 = 1$ on R_0 and smoothly become 0 in the ϵ^{10}-neighborhood of R_0. Our claim is in R_t ω is always 1 for δ small enough.

By the key lemma 1.5.2 and (1.51), we know that u_1 is negative for small δ. Hence both $a(t)$ and $b(t)$ are decreasing functions of time. And by (1.52), near the diagonal $x_1 = x_2$ for $|x| < \delta$ we have

$$\frac{x_1(\log(\delta^{-1}) - C)}{x_2(\log(\delta^{-1}) + C)} \leqslant \frac{-u_1(x_1, x_2)}{u_2(x_1, x_2)} \leqslant \frac{x_1(\log(\delta^{-1}) + C)}{x_2(\log(\delta^{-1}) - C)}. \tag{1.57}$$

This means that the vector field u is directed out of the region R_t on the diagonal. In addition, by the definition of $a(t)$ and $b(t)$, the fluid trajectories starting at the points outside of R_0 cannot enter R_t at any positive time through the vertical segments $\{(a(t), x_2) \in D^+, x_2 < a(t)\}$ and $\{(b(t), x_2) \in D^+, x_2 < b(t)\}$. Therefore, trajectories originating outside R_0 will not enter R_t at any time. This means that $\omega = 1$ in R_t.

We call $\Lambda(x_1, x_2, t) = \frac{4}{\pi}\int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4}\omega(y)dy_1dy_2$. By the key lemma 1.5.2 we have

$$u_1^l(b(t), t) \geqslant -b(t)\Lambda(b(t), x_2(t)) - Cb(t).$$

If $x_2 \leqslant 0$, then $x_2 \geqslant f(x_1) \geqslant -Cx_1^2$. Otherwise if $x_2 > 0$, $x_2 \leqslant x_1$. By an estimate similar to (1.51) and the fact $x_2 \leqslant b(t)$ in $R(t)$, we know

$$\begin{aligned}
|\Lambda(b(t), x_2(t))| &\leqslant |\Lambda(b(t), b(t))| + \left| \frac{4}{\pi} \int_{b(t)}^{2} \int_{x_2}^{b(t)} \frac{y_1 y_2}{|y|^4} dy_1 dy_2 \right| \\
&\leqslant |\Lambda(b(t), b(t))| + \left| \frac{4}{\pi} \int_{b(t)}^{2} \int_{f(b(t))}^{b(t)} \frac{y_1 y_2}{|y|^4} dy_1 dy_2 \right| \\
&\leqslant |\Lambda(b(t), b(t))| + \left| \frac{4}{\pi} \int_{b(t)}^{2} \int_{0}^{b(t)} \frac{y_1 y_2}{|y|^4} dy_1 dy_2 \right| \\
&\leqslant |\Lambda(b(t), b(t))| + \left| \frac{2}{\pi} \int_{b(t)}^{2} y_1 \left(\frac{1}{y_1^2} - \frac{1}{y_1^2 + b(t)^2} \right) dy_1 \right| \\
&\leqslant |\Lambda(b(t), b(t))| + C,
\end{aligned}$$

for some constant $C \geqslant 0$. Therefore we get

$$u_1^l(b(t), t) \geqslant -b(t)\Lambda(b(t), b(t)) - Cb(t). \tag{1.58}$$

And by a similar estimate we also have

$$u_1^u(a(t), t) \leqslant -a(t)\Lambda(a(t), 0) + Ca(t).$$

Observe that by geometry of the regions involved we also have

$$\Lambda(a(t), 0) \geqslant \frac{4}{\pi} \int_{R_t} \frac{y_1 y_2}{|y|^4} dy_1 dy_2 + \Lambda(b(t), b(t)).$$

Since $\omega = 1$ on R_t and if ϵ is sufficiently small,

$$\int_{R_t} \frac{y_1 y_2}{|y|^4} dy_1 dy_2 \geqslant \int_{\frac{\pi}{100}}^{\frac{\pi}{4}} \int_{\frac{a(t)}{\cos\phi}}^{\frac{b(t)}{\cos\phi}} \frac{\sin 2\phi}{2r} dr d\phi \geqslant C(-\log a(t) + \log b(t)) - C.$$

As a result,

$$u_1^u(a(t), t) \leqslant -a(t) \left(-C(\log a(t) - \log b(t)) + \Lambda(b(t), b(t)) \right) + Ca(t). \tag{1.59}$$

Then from the estimates (1.6.4) and (1.59) we know $a(t)$ and $b(t)$ are monotone decaying in time, and by finiteness of $\|u\|_{L^\infty}$ these function are Lipschitz in t. So we have sufficient regularity to do the following calculations

$$\frac{d}{dt}\log(b(t)) \geqslant -\Lambda(b(t), b(t)) - C,$$

and

$$\frac{d}{dt}\log(a(t)) \leqslant C(\log(a(t)) - \log(b(t))) - \Lambda(b(t), b(t)) + C.$$

Hence, by subtraction we have

$$\frac{d}{dt}(\log a(t) - \log b(t)) \leqslant C(\log a(t) - \log b(t)) + 2C.$$

By Gronwall's inequality we get $\log a(t) \leqslant (9\epsilon + C)\exp(\frac{t}{C})$, and by choosing ϵ small enough, we have $a(t) \leqslant \epsilon^{8\exp(Ct)}$. Note that the first coordinate of the characteristic originating at the point on $\partial\Omega$ near the origin with $x_1 = \epsilon^{10}$, does not exceed $a(t)$ by the definition of $a(t)$. To get (1.31), we only need to choose the initial data ω_0 such that $\|\nabla\omega_0\| \lesssim \epsilon^{-10}$. Thus, by the mean value theorem applying to ω between the origin and the point $(a(t), a(t))$, we get the desired lower bound with $\|\omega_0\|_{L^\infty} = 1$. $\square$

1.4 Stability of blow-up for a 1D model of 3D Euler equation

The results in this section com from a joint work with Tam Do and Alexander Kiselev [32].

1.4.1 Derivation of model equations and the main results

Recall we would like to have a model for 3d axisymmetric Euler equation:

$$\partial_t\left(\frac{\omega^\theta}{r}\right)+u^r\left(\frac{\omega^\theta}{r}\right)_r+u^z\left(\frac{\omega^\theta}{r}\right)_z=-\left(\frac{(ru^\theta)^2}{r^4}\right)_z, \tag{1.60}$$

$$\partial_t(ru^\theta)+u^r(ru^\theta)_r+u^z(ru^\theta)_z=0. \tag{1.61}$$

To obtain a simplified model, in [12] they denote $\frac{\omega^\theta}{r}=\omega,(ru^\theta)^2=\theta,r=y,z=x$ and let $r=1$, which draws analogy with the 2D Boussinesq system in the half-space $\mathbb{R}\times(0,\infty)$

$$\omega_t+u^x\omega_x+u^y\omega_y=\theta_x$$

$$\theta_t+u^x\theta_x+u^y\theta_y=0$$

where $u=(u^x,u^y)$ and is derived from ω by the Biot-Savart law. Then if we restrict the system to the boundary $\{(x,y):y=0\}$ so we have $u^y=0$. To derive a Biot-Savart law for the system, ω is assumed to be constant in y in a strip close to the boundary of width $a>0$, which leads to a law defined by convolution with the following kernel:

$$k(x_1)=\int_0^a\left.\frac{\partial}{\partial x_2}\right|_{x_2=0}G((x_1,x_2),(0,y_2))\,dy_2$$

where G is the Green's function of Laplacian in the upper half-plane. We know

$$G(z,w)=\frac{1}{2\pi}\log|z-w|-\frac{1}{2\pi}\log|z-w^*|,\quad w^*=(w_1,-w_2),$$

by a simple calculation one gets

$$u(x)=\tilde{k}*\omega(x), \tag{1.62}$$

where

$$\tilde{k}(x)=\frac{1}{\pi}\log\frac{|x|}{\sqrt{x^2+a^2}}. \tag{1.63}$$

In [12], the authors discard the smooth part of $\tilde{k}$ ($\frac{1}{\pi}\log(\sqrt{x^2+a^2})$), in this paper we will consider $\tilde{k}$ directly or even more general perturbed kernels.

Based on the recent numerical result about potential singularity profile for 3d axisymmetric Euler equation ([45]), we are in particular interested in the case when ω is periodic in $x-$variable. This requires a further simplification of the Biot-Savart law between u and ω, which we will do it in next section. It turns out that this periodic assumption is not crucial, and we can do the same estimate for nonperiodic ω, which we will postpone to the appendix.

The system (1.8), (1.9), (1.12) is locally well posed and possess a Beale-Kato-Majda type criterion. We formalize this below.

Proposition 1.4.1. *(Local existence and Blow-up criteria) Suppose* $(\omega_0,\theta_0)\in H^m(\mathbb{S}^1)\times H^{m+1}(\mathbb{S}^1)$ *(or* $H^m(\mathbb{R})\times H^{m+1}(\mathbb{R})$*) where* $m\geqslant 2$*. Then there exists* $T=T(\omega_0,\theta_0)>0$ *such that there exists a unique classical solution* (ω,θ) *of* (1.8), (1.9), (1.12) *and* $(\omega,\theta)\in C([0,T];H^m\times H^{m+1})$*. In particular, if* T^* *is a maximal time of existence then*

$$\lim_{t\nearrow T^*}\int_0^t \|u_x(\cdot,\tau)\|_{L^\infty}\,d\tau=\infty. \tag{1.64}$$

The proof of the proposition is relatively standard. A short discussion of this topic can be found in [12]. A similar statement is also proved in detail in [13].

1.5 Periodic Case

In this section, we prove finite time blow-up of the system with the kernel given by (1.12). From now on, we will refer to the kernel given by (1.11) as the Hou-Lou kernel and denote it u_{HL}. In addition, we will consider solutions which are mean zero. Let us first quickly derive an expression for (1.12) in the case when the solution is periodic with period L. We periodize the kernel associated with our velocity

$$
\begin{aligned}
u(x,t) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\omega(y)\log\frac{|x-y|}{\sqrt{(x-y)^2+a^2}}\,dy = \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\int_0^L \omega(y)\log\frac{|x-y+nL|}{\sqrt{(x-y+nL)^2+a^2}}\,dy\\
&= \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\int_0^L \omega(y)\log|x-y+nL|\,dy\\
&\quad - \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^L \omega(y)\left(\log((x+ia-y)+nL)+\log((x-ia-y)+nL)\right)\,dy\\
&= \frac{1}{\pi}\int_0^L \omega(y)\log\left|(x-y)\prod_{n=1}^{\infty}\left(1-\frac{(\mu(x-y))^2}{\pi^2n^2}\right)\right|\,dy\\
&\quad - \frac{1}{2\pi}\int_0^L \omega(y)\log\left|(x+ia-y)\prod_{n=1}^{\infty}\left(1-\frac{(\mu(x+ia-y))^2}{\pi^2n^2}\right)\right|\,dy\\
&\quad - \frac{1}{2\pi}\int_0^L \omega(y)\log\left|(x-ia-y)\prod_{n=1}^{\infty}\left(1-\frac{(\mu(x-ia-y))^2}{\pi^2n^2}\right)\right|\,dy\\
&= \frac{1}{\pi}\int_0^L \omega(y)\log|\sin[\mu(x-y)]|\,dy - \frac{1}{2\pi}\int_0^L \omega(y)\log|\sin(\mu(x-ia-y))\sin(\mu(x+ia-y))|\,dy
\end{aligned}
$$

where we set $\mu = \pi/L$. By a quick computation we have,

$$
\begin{aligned}
\sin\mu(x-ia)\sin\mu(x+ia) &= \frac{e^{i\mu(x-ia)}-e^{-i\mu(x-ia)}}{2i}\,\frac{e^{i\mu(x+ia)}-e^{-i\mu(x+ia)}}{2i}\\
&= \frac{e^{2\mu a}+e^{-2\mu a}}{4} - \frac{e^{2i\mu x}+e^{-2i\mu x}}{4}\\
&= \frac{1}{2}(\cosh(2\mu a)-\cos(2\mu x)) = \frac{1}{2}(\cosh(2\mu a)-1)+\sin^2(\mu x)
\end{aligned}
$$

By slight abuse of notation let us rename the quantity $(1/2)(\cosh(2\mu a)-1)$ to be our new a. We think of a as being small though our estimates later will be true for arbitrary positive a. Combining the above calculations, our velocity u can be now written as

$$
u(x) = \frac{1}{2\pi}\int_0^L \omega(y)\left(\log|\sin^2[\mu(x-y)]| - \log|\sin^2[\mu(x-y)]+a|\right)\,dy. \tag{1.65}
$$

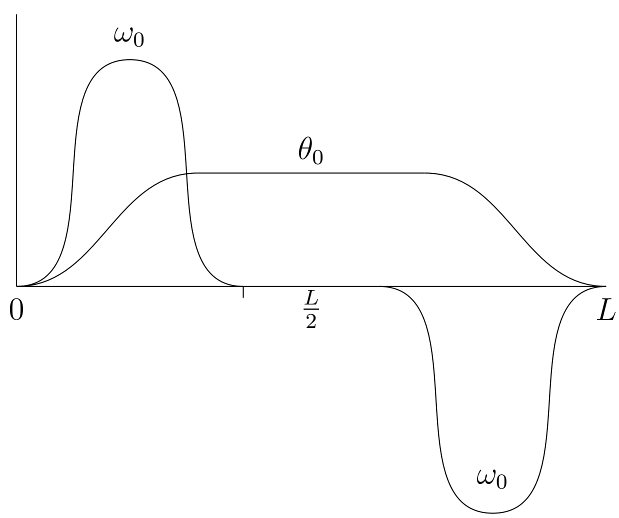
The main result of this section is the following

Theorem 1.5.1. *There exists mean zero initial data such that solutions to* (1.8) *and* (1.9), *with velocity given by* (1.65) *blow up in finite time i.e. there exists a time* T^* *such that we have* (1.64).

We will consider the following type of initial data:

- θ_{0x}, ω_0 smooth odd periodic with period L
- $\theta_{0x}, \omega_0 \geqslant 0$ on $[0, \frac{1}{2}L]$.
- $\theta_0(0) = 0$
- $\|\theta_0\|_\infty \leqslant M$

This can be visualized as follows:



From **Proposition 1.4.1** one has the local well-posedness for our system((1.8)(1.9)(1.65)). By locally well posedness and the transport structure of the system, all the above properties for our choice of initial data will be propagated in time up until possible blow-up time.

The proof of singularity formation will follow by contradiction. This argument is similar to the blow-up argument in the nonlinear Schrödinger equation ([38] [85]). The motivation for the choice of initial data above is the following possible blow-up scenario: $u \leqslant 0$ on $[0, L/2]$ and θ will be pushed towards the origin by the flow which also causes ω to be pushed towards the origin while increasing it's L^∞ norm until there is gradient blow-up at the origin. Our argument is in a similar spirit to [21] where the authors consider the quantity

$$\int_0^{x_0} \frac{\omega(x,t)}{x} dx.$$

However, we are not able to get an estimate of this quantity. Instead, intuitively the movement of the bump of ω will lead a fast movement of θ, which may make θ_x blow up at the origin. Under this intuition, in the proof of blowup we track the quantity

$$\int_0^{\frac{L}{2}} \theta(x,t) \cot(\mu x) dx.$$

Using that our initial data is also odd with respect to $x = \frac{L}{2}$ we can write u as

$$\begin{aligned} u(x) &= \frac{1}{\pi} \left[\int_0^{L/2} + \int_{L/2}^{L} \right] \omega(y) \left(\log|\sin^2[\mu(x-y)]| - \log|\sin^2[\mu(x-y)] + a| \right) dy \\ &= \frac{1}{\pi} \int_0^{L/2} \left(\log \left| \frac{\sin^2 \mu(x-y)}{\sin^2 \mu(x+y)} \right| + \log \left| \frac{\sin^2 \mu(x+y) + a}{\sin^2 \mu(x-y) + a} \right| \right) \omega(y)\, dy \end{aligned}$$

Define

$$F(x,y,a) = \frac{\tan \mu y}{\tan \mu x} \left(\log \left| \frac{\sin^2 \mu(x-y)}{\sin^2 \mu(x+y)} \right| + \log \left| \frac{\sin^2 \mu(x+y) + a}{\sin^2 \mu(x-y) + a} \right| \right) \tag{1.66}$$

and so our corresponding velocity u for the system (1.8) and (1.9) can be written in the following form, which will be useful in the proof

$$u(x)\cot(\mu x) = \frac{1}{\pi}\int_0^{L/2} F(x,y,a)\omega(y)\cot(\mu y)\,dy \tag{1.67}$$

The majority of this section will be devoted to proving properties of F that will allow for a proof of blow-up analogous to the HL model. These properties are contained in the following lemma.

Lemma 1.5.2. **(a)** *There exists a positive constant* C *depending on* a *such that* $F(x,y,a) \leqslant -C < 0$ *for* $0 < x < y < L/2$.

(b) *For any* $0 < y < x < \frac{L}{2}$, $F(x,y,a)$ *is increasing in* x.

(c) *For any* $0 < x, y < \frac{L}{2}$, $\cot(\mu y)(\partial_x F)(x,y,a) + \cot(\mu x)(\partial_x F)(y,x,a)$ *is positive.*

Note that F is not symmetric in x and y. Define

$$K(x,y) = \frac{\tan\mu y}{\tan\mu x}\left(\log\left|\frac{\sin\mu(x+y)}{\sin\mu(x-y)}\right|\right),$$

then

$$F(x,y,a) = -2K(x,y) + \frac{\tan\mu y}{\tan\mu x}\left(\log\left|\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right|\right). \tag{1.68}$$

The term $K(x,y)$ arises from the HL model and we view as the main contributor from F in regards to blow-up. In order to show lemma 1.5.2, we need the following technical lemma for $K(x,y)$:

Lemma 1.5.3. *For simplicity, let's write* $K(x,y)$ *in the following form:*

$$K(x,y) = s\log\left|\frac{s+1}{s-1}\right|, \quad \text{with} \quad s = \frac{\tan(\mu y)}{\tan(\mu x)}. \tag{1.69}$$

Then it has the following properties:

(a) $K(x,y) \geqslant 0$ *for all* $x, y \in (0, \frac{1}{2}L)$ *with* $x \neq y$

(b) $K(x,y) \geqslant 2$ *and* $K_x(x,y) \geqslant 0$ *for all* $0 < x < y < \frac{1}{2}L$

(c) $K(x,y) \geqslant 2s^2$ *and* $K_x(x,y) \leqslant 0$ *for all* $0 < y < x < \frac{1}{2}L$

The detailed proof of lemma 1.5.3 can be found in [12].

Proof of Lemma (1.5.2)*(a).* First, it is easy to see that F is non-positive. Indeed

$$\left|\frac{\sin^2 \mu(x-y)}{\sin^2 \mu(x+y)}\right| \left|\frac{\sin^2 \mu(x+y)+a}{\sin^2 \mu(x-y)+a}\right| = \left|\frac{1+\frac{a}{\sin^2 \mu(x+y)}}{1+\frac{a}{\sin^2 \mu(x-y)}}\right| \leqslant 1 \tag{1.70}$$

because $\sin^2 \mu(x-y) \leqslant \sin^2 \mu(x+y)$.

For the better upper bound, we first consider the region $0 < x < y < L/4$. For the region $L/4 < x < y < L/2$, if we take $x^* = \frac{L}{2} - x$, $y^* = \frac{L}{2} - y$, then $0 < y^* < x^* < L/4$, this means the argument for this region would be the same as the region $0 < x < y < L/4$ by changing the name of the variables. Hence, let us only consider the previous region. We divide our estimate into 4 separate cases. Let $a^* = \min\{a, \frac{1}{16}\}$.

Case 1: $\frac{\sqrt{a^*}}{\pi}L = \frac{\sqrt{a^*}}{\mu} < x < y < L/4$

In this domain we have $\sin \mu y > \sin \mu x > \frac{\sin(\frac{\pi}{4})}{\frac{\pi}{4}}\mu x > \frac{1}{\sqrt{2}}\mu x > \frac{1}{\sqrt{2}}\sqrt{a^*}$, $\cos \mu x > \cos \mu y > \frac{1}{\sqrt{2}}$, hence

$$\sin^2 \mu(x-y) = \sin^2 \mu(x+y) - 4 \sin \mu x \sin \mu y \cos \mu x \cos \mu y < \sin^2 \mu(x+y) - a^*,$$

so

$$F(x,y,a) \leqslant \log\left|\frac{\sin^2 \mu(x-y)}{\sin^2 \mu(x+y)}\right| + \log\left|\frac{\sin^2 \mu(x+y)+a^*}{\sin^2 \mu(x-y)+a^*}\right| = \log\left|\frac{1+\frac{a^*}{\sin^2 \mu(x+y)}}{1+\frac{a^*}{\sin^2 \mu(x-y)}}\right| \tag{1.71}$$

$$\leqslant \log\left|\frac{1+\frac{a^*}{\sin^2 \mu(x+y)}}{1+\frac{a^*}{\sin^2 \mu(x+y)-a^*}}\right| \leqslant -C_0(a) < 0 \tag{1.72}$$

where $C_0(a)$ is a positive constant independent of x, y. In the last step we use the fact that the function $\left(1+\frac{a^*}{z}\right)\left(1+\frac{a^*}{z-a^*}\right)^{-1} = 1-\frac{(a^*)^2}{z^2}$ is increasing in z for $a^* < z < 1$ and fixed a^*.

Case 2: $0 < x < y < \frac{\sqrt{a^*}}{\mu} < L/4$

From lemma 1.5.3 (b), we know

$$-4 \geqslant -2K(x,y) = \frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\tan\mu y+\tan\mu x}{\tan\mu y-\tan\mu x}\right|^2 = \frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right|.$$

so if we can show the contribution from the other part of $F(x,y,a)$ is bounded above by some constant less than 4, we are done. Expanding, we have that second term in (1.68) is equal to

$$\frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu x\cos^2\mu y+2\sin\mu x\cos\mu y\sin\mu y\cos\mu x+\sin^2\mu y\cos^2\mu x+a}{\sin^2\mu x\cos^2\mu y-2\sin\mu x\cos\mu y\sin\mu y\cos\mu x+\sin^2\mu y\cos^2\mu x+a}\right|. \quad (1.73)$$

Since $0 < y < \frac{\sqrt{a^*}}{\mu} \leqslant \frac{\sqrt{a}}{\mu}$, we know $\sin^2\mu y\cos^2\mu x < \sin^2\sqrt{a}\cdot 1 < a$. Then we have that (1.73) is bounded above by

$$\frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu x\cos^2\mu y+2\sin\mu x\cos\mu y\sin\mu y\cos\mu x+2\sin^2\mu y\cos^2\mu x}{\sin^2\mu x\cos^2\mu y-2\sin\mu x\cos\mu y\sin\mu y\cos\mu x+2\sin^2\mu y\cos^2\mu x}\right| = s\log\left|\frac{2s+\frac{1}{s}+2}{2s+\frac{1}{s}-2}\right| \quad (1.74)$$

where $s = \frac{\tan\mu y}{\tan\mu x}$. As a function of s, by direct calculation we find the derivative of the right hand side of (1.74) is

$$\frac{4s-8s^3}{1+4s^4}+\log\left|\frac{2s+\frac{1}{s}+2}{2s+\frac{1}{s}-2}\right|. \quad (1.75)$$

By taking the derivative of (1.75), we find the second derivative of (1.74) is

$$-\frac{8(4s^4+4s^2-1)}{(4s^4+1)^2},$$

which is negative for $s > 1$. And we know

$$\lim_{s\to\infty} \frac{4s-8s^3}{1+4s^4} + \log\left|\frac{2s+\frac{1}{s}+2}{2s+\frac{1}{s}-2}\right| = 0$$

which means the right hand side of (1.74) is increasing in s for $s > 1$ and

$$\lim_{s\to\infty} s\log\left|\frac{2s+\frac{1}{s}+2}{2s+\frac{1}{s}-2}\right| = 2.$$

Case 3: $\frac{\sqrt{a^*}}{2\mu} < x < \frac{\sqrt{a^*}}{\mu} < y < L/4$

In this case, because we know x is away from 0, $s = \dfrac{\tan \mu y}{\tan \mu x} \leqslant C_1(a)$ for some constant depending on a. Also, $\cos^2 \mu y \sin^2 \mu x \leqslant 1 \cdot \sin^2 \sqrt{a} \leqslant a$. Then (1.73) is bounded above by

$$s\log\left|\frac{s+2+\frac{2}{s}}{s-2+\frac{2}{s}}\right|. \tag{1.76}$$

Similarly to the previous case, the second derivative of (1.76) is negative for $s > 1$ and the limit of the first derivative of (1.76) as s goes to infinity is zero, which means (1.76) monotonically increases to 4 as $s \to \infty$ for $s > 1$. However, since s is bounded above, the expression above can be bounded by some constant $C_2(a)$ which is strictly less than 4.

Case 4: $0 < x < \frac{\sqrt{a^*}}{2\mu} < \frac{\sqrt{a^*}}{\mu} < y < L/4$

On the set $A = \{(x, y) : 0 \leqslant x \leqslant \frac{\sqrt{a^*}}{2\mu}, \frac{\sqrt{a^*}}{\mu} \leqslant y \leqslant L/4\}$, $F(x, y, a)$ is a continuous negative function (since $|x - y|$ has a positive lower bound and points where $x = 0$ are removable singularities). Since $F \neq 0$ on A and A is compact, F achieves a maximum $C_3(a)$ which is strictly less than 0.

This completes the analysis for the region $0 < x < y < L/4$. Now, we are left the domain $0 < x < L/4 < y < L/2$.

This case is simpler and the analysis is divided in the following two cases. First, suppose $0 < L/8 < x < L/4 < y < 3L/8 < L/4$ Then $\frac{3\pi}{8} < \mu(x+y) < \frac{5\pi}{8}$ and $0 < \mu(y-x) < \frac{\pi}{4}$ so there exists $\epsilon > 0$ such that $\sin^2\mu(x+y) \geqslant \frac{1}{2}+\epsilon$. However, $\sin^2\mu(x-y) < \frac{1}{2}$. From this, we get $\sin^2\mu(x+y) - \sin^2\mu(x-y) \geqslant \epsilon^*$ for some constant ϵ^*, which means (1.71) follows if we replace the a^* by ϵ^*. Then we get the desired estimate. If x and y are not in this region, there exists a constant $c > 0$ such that $y - x > c > 0$, then again by the same argument as in the **Case 4** and we get the desired inequality.

This completes the proof of (a).

□

Proof of 1.5.2(b). We compute directly and get

$$\begin{aligned}
\cot(\mu y)(\partial_x F)(x,y,a) &= -\mu\csc^2(\mu x)\left(\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right)\\
&+\mu\cot(\mu x)\left[\frac{2\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)} - \frac{2\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)+a}\right]\\
&-\mu\cot(\mu x)\left[\frac{2\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)} - \frac{2\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)+a}\right]\\
&= -\mu\csc^2(\mu x)\left(\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right)\\
&+\mu\cot(\mu x)\left[\frac{2a\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)(\sin^2\mu(x-y)+a)} - \frac{2a\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)(\sin^2\mu(x+y)+a)}\right]\\
&= I + II.
\end{aligned}$$

For I, by the same calculation as (1.70), we know it is positive. For II, when $x > y$, can be expressed as

$$\cot(\mu x)(f(x-y) - f(x+y)),$$

where $f(t) = \frac{\cos(\mu t)}{\sin(\mu t)(\sin^2(\mu t)+a)}$. It is easy to see that whenever $0 < y < x < \frac{L}{2}$, $\cos\mu(x-y) \geqslant \cos\mu(x+y)$, $\sin\mu(x-y) \leqslant \sin\mu(x+y)$. This means $f(x-y) \geqslant f(x+y)$, which also means $II \geqslant 0$. This completes the proof of (b). □

Proof of 1.5.2(c). Now, for the final part of the lemma. First of all, we set

$$\begin{aligned}
G(x,y,a) &= \cot(\mu y)(\partial_x F)(x,y,a) + \cot(\mu x)(\partial_x F)(y,x,a) \\
&= -\mu(\csc^2(\mu x) + \csc^2(\mu y)) \left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right] \\
&+ \mu(\cot(\mu x) - \cot(\mu y)) \frac{2a\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)(\sin^2\mu(x-y)+a)} \\
&- \mu(\cot(\mu x) + \cot(\mu y)) \frac{2\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)(\sin^2\mu(x+y)+a)}. \\
&= -\mu(\cot^2(\mu x) + \cot^2(\mu y) + 2) \left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right] \\
&- \mu\frac{2a\cos\mu(x-y)}{(\sin^2\mu(x-y)+a)\sin(\mu x)\sin(\mu y)} - \mu\frac{2a\cos\mu(x+y)}{(\sin^2\mu(x+y)+a)\sin(\mu x)\sin(\mu y)} \\
&= -\mu(\cot^2(\mu x) + \cot^2(\mu y) + 2) \left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right] \\
&- 2\mu\cot(\mu x)\cot(\mu y)\left[\frac{a}{\sin^2\mu(x-y)+a} + \frac{a}{\sin^2\mu(x+y)+a}\right] \\
&- 2\mu\left[\frac{a}{\sin^2\mu(x-y)+a} - \frac{a}{\sin^2\mu(x+y)+a}\right]
\end{aligned}$$

Now our aim is to prove the positivity of $G(x,y,a)$. Notice that when $a = 0$, $G(x,y,a) = 0$, as a consequence, to prove the positivity of $G(x,y,a)$, the only thing we need to show is that this function is increasing in a for any x, y in the domain. On the other hand,

$$\begin{aligned}
\frac{1}{\mu}\partial_a G(x,y,a) &= (\cot^2(\mu x)+\cot^2(\mu y)+2)\left[\frac{1}{\sin^2\mu(x-y)+a}-\frac{1}{\sin^2\mu(x+y)+a}\right]\\
&-2\cot(\mu x)\cot(\mu y)\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2}+\frac{\sin^2\mu(x+y)}{(\sin^2\mu(x-y)+a)^2}\right]\\
&-2[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2}-\frac{\sin^2\mu(x+y)}{(\sin^2\mu(x+y)+a)^2}]\\
&=(\cot^2(\mu x)+\cot^2(\mu y)+2)\frac{\sin^2\mu(x-y)-\sin^2\mu(x+y)}{(\sin^2\mu(x-y)+a)(\sin^2\mu(x+y)+a)}\\
&-2\cot(\mu x)\cot(\mu y)\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2}+\frac{\sin^2\mu(x+y)}{(\sin^2\mu(x-y)+a)^2}\right]\\
&-2\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2}-\frac{\sin^2\mu(x+y)}{(\sin^2\mu(x+y)+a)^2}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mu}((\sin^2\mu(x-y)+a)(\sin^2\mu(x+y)+a))^2\partial_a G(x,y,a)\\
&=(\cot^2(\mu x)+\cot^2(\mu y)+2)(\sin^2\mu(x+y)-\sin^2\mu(x-y))(\sin^2\mu(x-y)+a)(\sin^2\mu(x+y)+a)\\
&-2\cot(\mu x)\cot(\mu y)\left[\sin^2\mu(x-y)(\sin^2\mu(x+y)+a)^2+\sin^2\mu(x+y)(\sin^2\mu(x-y)+a)^2\right]\\
&-2\left[\sin^2\mu(x-y)(\sin^2\mu(x+y)+a)^2-\sin^2\mu(x+y)(\sin^2\mu(x-y)+a)^2\right].
\end{aligned}$$

It is easy to see this is a quadratic polynomial in a of the form $A_2x^2+A_1x+A_0$. We will explicitly compute A_2, A_1, and A_0 and show each term is non-negative. For the second order term we get

$$A_2 = (\cot^2(\mu x) + \cot^2(\mu y) + 2)(\sin^2 \mu(x-y) - \sin^2 \mu(x+y))$$

$$- 2\cot(\mu x)\cot(\mu y)[\sin^2 \mu(x-y) + \sin^2 \mu(x+y)]$$

$$- 2[\sin^2 \mu(x-y) - \sin^2 \mu(x+y)].$$

$$= (\cot^2(\mu x) + \cot^2(\mu y))(\sin^2 \mu(x+y) - \sin^2 \mu(x-y))$$

$$- 2\cot(\mu x)\cot(\mu y)[\sin^2 \mu(x-y) + \sin^2 \mu(x+y)].$$

This means

$$\tan(\mu x)\tan(\mu y)A_2 =(\frac{\tan(\mu x)}{\tan(\mu y)} + \frac{\tan(\mu y)}{\tan(\mu x)})(\sin^2 \mu(x+y) - \sin^2 \mu(x-y))$$

$$- 2[\sin^2 \mu(x-y) + \sin^2 \mu(x+y)].$$

If we set $\frac{\tan(\mu x)}{\tan(\mu y)} = s$, wel have

$$\frac{\tan(\mu x)\tan(\mu y)}{\cos(\mu y)\cos(\mu x)\sin(\mu y)\sin(\mu x)}A_2 = (s+\frac{1}{s})\cdot 4 - 2[2\cdot(s+\frac{1}{s})] = 0.$$

This means as long as $0 < x, y < \frac{L}{2}$, $A_2 = 0$. Similarly, for coefficient of the first order term A_1, we have

$$A_1 = (\cot^2(\mu x) + \cot^2(\mu y) + 2)(\sin^2 \mu(x+y) - \sin^2 \mu(x-y))(\sin^2 \mu(x+y) + \sin^2 \mu(x-y))$$

$$- 2\cot(\mu x)\cot(\mu y)[2\sin^2 \mu(x-y)\sin^2 \mu(x+y) + 2\sin^2 \mu(x+y)\sin^2 \mu(x-y)]$$

$$- 2[2\sin^2 \mu(x-y)\sin^2 \mu(x+y) - 2\sin^2 \mu(x+y)\sin^2 \mu(x-y)]$$

$$\geqslant (\cot^2(\mu x) + \cot^2(\mu y))[\sin^4 \mu(x+y) - \sin^4 \mu(x-y)]$$

$$- 8\cot(\mu x)\cot(\mu y)[\sin^2 \mu(x-y)\sin^2 \mu(x+y)].$$

Again, by setting $\frac{\tan(\mu x)}{\tan(\mu y)} = s$, we get

$$\frac{\tan(\mu x)\tan(\mu y)}{\cos(\mu x)\cos(\mu y)\sin(\mu x)\sin(\mu y)}A_1 \geqslant (s+\frac{1}{s})\cdot 4\cdot 2(s+\frac{1}{s}) - 8(s+\frac{1}{s}-2)(s+\frac{1}{s}+2) \geqslant 32.$$

Lastly, for the coefficient of the constant term A_0, we have

$$
\begin{aligned}
A_0 &= (\cot^2(\mu x) + \cot^2(\mu y) + 2)(\sin^2\mu(x+y) - \sin^2\mu(x-y))\sin^2\mu(x+y)\sin^2\mu(x-y) \\
&\quad - 2\cot(\mu x)\cot(\mu y)[\sin^2\mu(x-y)\sin^2\mu(x+y)(\sin^2\mu(x+y) + \sin^2\mu(x-y))] \\
&\quad - 2\sin^2\mu(x-y)\sin^2\mu(x+y)[\sin^2\mu(x+y) - \sin^2\mu(x-y)] \\
&= (\cot^2(\mu x) + \cot^2(\mu y))(\sin^2\mu(x+y) - \sin^2\mu(x-y))\sin^2\mu(x+y)\sin^2\mu(x-y) \\
&\quad - 2\cot(\mu x)\cot(\mu y)\sin^2\mu(x-y)\sin^2\mu(x+y)[\sin^2\mu(x+y) + \sin^2\mu(x-y)].
\end{aligned}
$$

For $x = y$, $A_0 = 0$. Otherwise, again by setting $s = \frac{\tan(\mu x)}{\tan(\mu y)}$ after computation we have

$$
\frac{\tan(\mu x)\tan(\mu y)}{\sin^2\mu(x-y)\sin^2\mu(x+y)\cos(\mu x)\cos(\mu y)\sin(\mu x)\sin(\mu y)}A_0 = (s+\frac{1}{s})\cdot 4 - 2\cdot(2s+\frac{2}{s}) = 0.
$$

In all, we have $\partial_a G(x,y,a) \geqslant 0$ for $0 < x, y < \frac{L}{2}$. This completes the proof. □

Remark 1.5.4. *One may notice that when $a \to \infty$, $\frac{1}{\mu}G(x,y,a)$ tends to*

$$
-(\cot^2(\mu x) + \cot^2(\mu y) + 2)\left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right)\right] - 4\cot(\mu x)\cot(\mu y). \tag{1.77}
$$

The positivity of this quantity is also proved by lemma 4.2 in [12], in which the authors use technical trigonometric inequalities. Our proof of the above lemma provides another approach to estimating this quantity.

With these lemmas at our disposal, we are ready to prove finite-time blow up.

Proof of Theorem 1.5.1.

Suppose we have a global smooth solution. We will show blow up of the following quantity:

$$
I(t) := \int_0^{L/2} \theta(x,t)\cot(\mu x)\,dx.
$$

thereby arriving at a contradiction since

$$|I(t)| \leqslant C\|\theta_{0x}\|_{L^\infty} \exp\left(\int_0^t \|u_x(\cdot,s)\|_{L^\infty}\,ds\right).$$

If I were to become infinite in finite time, we would be able to use Beale-Kato-Majda type condition for the system as stated in equation (1.64) from which we can conclude blow-up. We first compute the derivative of $I(t)$:

$$\frac{d}{dt}I(t) = -\frac{1}{\pi}\int_0^{L/2} \theta_x(x) \int_0^{L/2} \omega(y)\cot(\mu y)F(x,y,a)\,dy\,dx.$$

By the negativity of F and part (a) of the lemma, the expression above is bounded below by

$$\frac{C}{\pi}\int_0^{L/2} \theta_x(x) \int_x^{L/2} \omega(y)\cot(\mu y)\,dy\,dx = \frac{C}{\pi}\int_0^{L/2} \theta(x)\omega(x)\cot(\mu x)\,dx := CJ(t),$$

where $J(t) = \frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)\cot(\mu x)\,dx$. Then,

$$\frac{d}{dt}(CJ(t)) = \frac{C}{\pi}\int_0^{L/2} \theta(x)\omega(x)\left(u(x)\cot(\mu x)\right)_x\,dx + \frac{C\mu}{2\pi}\int_0^{L/2} \theta^2(x)\csc^2(\mu x)\,dx \quad (1.78)$$

By Cauchy-Schwarz inequality, the second integral is bounded below by $\frac{C}{L^2}I(t)^2$ for some constant C. The first integral is given by

$$\frac{C}{\pi}\int_0^{L/2} \theta_y(y)\left[\int_y^{L/2} \omega(x)\left(u(x)\cot(\mu x)\right)_x\,dx\right]dy \quad (1.79)$$

Observe that since θ is non-decreasing on $[0, L/2]$, the expression (1.79) is positive if we can show the integral in the brackets is positive as well. This is our next task. For $x, y \in [0, \frac{1}{2}L]$, $\omega(x)$ can be decomposed as

$$\omega(x) = \omega(x)\chi_{[0,y]}(x) + \omega(x)\chi_{[y,\frac{1}{2}L]}(x) =: \omega_\ell(x) + \omega_r(x).$$

Then we can decompose the integral:

$$\int_y^{L/2} \omega(x)[u(x)\cot(\mu x)]_x \, dx = \frac{1}{\pi}\int_0^{L/2} \omega_r(x) \int_0^{L/2} \omega_\ell(y)\cot(\mu y)(\partial_x F)(x,y,a)\, dy\, dx$$
$$+ \frac{1}{\pi}\int_0^{L/2} \omega_r(x) \int_0^{L/2} \omega_r(y)\cot(\mu y)(\partial_x F)(x,y,a)\, dy\, dx$$

By positivity of ω on $[0, \frac{1}{2}L]$ and part (b) of the key lemma, the first integral is positive. By using symmetry, the second integral is equal to

$$\frac{1}{2\pi}\int_0^{L/2}\int_0^{L/2} \omega_r(x)\omega_r(y)G(x,y,a)\, dy\, dx$$

where $G(x,y,a) = \cot(\mu y)(\partial_x F)(x,y,a) + \cot(\mu x)(\partial_x F)(y,x,a)$. However, by part (c) of the lemma, this is positive. Together with (1.78) and (1.79) we have:

$$\frac{d^2}{dt^2} I \geqslant CI^2, \tag{1.80}$$

for some constant C. To close the proof, we only need the following lemma:

Lemma 1.5.5. *Suppose $I(t)$ solve the following initial value problem:*

$$\frac{d}{dt} I(t) \geqslant C \int_0^t I^2(s) ds, I(0) = I_0. \tag{1.81}$$

Then there exists $T = T(C, I_0)$ so that $\lim_{t\to T} I(t) = \infty$.

Moreover, for fixed C and any $\epsilon > 0$, there is an $A > 0$, so that for any $I_0 \geqslant A$, the blow up time $T < \epsilon$.

The proof of this lemma is straightforward, and one can also find the proof in [12].

1.6 Perturbation

In this section, we consider our system (1.8) and (1.9) but with a Biot-Savart which is a perturbation of the Hou-Lou kernel. We will work with periodic solutions with period L. The velocity u is given by the following choice of Biot-Savart Law

$$u(x) = \frac{1}{\pi}\int_0^L (\log|\sin[\mu(x-y)]| + f(x,y))\,\omega(y)\,dy, \quad \mu := \pi/L \tag{1.82}$$

$$:= u_{HL}(x) + u_f(x) \tag{1.83}$$

where f is a smooth function whose precise properties we will specify later. We view f as a perturbation and we will show solutions to the system (1.8) and (1.9) with (1.82) can still blow-up in finite time. As with the previous system (1.8), (1.9), (1.12), we still have a local-well-posedness result akin to Proposition (1.4.1) holds here. In particular, if T^* is a maximal time of existence of a solution then we must have

$$\lim_{t\nearrow T^*}\int_0^t \|u_x(\cdot,\tau)\|_{L^\infty}\,d\tau = \infty \tag{1.84}$$

We show such a time can exist below.

Theorem 1.6.1. *Let $f \in C^2(\mathbb{R}^2)$, periodic with period L such that $f(x,y) = f(-x,-y)$ for all x, y. Then there exists initial data ω_0, θ_0 such that solutions of* (1.8) *and* (1.9)*, with velocity given by (1.82), blow up in finite time. Again, that means there exists a time T^* such that we have (1.84).*

We will consider the following type of initial data:

- θ_{0x}, ω_0 smooth odd periodic with period L
- $\theta_{0x}, \omega_0 \geqslant 0$ on $[0, \frac{1}{2}L]$.

- $\theta_0(0) = 0$
- $(\operatorname{supp}\theta_{0x} \cup \operatorname{supp}\omega_0) \cap [0, \frac{1}{2}L] \subset [0, \epsilon]$
- $\|\theta_0\|_\infty \leqslant M$

We will make the choice of specific ϵ below and unless stated otherwise, ϵ will always refer to the ϵ defined in the way above. Observe that by the assumptions, ω_0 and θ_{0x} are also odd with respect to $\frac{1}{2}L$. By the following lemma 1.6.4, we can choose ϵ such that the mass of ω near the origin gets closer to the origin leading to a scenario where blow-up can be achieved.

Remark 1.6.2. *With the choice of* $f(x,y) = \log\sqrt{\sin^2\mu(x-y)+a}$, *we have the kernel from the previous section. However, in the previous section, we proved blow-up for a larger class of initial data.*

Remark 1.6.3. *Comparing with the construction in the previous section, one can find that the initial condition for the blow up of general pertubation is more restrictive, we require small-supported initial data in order to achieve singularity formation.*

Lemma 1.6.4. *With the initial data* ω_0 *and* θ_0 *as given above, we can choose* ϵ_1 *small such that for* $\epsilon < \epsilon_1$, $u(x) < 0$ *for* $x \leqslant \epsilon$ *where* u *is defined as* (1.82).

Proof. By periodicity and support property of ω,

$$\begin{aligned} u(x) &= \frac{1}{\pi}\int_0^{L/2}\left(\log\left|\frac{\tan(\mu x)-\tan(\mu y)}{\tan(\mu x)+\tan(\mu y)}\right| + f(x,y) - f(x,-y)\right)\omega(y)\,dy \\ &= \frac{1}{\pi}\int_0^{\epsilon}\left(\log\left|\frac{\tan(\mu x)-\tan(\mu y)}{\tan(\mu x)+\tan(\mu y)}\right| + f(x,y) - f(x,-y)\right)\omega(y)\,dy. \end{aligned}$$

By the mean value theorem, for $0 \leqslant y \leqslant \epsilon$, $|f(x,y) - f(x,-y)| \leqslant 2\epsilon \|f\|_{C^1}$. By the singularity of the HL kernel when $x = y = 0$, we can choose ϵ_1 such that the expression in the parentheses is negative for $0 < x, y \leqslant \epsilon$. $\square$

We will also need the following lemma controlling the integral of ω over half the period.

Lemma 1.6.5. *There exists $\epsilon_2 > 0$ such that for $\epsilon < \epsilon_2$, with ω_0 and θ_0 as chosen above, solutions of* (1.8), (1.9), (1.82) *satisfy*

$$\int_0^{L/2} \omega(y,t)\, dy \leqslant Mt.$$

Proof. Integrating both sides of (1.8) and integrating by parts we get

$$\int_0^{L/2} \omega_t(y,t)\, dy = \int_0^{L/2} u_x(y)\omega(y,t)\, dy + \int_0^{L/2} \theta_x(y,t)\, dy \leqslant M + \int_0^{L/2} u_x(y)\omega(y,t)\, dy$$

If we can show the remaining integral on the right is negative, we are done. The integral can be written as

$$\frac{1}{\pi}\int_0^{L/2}\int_0^{L/2} \left(\cot[\mu(x-y)] - \cot[\mu(x+y)] + f_x(x,y) - f_x(x,-y)\right)\omega(x)\omega(y)\, dy\, dx$$

By symmetry, the integral with $\cot[\mu(x-y)]$ is 0 and using the support property of ω, the above line is equal to

$$\frac{1}{\pi}\int_0^{\epsilon}\int_0^{\epsilon} \left(-\cot[\mu(x+y)] + f_x(x,y) - f_x(x,-y)\right)\omega(x)\omega(y)\, dy\, dx$$

Since f is smooth and ω is positive, we can make ϵ_2 smaller so that the expression in the parentheses above in the integrand is negative. $\square$

Now, so we can take advantage of our lemmas, we choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ for the support of our initial data.

Proof of Theorem 1.6.1. Throughout, $C(f)$ will be a positive constant that only depends on f and not ω_0. We will show that

$$I(t) := \int_0^{L/2} \theta(x,t)\cot(\mu x)\,dx \tag{1.85}$$

must blow-up. Taking time derivative of I and using lemma 1.5.3, we get

$$\begin{aligned}\frac{d}{dt}I(t) &= -\int_0^{L/2} u(x)\theta_x(x)\cot(\mu x)\,dx\\ &= \frac{1}{\pi}\int_0^{L/2}\theta_x(x)\int_0^{L/2}\omega(y)\cot(\mu y)K(x,y)\,dy\,dx\\ &\quad + \int_0^{L/2}\theta_x(x)\left(u_f(x)\cot(\mu x)\right)\,dx \geqslant J(t) + \int_0^{L/2}\theta_x(x)\left(u_f(x)\cot(\mu x)\right)\,dx\end{aligned}$$

where, using the same notation as before,

$$J(t) = \frac{2}{\pi}\int_0^{L/2}\theta(x)\omega(x)\cot(\mu x)\,dx$$

Now, we would like to bound the extra term arising because of f. Since f is smooth and ω is supported near the origin,

$$|u(x)\cot(\mu x)| = \left|\int_0^{\epsilon}\left[\cot(\mu x)(f(x,y)-f(x,-y))\right]\omega(y)\,dy\right| \leqslant C(f)\cdot\left(\int_0^{L/2}\omega(y)dy\right).$$

Therefore, we have

$$\frac{d}{dt}I(t) \geqslant J(t) - C(f)M\left(\int_0^{L/2}\omega(y)dy\right) \geqslant J(t) - C(f)M^2 t \tag{1.86}$$

Now, we derive a differential inequality for $J(t)$.

$$\begin{aligned}\frac{d}{dt}J(t) &= \frac{2}{\pi}\int_0^{L/2} -(\theta(x)\omega(x))_x u(x)\cot(\mu x) + \theta_x(x)\theta(x)\cot(\mu x)\,dx\\ &= \frac{2}{\pi}\int_0^{L/2}\theta(x)\omega(x)(u(x)\cot(\mu x))_x\,dx + \frac{\mu}{\pi}\int_0^{L/2}\theta^2(x)\csc^2(\mu x)\,dx\end{aligned}$$

As before, by Cauchy-Schwarz inequality the second integral is bounded below by $\frac{2}{L^2}I(t)^2$. We split the first integral into two parts:

$$\frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)(u_{HL}(x)\cot(\mu x))_x\,dx + \frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)(u_f(x)\cot(\mu x))_x\,dx.$$

By the arguments in the proof of theorem 1.5.1, the first integral is positive. For the second integral, we integrate by parts and get

$$\frac{2}{\pi}\int_0^{L/2} \theta_y(y)\left[\int_y^{L/2} \omega(x)(u_f(x)\cot(\mu x))_x\,dx\right]dy \tag{1.87}$$

Using the smoothness and boundedness of f,

$$|\partial_x(u_f(x)\cot(\mu x))| = \left|\int_0^{\epsilon} \partial_x\left[\cot(\mu x)(f(x,y)-f(x,-y))\right]\omega(y)\,dy\right| \tag{1.88}$$

Now let $h(x,y) = \cot(\mu x)(f(x,y)-f(x,-y))$. Then it is easy to see that $h \in C^1$ when $f \in C^2$, which also means $\partial_x h(x,y)$ is bounded above. This mean the right hand side of (1.88) can be bounded above by

$$\leqslant C(f)\cdot\left(\int_0^{L/2}\omega(y)dy\right).$$

Inserting this estimate into (1.87), we get that it is bounded below by

$$-C(f)M\left(\int_0^{L/2}\omega(y)dy\right)^2.$$

Putting things together, we get

$$\frac{d}{dt}J(t) \geqslant \frac{2}{L^2}I(t)^2 - C(f)M\left(\int_0^{L/2}\omega(y)dy\right)^2 \geqslant \frac{2}{L^2}I(t)^2 - C(f)M^3t^2 \tag{1.89}$$

Now, we will show the differential inequalities we have shown will lead to blow-up. By (1.86) and (1.89) we have

$$\begin{aligned}\frac{d}{dt}I(t) &\geqslant \frac{2}{L^2}\int_0^t I^2(s)\,ds + J(0) - c(f)M^2t - C(f)M^3\frac{t^3}{3}\\ &\geqslant \frac{2}{L^2}\int_0^t I^2(s)\,ds - c(f)M^2t - C(f)M^3\frac{t^3}{3}\end{aligned} \tag{1.90}$$

We claim that one can choose $I(0)$ large enough so that the effect of the negative terms is minimized. By a rather crude estimate we have

$$\frac{d}{dt}I(t) \geqslant -c(f)M^2 t - C(f)M^3\frac{t^3}{3}.$$

After integration, this implies

$$I(t) \geqslant I(0) - C(f)M^2\left(\frac{t^2}{2} + M\frac{t^4}{12}\right). \tag{1.91}$$

Now fix a time, say 1, and we'll show $I(0)$ can be chosen large enough so that $I(t)$ blows up before time 1. We first choose $I(0) \geqslant C(f)M^2\left(\frac{1}{2} + M\frac{1}{12}\right)$, then for $t < 1$,

$$\frac{1}{L^2}\int_0^t I^2(s)\,ds \geqslant \frac{t}{L^2}\left[I(0) - C(f)M^2\left(\frac{1}{2} + \frac{M}{12}\right)\right]^2$$

Choose $I(0)$ such that

$$I(0) \geqslant C(f)M^2\left(\frac{1}{2} + \frac{M}{12}\right) + L\sqrt{c(f)M^2 + C(f)\frac{M^3}{3}}$$

Then, for $0 < t < 1$, with this choice of $I(0)$, using (1.90)

$$\begin{aligned}\frac{d}{dt}I(t) &\geqslant \frac{1}{L^2}\int_0^t I(s)^2\,ds + t\left(c(f)M^2 + C(f)\frac{M^3}{3}\right) - c(f)M^2 t - C(f)M^3\frac{t^3}{3}\\ &\geqslant \frac{1}{L^2}\int_0^t I(s)^2\,ds\end{aligned}$$

By perhaps making $I(0)$ a little larger, if needed, we can show $I(t)$ becomes infinite before time 1 by lemma 1.5.5. □

Chapter 2

Mixing by Fluid Flow

2.1 Introduction

In the previous chapter, we discuss the possible singularity formation of nonlinear nonlocal differential equations. In this chapter, we consider the problem in the other direction, which is how to regularize the solution to a differential equation. We finally show that the optimal mixing flow are able to prevent the singularity formation. Hence, in this chapter, we start from studying the term "mixing", then try to take advantage of it.

2.1.1 Mixing

The mixing of tracer particles by fluid flows is ubiquitous in nature, and have applications ranging from weather forecasting to food processing. An important question that has attracted attention recently is to study "how well" tracers can be mixed under a constraint on the advecting velocity field, and what is the optimal choice of the "best mixing" velocity field (see [83] for a recent review).

Our aim in section 2.2 is to study how well passive tracers can be mixed under an *enstrophy constraint* on the advecting fluid. By passive, we mean that the tracers provide no feedback to the advecting velocity field. Further, we assume that diffusion of the tracer particles is weak and can be neglected on the relevant time scales. Mathematically, the density of such tracers (known as passive scalars) is modeled by the transport equation

$$\partial_t \theta(x,t) + u \cdot \nabla \theta = 0, \quad \theta(x,0) = \theta_0(x). \tag{2.1}$$

To model stirring, the advecting velocity field u is assumed to be incompressible. For simplicity we study (2.1) with periodic boundary conditions (with period 1), mean zero initial data, and assume that all functions in question are smooth.

The first step is to quantify "how well" a passive scalar is mixed in our context. For *diffusive* passive scalars, the decay of the variance is a commonly used measure of mixing (see for instance [17, 34, 79, 84] and references there in). But for diffusion free scalars the variance is a conserved and does not change with time. Thus, following [62] we quantify mixing using the H^{-1}-Sobolev norm: the *smaller* $\|\theta\|_{H^{-1}}$, the better mixed the scalar θ is.

The reason for using a negative Sobolev norm in this context has its roots in [34, 62, 70, 79]. The motivation is that if the flow generated by the velocity field is mixing in the ergodic theory sense, then any advected quantity (in particular θ) converges to 0 weakly in L^2 as $t \to \infty$. This can be shown to imply that $\|\theta(\cdot,t)\|_{H^s} \to 0$ for all $s < 0$, and conversely, if $\|\theta(\cdot,t)\|_{H^s} \to 0$ for some $s < 0$ then $\theta(x,t)$ converges weakly to zero. Thus any negative Sobolev norm of θ can in principle be used to quantify its mixing properties. In two dimensions the choice of using the H^{-1} norm in particular was suggested by Lin et. al. [62] as it scales like the area dominant unmixed regions; a natural length scale associated with the system. We will work with the same Sobolev norm in any dimension d; the ratio of H^{-1} norm to L^2 norm has a dimension of length, and since the L^2 norm of $\theta(x,t)$ is conserved, the H^{-1} norm provides a natural length scale associated with the mixing process.

The questions we study in section 2.2 are motivated by recent work of Lin et. al. [62]. In [62], the authors address two questions on the two dimensional torus:

- The time decay of $\|\theta(t)\|_{H^{-1}}$, given the *fixed energy* constraint $\|u(t)\|_{L^2} = U$.
- The time decay of $\|\theta(t)\|_{H^{-1}}$ given a *fixed enstrophy* constraint of the form $\|\nabla u(t)\|_{L^2} = F$.

In the first case the authors prove a lower bound for $\|\theta(\cdot,t)\|_{H^{-1}(\mathbb{T}^2)}$ that is linear in t, with negative slope. This suggests that it may be possible to "mix perfectly in finite time"; namely choose u in a manner that drives $\|\theta(\cdot,t)\|_{H^{-1}}$ to zero in finite time. This was followed by an explicit example in [66] exhibiting finite time perfect mixing, under a finite energy constraint. This example uses an elegant "slice and dice" construction, which requires the advecting velocity field to develop finer and finer scales. Thus, while their example maintains a fixed energy constraint, the enstrophy ($\|\nabla u\|_{L^2}$) explodes. Together with the numerical analysis in [62, 66] this suggests that finite time perfect mixing by an enstrophy constrained incompressible flow might be impossible. Our main theorem in section 2.2 settles this affirmatively. A simplified version of the main theorem of section 2.2 can be stated as follows:

Theorem 2.1.1. *Let u be any incompressible flow satisfies $\|\nabla u(t)\|_{L^2} \leqslant F$ for all t. Let θ be the solution to (2.1), then $\|\theta(\cdot,t)\|_{H^{-1}} \geqslant c_1 \exp(-Ct)$ for some constants c_1 and C, where the decay rate C only depends on $\|\theta_0\|_{L^\infty}$, F and the size of super level set of θ_0.*

2.1.2 Preventing blow-up by mixing

After the previous project, there are two questions one may ask: Is this exponential lower bound sharp? Can this give some insight into the process of mixing be useful in other PDE models of natural processes?

For the first question, the numerical experiments in [48] suggest this lower bound should be sharp. Recently, in [88], Yao Yao and Andrej Zlatoš proved that the exponential lower bound is sharp by constructing a family of specific flows. A different construction leading to similar results was given in [2].

The second question is more open ended and so a bit harder to answer. In section 2.3, we study the role of mixing in systems where chemotaxis is present.

Chemotaxis is ubiquitous in biology and ecology. This term is used to describe motion where cells or species sense and attempt to move towards higher (or lower) concentration of some chemical. The first mathematically rigorous studies of chemotaxis effects have been by Patlak [76] and Keller-Segel [51], [52]. The latter work involved derivation and first analysis of Keller-Segel system, the most studied model of chemotaxis. Keller-Segel equation describes a population of bacteria or mold that secrete a chemical and are attracted by it. In one version of the simplified parabolic-elliptic form, this equation can be written in $\mathbb{R}^d$ as (see e.g. [77])

$$\partial_t \rho - \Delta \rho + \nabla \cdot (\rho \nabla (-\Delta)^{-1} \rho) = 0, \quad \rho(x, 0) = \rho_0(x). \tag{2.2}$$

The last term on the left hand side describes attraction of ρ by the chemical whose concentration is given by $c(x, t) = (-\Delta)^{-1} \rho(x, t)$. The literature on the Keller-Segel equation is enormous. It is known that in dimensions larger than one, solutions to (2.2) can concentrate finite mass in a measure zero region and so blow up in finite time. We refer to [77], [40], and [41] for more details and further references.

Typically, chemotactic processes take place in fluid, and often the agents involved in chemotaxis are also advected by the ambient flow. Some of the examples involve monocytes using chemokine signalling to concentrate at a source of infection (see e.g. [29,82]), sperm and eggs of marine animals practicing broadcast spawning in the ocean (see e.g. [14, 72]), and other numerous instances in biology and ecology. Our goal in this section is to study the possible effects resulting from interaction of chemotactic and fluid transport processes. Of particular interest to us is the possibility of suppression of finite time blow up due to the mixing effect of fluid flow. The problem of chemotaxis in fluid flow has been studied before; for example, in a setting similar to ours [54] studied the effect of chemotaxis and fluid advection on the efficiency of absorbing reaction. Moreover, in a series of papers [64], [36], [30], [63], [65] a very interesting problem coupling chemotactic density with fluid mechanics equation actively forced by this density has been considered in a variety of different settings. The active coupling makes the system more challenging to analyze, but in some cases intriguing results involving global existence of weak solutions (the definition of which implies lack of the δ-function blow up) have been proved. These results, however, apply either in the setting where the initial data is small (see e.g. [65]) or close to constant [36], or in the systems where both chemotactic equation and the fluid equation have globally regular solutions if not coupled. In other words, to the best of our knowledge, there have been no rigorous results providing an example of suppression of the chemotactic explosion by fluid flow; only results showing that presence of fluid flow does not lead to blow up for the initial data that would not blow up without the flow.

In section 2.3, our main focus will be on the question whether incompressible fluid flow can arrest the finite time blow up phenomenon which is the key signature of Keller-Segel model. There are two possible fluid flow effects that can be helpful in finite time blow up prevention. The first applies in infinite regions, where strong flow can help diffusion quickly spread the density so thin that chemotactic effects become weak. The second effect is more universal and subtle to analyze, and involves mixing in a finite volume. In this case, the concentration may remain significant, but the flow is constantly mixing the density and preventing chemotaxis from building a concentration peak. We are primarily interested in the mixing effect, and so will consider a finite region setting. It will also be convenient for us to adopt periodic boundary conditions and to consider the Keller-Segel equation with advection on a torus. This is not essential, and many of our results also apply on a finite region with Neumann, Dirichlet or Robin boundary conditions.

Let us now briefly state our main result in section 2.3. Since we are working on $\mathbb{T}^d$, we will define the concentration of the chemical by factoring out a constant background: $c(x,t) = (-\Delta)^{-1}(\rho(x,t) - \overline{\rho})$. Here $\rho(x,t) \in L^2$ is the species density, and $\overline{\rho}$ is its mean over $\mathbb{T}^d$. The inverse Laplacian can be defined on the Fourier side, or by an appropriate convolution as will be discussed below. Consider the equation

$$\partial_t \rho + (u \cdot \nabla)\rho - \Delta\rho + \nabla \cdot (\rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho})) = 0, \quad \rho(x,0) = \rho_0(x), \quad x \in \mathbb{T}^d. \tag{2.3}$$

We will prove the following theorem in section 2.3.

Theorem 2.1.2. *Given any initial data $\rho_0 \geqslant 0$, $\rho_0 \in C^\infty(\mathbb{T}^d)$, $d = 2$ or 3, there exist smooth incompressible flows u such that the unique solution $\rho(x,t)$ of (2.19) is globally regular in time.*

2.2 Lower bounds on the mix norm of scalars advected by enstrophy-constrained flows

The results in this section come from a joint work with Gautam Iyer and Alexander Kiselev [48].

2.2.1 Main result and discussion

First let us state our main theorem in this section.

Theorem 2.2.1. *Let u be a smooth (time dependent) incompressible periodic vector field on the d-dimensional torus, and let θ solve* (2.1) *with periodic boundary conditions and L^∞ initial data θ_0. For any $p > 1$ and $\lambda \in (0,1)$ there exists a length scale $r_0 = r_0(\theta_0, \lambda)$, an explicit constant $\varepsilon_0 = \varepsilon_0(\lambda, d)$, and a constant $c = c(d,p)$ such that*

$$\|\theta(t)\|_{H^{-1}} \geqslant \varepsilon_0 r_0^{d/2+1} \|\theta_0\|_{L^\infty} \exp\Big(\frac{-c}{m(A_\lambda)^{1/p}} \int_0^t \|\nabla u(s)\|_{L^p} \, ds \Big). \tag{2.4}$$

Here A_λ is the super level set $\{\theta_0 > \lambda \|\theta_0\|_{L^\infty}\}$.

In particular, if the instantaneous enstrophy constraint $\|\nabla u\|_{L^2} \leqslant F$ is enforced, then $\|\theta(t)\|_{H^{-1}}$ decays at most exponentially with time.

Before commenting on the r_0 and $m(A_\lambda)$ dependence, we briefly mention some applications. There are many physical situations where $\int_0^t \|\nabla u(s)\|_{L^2} \, ds$ is well controlled. Some examples are when u satisfies the incompressible Navier-Stokes equations with $\dot{H}^{-1}$ forcing [16, 33], the 2D incompressible Euler equations [4] or a variety of active scalar equations including the critical surface quasi-geostrophic equation [9, 10, 20, 53]. In each of these situations the passive scalars can not be mixed perfectly in finite time.

More precisely, a lower bound for the H^{-1}-norm of the scalar density can be read off using (2.4) and the appropriate control on $\|\nabla u\|_{L^2}$.

We also mention that the proof of this theorem is not based on energy methods. Instead, the main idea is to relate the notion of "mixed to scale δ" to the H^{-1} norm, and use recent progress by Crippa and DeLellis [23] towards Bressan's rearrangement cost conjecture [7]. Some of these ideas were already suggested in [66].

We defer the proof of Theorem 2.2.1 to Section 2.2.2, and pause to analyze the dependence of the bound in (2.4) on r_0 and $m(A_\lambda)$.

The length scale r_0 is morally the scale at which the super level set A_λ is "unmixed"; a notion that is made precise later. Our proof, however, imposes a slightly stronger condition: namely, our proof will show that r_0 can be any length scale such that "most" of the super level set A_λ occupies "most" of the union of disjoint balls of radius *at least* r_0. While we are presently unable to estimate r_0 in terms of a tangible norm of θ_0, we remark that we at least expect a connection between r_0 and the ratio of the measure of A_λ to the perimeter of A_λ (see [81] for a related notion).

On the other hand, we point out that the pre-factor in (2.4) can be improved at the expense of the decay rate. To see this, suppose for some $\kappa \in [0, 1/2)$ there exists N disjoint balls of radius at least r_1 such that the fraction of each of these balls occupied by A_λ is *at least* $1 - \kappa$. Then our proof will show that (2.4) in Theorem 2.2.1 can be replaced by

$$\|\theta(t)\|_{H^{-1}} \geqslant \varepsilon_0 r_1^{d/2+1} \|\theta_0\|_{L^\infty} \exp\Big(\frac{-c}{(N r_1^d)^{1/p}} \int_0^t \|\nabla u(s)\|_{L^p} \, ds \Big). \tag{2.4$'$}$$

In this case, if $\theta_0 \in C^1$, the mean value theorem will guarantee that we can choose $N = 1$ and $r_1 > \frac{\|\theta_0\|_{L^\infty}}{C\|\nabla\theta_0\|_{L^\infty}}$ for a purely dimensional constant C.

Next we turn to the exponential decay rate. The dependence of this on $m(A_\lambda)$ is natural. To see this, suppose momentarily that θ_0 only takes on the values ± 1 or 0 representing two insoluble, immiscible fluids which are injected into a large fluid container. Physical intuition suggests that the less the amount of fluid that is injected, the faster one can mix it. Indeed, this is reflected in (2.4) as in this case $m(A_\lambda) = \frac{1}{2} m(\operatorname{supp}(\theta_0))$; so the smaller the support of the initial data, the worse the lower bound (2.4) is. We mention that a bound similar to (2.4) was proved in [78] using optimal transport and ideas from [6]. In [78], however, the author only considers bounded variation "binary phase" initial data, where the two phases occupy the entire region; consequently the result does not capture the dependence of the decay rate on the initial data.

Notational convention, and plan of this section.

We will assume throughout this section that $d \geqslant 2$ is the dimension, and $\mathbb{T}^d$ is the d-dimensional torus, with side length 1. All periodic functions are assumed to be 1-periodic, and we use m to denote the Lebesgue measure on $\mathbb{T}^d$. We will use $\|f\|_{H^s}$ to denote the *homogeneous* Sobolev norms.

The rest of this section is organized as follows: In Section 2.2.2 we describe the notion of δ-mixed data, and prove Theorem 2.2.1, modulo a few Lemmas. In Section 2.2.3 we prove the required lemmas.

2.2.2 Rearrangement Costs and the Proof of the Main Theorem.

We devote this section to the proof of Theorem 2.2.1. The idea behind the proof is as follows. First, if $\|\theta\|_{H^{-1}}$ is small enough, then its super-level sets are mixed to certain scales (Lemma 2.2.4 below). Second, any flow that starts with an "unmixed" set and mixes it to scale δ has to do a minimum amount of work [7, 23]. Putting these together yields Theorem 2.2.1.

We begin by describing the notion of "mixed to scale δ", and relate this to the H^{-1} Sobolev norm.

Definition 2.2.2. *Let $\kappa \in (0, \frac{1}{2})$ be fixed. For $\delta > 0$, we say a set $A \subseteq \mathbb{T}^d$ is δ-semi-mixed if*

$$\frac{m\big(A \cap B(x,\delta)\big)}{m(B(x,\delta))} \leqslant 1 - \kappa \quad \text{for every } x \in \mathbb{T}^d.$$

If additionally A^c is also δ-semi-mixed, then we say A is δ-mixed (or mixed to scale δ).

Remark 2.2.3. *The parameters δ and κ measures the scale and "accuracy" respectively. The key parameter here is the scale δ, and the accuracy parameter $\kappa \in (0, 1/2)$ only plays an auxiliary role. Given a specific initial distribution to mix, κ can be chosen to optimize the bound.*

Note that the notion of a set being mixed here is the same as that of Bressan [7]. A set being semi-mixed is of course a weaker notion.

One relation between δ-semi-mixed and negative Sobolev norms is as follows.

Lemma 2.2.4. *Let $\lambda \in (0, 1]$ and $\theta \in L^\infty(\mathbb{T}^d)$. Then for any integer $n > 0$, $\kappa \in (0, \frac{\lambda}{1+\lambda})$*

there exists an explicit constant $c_0 = c_0(d, \kappa, \lambda, n)$ *such that*

$$\|\theta\|_{H^{-n}} \leqslant \frac{\|\theta\|_{L^\infty} \delta^{n+d/2}}{c_0} \implies A_\lambda \text{ is } \delta\text{-semi-mixed.}$$

Here A_λ *is the super level set defined by* $A_\lambda \stackrel{def}{=} \{\theta > \lambda \|\theta\|_{L^\infty}\}$.

Our interest in this Lemma is mainly when $n = 1$. Note that while Lemma 2.2.4 guarantees the super level sets A_λ are δ-semi-mixed, they need not be δ-mixed. Indeed if A_λ is very small, its complement won't be δ-semi-mixed. Also, we remark that the converse of Lemma 2.2.4 need not be true. For example the function

$$f(x) = \sin(2\pi x) + 10 \sin(2\pi n x)$$

has $\|f\|_{H^{-1}(\mathbb{T}^1)} = O(1)$, and the super level set $\{f > 5\}$ is certainly semi-mixed to scale $1/n$ (see also [62]).

The proof of Lemma 2.2.4 follows from a duality and scaling argument. For clarity of presentation we postpone the proof to Section 2.2.3. Returning to Theorem 2.2.1, the main ingredient in its proof is a lower bound on the "amount of work" required to mix a set to fine scales. This notion goes back to a conjecture of Bressan for which a $500 prize was announced [8].

Conjecture 2.2.5 (Bressan '03 [7])**.** *Let* H *to be the left half of the torus, and* Ψ *be the flow generated by an incompressible vector field* u*. If after time* T *the image of* H *under the flow* Ψ *is* δ*-mixed, then there exists a constant* C *such that*

$$\int_0^T \|\nabla u(\cdot, t)\|_{L^1} \, dt \geqslant \frac{|\ln \delta|}{C}. \tag{2.5}$$

We refer the reader to [7] for the motivation of the lower bound (2.5) and further discussion. To the best of our knowledge, this conjecture is still open. However, Crippa and De Lellis [23] made significant progress towards the resolution of this conjecture.

Theorem 2.2.6 (Crippa, De Lellis '08 [23]). *Using the same notation as in Conjecture 2.2.5, for any $p > 1$ there exists a finite positive constant C_p such that*

$$\int_0^T \|\nabla u(\cdot, t)\|_{L^p}\, dt \geqslant \frac{|\ln \delta|}{C_p}. \tag{2.6}$$

For our purposes we will need two extensions of Theorem 2.2.6. First, we will need to start with sets other than the half torus. Second, we will need lower bounds for the work done to *semi*-mix sets to small scales. Note that in order for a flow to δ-mix a set A, it has to both δ-semi-mix A and δ-semi-mix A^c. Generically each of these steps should cost comparable amounts, and hence a semi-mixed version of Theorem 2.2.6 should follow using techniques in [23]. We state this as our next lemma.

Lemma 2.2.7. *Let Ψ be the flow map of an incompressible vector field u. Let $A \subset \mathbb{T}^d$ be any measurable set and let $p > 1$. There exist constants $r_0 = r_0(A)$ and $a = a(d, \kappa, p) > 0$, such that if for some $\delta < r_0/2$ and $T > 0$ the set $\Psi_T(A)$ is δ-*semi-*mixed, then*

$$\int_0^T \|\nabla u(\cdot, t)\|_{L^p}\, dt \geqslant \frac{m(A)^{1/p}}{a} \Big| \ln \frac{2\delta}{r_0} \Big|. \tag{2.7}$$

Morally the constant r_0 above should be a length scale at which set A is not semi-mixed. Our proof, however, uses a condition on r_0 which is slightly stronger than only requiring that A is not semi-mixed to scale r_0. Namely, we will require "most" of A to occupy "most" of the union of disjoint balls of radius at least r_0. Deferring the proof of Lemma 2.2.7 to Section 2.2.3, we prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Replacing θ with $\theta/\|\theta\|_{L^\infty}$, we may without loss of generality assume $\|\theta_0\|_{L^\infty} = 1$. Fix $0 < \lambda \leqslant 1$, and $\kappa \in (0, \frac{\lambda}{1+\lambda})$. Let a be the constant from Lemma 2.2.7, and c_0 the constant from Lemma 2.2.4 with $n = 1$. Choose

$$\delta = \Big(c_0 \|\theta(t)\|_{H^{-1}} \Big)^{\frac{2}{d+2}}.$$

Then certainly $\|\theta(t)\|_{H^{-1}} \leqslant \delta^{d/2+1}/c_0$ and by Lemma 2.2.4 the super level set $\{\theta(t) > \lambda\}$ is δ-semi-mixed.

Now, since θ satisfies the transport equation (2.1), we know $\{\theta(t) > \lambda\} = \Psi_t(A_\lambda)$, where Ψ is the flow of the vector field u. Thus, Lemma 2.2.7 now implies

$$\delta \geqslant \frac{r_0}{2} \exp\Big(\frac{-a}{m(A_\lambda)^{1/p}} \int_0^t \|\nabla u\|_{L^p}\Big).$$

Consequently

$$\|\theta(t)\|_{H^{-1}} = \frac{\delta^{d/2+1}}{c_0} \geqslant \frac{r_0^{d/2+1}}{c_0 2^{d/2+1}} \exp\Big(\frac{-da}{m(A_\lambda)^{1/p}} \int_0^t \|\nabla u\|_{L^p}\Big),$$

finishing the proof. □

2.2.3 Proofs of Lemmas.

We devote this section to the proofs of Lemmas 2.2.4 and 2.2.7.

Proof of Lemma 2.2.4. Suppose for the sake of contradiction that A_λ is not δ-semi-mixed. Then by definition, there exists $x \in \mathbb{T}^d$ such that

$$m(A_\lambda \cap B(x,\delta)) \geqslant (1-\kappa)m(B(x,\delta)) = (1-\kappa)\pi(d)\delta^d. \tag{2.8}$$

Here $\pi(d)$ is the volume of d-dimensional unit ball.

By duality

$$\|\theta\|_{H^{-n}} = \sup_{f \in H^n} \frac{1}{\|f\|_{H^n}} \Big| \int_{\mathbb{T}^d} \theta(x) f(x)\,dx \Big|. \tag{2.9}$$

We choose $f \in H^n$ to be a function which is identically equal to 1 in $B(x,\delta)$, and which vanishes outside $B(x,(1+\varepsilon)\delta)$ for some small $\varepsilon > 0$. A direct calculation shows that we can arrange

$$\|f\|_{H^n} \leqslant c_1(d) \cdot \varepsilon^{-n+\frac{1}{2}} \cdot \delta^{-n+\frac{d}{2}},$$

for some (explicit) constant c_1 depending only on the dimension.

On the other hand using (2.8) gives

$$\int_{\mathbb{T}^d} \theta(x) f(x) dx \geqslant \pi(d) \|\theta\|_{L^\infty} \delta^d \big((1-\kappa)\lambda - \kappa - c_2(d)\varepsilon \big), \tag{2.10}$$

for some (explicit) dimensional constant $c_2(d)$. Choosing $\varepsilon = \frac{\lambda-(1+\lambda)\kappa}{2c_2(d)}$ and using (2.9) we obtain

$$\|\theta\|_{H^{-n}} \geqslant \frac{\|\theta\|_{L^\infty} \delta^{-n+\frac{d}{2}}}{c_0(d,\kappa,\lambda,n)}$$

as desired. □

Remark. *Observe* $c_0 = c_0'(d,n)(\lambda - (1+\lambda)\kappa)^{n-\frac{1}{2}}$.

Now we turn to Lemma 2.2.7. For this, we need a result from [23] which controls the Lipshitz constant of the Lagrangian map except on a set of small measure.

Proposition 2.2.8 (Crippa DeLellis '08 [23]). *Let* $\Psi(t,x)$ *be the flow map of the (incompressible) vector field* u. *For every* $p > 1$, $\eta > 0$, *there exists a set* $E \subset \mathbb{T}^d$ *and a constant* $c = c(d,p)$ *such that* $m(E^c) \leqslant \eta$ *and for any* $t \geqslant 0$ *we have*

$$\operatorname{Lip}(\Psi^{-1}(t,\cdot)|_{E^c}) \leqslant \exp\Big(\frac{c}{\eta^{\frac{1}{p}}} \int_0^t \|\nabla u(s)\|_{L^p} \, ds \Big). \tag{2.11}$$

Here

$$\operatorname{Lip}(\Psi^{-1}(t,\cdot)|_{E^c}) \stackrel{def}{=} \sup_{\substack{x,y\in E^c \\ x\neq y}} \frac{|\Psi^{-1}(t,x) - \Psi^{-1}(t,y)|}{|x-y|}$$

is the Lipshitz constant of Ψ^{-1} *on* E^c.

The proof of Proposition 2.2.8 is built upon the simple observation [3] that for a passive scalar $\theta(x,t)$ and smooth advecting velocity u one has the inequality

$$\int \log_+ \big| \nabla\theta\big(t, \Psi(t,x)\big) \big| \, dx \leqslant \int_0^t \int \big| \nabla u\big(t, \Psi(t,x)\big) \big| \, dx. \tag{2.12}$$

This can be proved by an elementary calculation. In fact, even the point wise bound

$$D \log |\nabla \theta| \leqslant |\nabla u|$$

is true, where $D = \partial_t + u \cdot \nabla$ is the material derivative. In the form (2.12), this inequality is not very useful. But it turns out that the more sophisticated maximal form of this inequality [3, 23] can be much more useful and is essentially what leads to Proposition 2.2.8. We refer the reader to [23] for the details of the proof.

We use Proposition 2.2.8 to prove Lemma 2.2.7 below.

Proof of Lemma 2.2.7. The main idea behind the proof is as follows: Suppose first r_0 is some large scale at which the set A is "not semi-mixed". Let $T > 0$ be fixed and suppose $\Psi_T(A)$ is δ-semi-mixed for some $\delta < r_0/2$. Since $\Psi_T(A)$ is δ-semi-mixed, there should be many points $\tilde{x} \in \Psi_T(A)$ and $\tilde{y} \in \Psi_T(A)^c$ such that $|\tilde{x} - \tilde{y}| < \delta$. Since A is "not semi-mixed" to scale r_0, there should be many points $\tilde{x}$ and $\tilde{y}$ so that we additionally have $|\Psi_T^{-1}(\tilde{x}) - \Psi_T^{-1}(\tilde{y})| \geqslant r_0/2$. This will force the Lipshitz constant of Ψ_T^{-1} to be at least $r_0/(2\delta)$ on a set of large measure. Combined with Proposition 2.2.8 this will give the desired lower bound on $\int_0^t \|\nabla u\|_{L^p}$.

We now carry out the details of the above outline. The first step in the proof is to choose the length scale r_0. Let $\varepsilon = \varepsilon(\kappa, d)$ be a small constant to be chosen later. We claim that there exists a natural number l and finitely many disjoint balls $B(x_1, r_1)$, ..., $B(x_l, r_l)$ such that

$$m\Big(\bigcup_{i=1}^{l} B(x_i, r_i)\Big) \geqslant \frac{m(A)}{2 \cdot 3^d} \quad \text{and} \quad \frac{m(A \cap B(x_j, r_j))}{m(B(x_j, r_j))} > 1 - \varepsilon \tag{2.13}$$

for every $j \in \{1, \dots l\}$.

To see this, note that the metric density of A is 1 almost surely in A. Thus, removing a set of measure 0 from A if necessary, we know that for every $x \in A$ there exists an $r \in (0,1]$ such that

$$\frac{m(A \cap B(x,r))}{m(B(x,r))} > 1 - \varepsilon.$$

Now choose $K \subset A$ compact with $m(K) > m(A)/2$. Since the above collection of balls is certainly a cover of K, we pass to a finite sub-cover. Applying Vitali's lemma to this sub-cover we obtain a disjoint sub-family $\{B(x_i, r_i) \mid i = 1, \ldots, l\}$ with $m(\cup B(x_i, r_i)) \geqslant m(K)/3^d$. This immediately implies (2.13). For convenience let $B_i = B(x_i, r_i)$, and choose $r_0 = \min\{r_1, \ldots, r_l\}$.

Now let $\eta > 0$ be another small parameter that will be chosen later. By Proposition 2.2.8 we know that there exists a set E with $m(E) \leqslant \eta$ such that the inequality (2.11) holds. Define the set

$$F = \Big\{x \in \mathbb{T}^d \;\Big|\; \frac{m(B(x,\delta) \cap E)}{m(B(x,\delta))} > \frac{\kappa}{2}\Big\} \tag{2.14}$$

Clearly $F \subset \{M\chi_E > \kappa/2\}$, where $M\chi_E$ is the maximal function of χ_E. Consequently,

$$m(F) \leqslant m\big(\{M\chi_E > \frac{\kappa}{2}\}\big) \leqslant \frac{2c_1}{\kappa} m(E)$$

for some explicit constant $c_1 = c_1(d)$. (It is well known that $c_1 = 3^d$ will suffice.)

Since Ψ_T is measure preserving we know $m(\Psi_T^{-1}(E \cup F)) \leqslant (1 + 2c_1/\kappa)\eta$. Thus choosing

$$\eta = \frac{\kappa}{\kappa + 2c_1}\Big(\frac{1}{4^d} - \varepsilon\Big)\sum_{i=1}^{l} m\big(B_i\big)$$

will guarantee

$$m(\Psi_T^{-1}(E \cup F)) \leqslant \Big(\frac{1}{4^d} - \varepsilon\Big)\sum_{i=1}^{l} m\big(B_i\big).$$

This implies that for some $i_0 \leqslant l$ we must have

$$m((B_{i_0} \cap A) - \Psi_T^{-1}(E \cup F)) \geqslant \Big(1 - \frac{1}{4^d}\Big) m\big(B_{i_0}\big). \tag{2.15}$$

By reordering, we may without loss of generality assume that $i_0 = 1$. Consequently, for

$$C_1 = \Big\{ x \in (B_1 \cap A) - \Psi_T^{-1}(E \cup F) \;\Big|\; d(x, B_1^c) > \frac{r_1}{2} \Big\}.$$

equation (2.15) implies

$$m(C_1) \geqslant \Big(\frac{1}{2^d} - \frac{1}{4^d}\Big) m(B_1).$$

Now, from the collection of open balls $\{B(\tilde{x}, \delta) \mid \tilde{x} \in \Psi_T(C_1)\}$ the Vitali covering lemma allows us to extract a finite disjoint collection $B(\tilde{x}_1, \delta)$, …, $B(\tilde{x}_n, \delta)$ such that

$$m\Big(\bigcup_1^n B(\tilde{x}_i, \delta)\Big) \geqslant \frac{m(C_1)}{5^d}.$$

Our goal is to find $\tilde{y}$ such that $\tilde{y} \in B(\tilde{x}_i, \delta) - E$ for some i, and $|\Psi_T^{-1}\tilde{y} - \Psi_T^{-1}x| > r_1/2$.

For convenience set $\tilde{B}_i = B(\tilde{x}_i, \delta)$. Since $\Psi_T(A)$ is δ-semi-mixed and $\tilde{x}_i \notin F$ we have

$$m(\Psi_T(A) \cap \tilde{B}_i) \leqslant (1 - \kappa) m(\tilde{B}_i) \quad \text{and} \quad m(E \cap \tilde{B}_i) \leqslant \frac{\kappa}{2} m(\tilde{B}_i). \tag{2.16}$$

Also, since Ψ_T is measure preserving and by the definition of B_1 we see

$$m\Big(\bigcup_{i=1}^n \tilde{B}_i \cap \Psi_T\big(B_1 - A\big)\Big) \leqslant m(B_1 - A) < \varepsilon m(B_1) \tag{2.17}$$

Using the fact that $\{\tilde{B}_i\}$ are all disjoint, summing (2.16) and using (2.17) gives

$$\begin{aligned} m(\bigcup_{i=1}^n \tilde{B}_i \cap E \cap \Psi_T(B_1)) &< \big(1 - \frac{\kappa}{2}\big) \sum_{i=1}^n m(\tilde{B}_i) + \varepsilon m(B_1) \\ &\leqslant \Big(1 - \frac{\kappa}{2} + \varepsilon 5^d \Big(\frac{1}{2^d} - \frac{1}{4^d}\Big)^{-1}\Big) \sum_{i=1}^n m(\tilde{B}_i). \end{aligned}$$

Thus choosing

$$\varepsilon < \frac{\kappa}{2\cdot 5^d}\Big(\frac{1}{2^d} - \frac{1}{4^d}\Big)$$

will guarantee

$$m\Big(\bigcup_{i=1}^n \tilde B_i \cap E \cap \Psi_T(B_1)\Big) < m\Big(\bigcup_{i=1}^n \tilde B_i\Big)$$

This in turn will guarantee that for some i we can find $\tilde y \in \tilde B_i - E - \Psi_T(B_1)$.

Now observe that

$$\tilde y, \tilde x_i \notin E, \qquad |\tilde y - \tilde x_i| < \delta, \qquad \text{and} \qquad |\Psi_T^{-1}(\tilde y) - \Psi_T^{-1}(\tilde x_i)| > \frac{r_1}{2}.$$

The last inequality above follows because $\Psi_T^{-1}(\tilde x_i) \in C_1$ and $\Psi_T^{-1}(\tilde y) \notin B_1$. This forces

$$\operatorname{Lip}(\Psi_T^{-1}|_{E^c}) \geqslant \frac{|\Psi_T^{-1}(\tilde y) - \Psi_T^{-1}(\tilde x_i)|}{|\tilde y - \tilde x_i|} > \frac{r_1}{2\delta} \geqslant \frac{r_0}{2\delta}.$$

Now using (2.11), and letting $a = a(d, \kappa, p)$ denote a constant that changes from line to line we obtain

$$\int_0^T \|\nabla u(t)\|_{L^p}\, dt \geqslant \frac{\eta^{\frac{1}{p}}}{a}\Big|\log\Big(\frac{r_0}{2\delta}\Big)\Big|. \tag{2.18}$$

Observe finally that

$$\eta = c_2 m\Big(\bigcup_{i=1}^l B_i\Big) \geqslant \frac{c_2 m(A)}{2\cdot 3^d}$$

for some explicit constant $c_2 = c_2(d, \kappa)$. Consequently (2.18) reduces to

$$\int_0^T \|\nabla u(t)\|_{L^p}\, dt \geqslant \frac{m(A)^{\frac{1}{p}}}{a}\Big|\log\Big(\frac{r_0}{2\delta}\Big)\Big|,$$

as desired. □

2.3 Preventing blow-up by mixing

The results in this section come from a joint work with Alexander Kiselev [57].

2.3.1 Main result and the interpretation

Let us first recall our main result. Since we are working on $\mathbb{T}^d$, we will define the concentration of the chemical by factoring out a constant background: $c(x,t) = (-\Delta)^{-1}(\rho(x,t) - \overline{\rho})$. Here $\rho(x,t) \in L^2$ is the species density, and $\overline{\rho}$ is its mean over $\mathbb{T}^d$. The inverse Laplacian can be defined on the Fourier side, or by an appropriate convolution as will be discussed below. Consider the equation

$$\partial_t \rho + (u \cdot \nabla)\rho - \Delta\rho + \nabla \cdot (\rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho})) = 0, \quad \rho(x,0) = \rho_0(x), \quad x \in \mathbb{T}^d. \tag{2.19}$$

We will prove the following theorem.

Theorem 2.3.1. *Given any initial data $\rho_0 \geqslant 0$, $\rho_0 \in C^\infty(\mathbb{T}^d)$, $d = 2$ or 3, there exist smooth incompressible flows u such that the unique solution $\rho(x,t)$ of (2.19) is globally regular in time.*

We will give more details on the choice of the flows later in the proof. The theorem certainly holds under weaker assumptions on the regularity of the initial data. In this section, for the sake of presentation, we do not make an effort to optimize the regularity conditions. The scheme of our proof and the kinds of the flow examples that we have will make the connection between mixing properties of the flow and its ability to suppress the chemotactic blow up quite explicit.

It is well known that the solutions to Keller-Segel equation can form singularities in finite time. The first rigorous proof of this result in the case where the domain is a two-dimensional disk was given by Jäger and Luckhaus [49]. Their proof is based on radial geometry and comparison principles. Nagai [74] has provided a proof of finite time blow up in more general bounded domains. We have not found a finite time blow up proof for the periodic case in the literature. Although it can be obtained by modification of the existing arguments, we will provide a short independent construction of such examples in the case of two spatial dimensions. This will imply that Theorem 2.3.1 indeed provides examples of the suppression of chemotactic explosion by fluid mixing.

We note that fluid advection has been conjectured to regularize singular nonlinear dynamics before. The most notable example is the case of the 3D Navier-Stokes and Euler equations. Constantin [15] has proved possibility of finite time blow up for the 3D Euler equation in $\mathbb{R}^3$ if the pure advection term in the vorticity formulation is removed from the equation. Hou and Lei have obtained numerical evidence for the finite time blow up in a system obtained from 3D Navier-Stokes equation by the removal of the pure transport terms [61]. In fact, finite time blow up has been also proved rigorously in some related modified model settings [43], [46], [42]. Of course, the proof of the global regularity of 3D Navier-Stokes remains an outstanding open problem, so whether 3D Navier-Stokes exhibits "advection regularization" is an open question. See also [60] for more discussion. As another example of related philosophy, we mention the paper [5] on the elliptic problem with "explosion" type reaction. There is no time variable and so no finite time blow up in this paper, but it shows that certain flows can significantly affect the "explosion threshold": namely, the value of the reaction coupling parameter beyond which there exist no regular positive solutions. To the best of our knowledge, the examples that we construct here are the first rigorous examples of the suppression of finite time blow up by fluid mixing in nonlinear evolution setting. It should be possible to extend our method to cover some other situations, which we will briefly discuss in the section 2.3.8.

The rest of this section is organized as follows. In section 2.3.2, we provide a construction of finite time blow up examples. In section 2.3.3 we prove the L^2 -based global regularity criterion that we will use. In section 2.3.4, we set up the strategy for controlling L^2 norm via H^1 norm. In section 2.3.5, we prove the key result on approximation of the solution of Keller-Segel equation by solution of pure advection equation on small time scales. In section 2.3.6, we prove that the relaxation enhancing flows of [17] suppress chemotactic blow up. We focus on the case of weakly mixing flows. In section 2.3.7 we outline another example of flows with such property, the Yao-Zlatoš efficient mixing flows. Finally, in section 2.3.8 we briefly discuss possible future extensions.

Throughout the section, C will stand for universal constants that may change from line to line.

2.3.2 Finite time blow up

In this section, our main result is a construction of examples where solutions to Keller-Segel equation without advection (2.2) blow up in finite time. As we mentioned in the introduction, similar results are well known in slightly different settings. The argument below is included for the sake of completeness. It is closely related to the construction of [74], but is simpler. The argument is essentially local and can be adapted to other situations as well. Although we will focus on the $d = 2$ case, some auxiliary results that remain valid in every dimension will be presented in more generality.

Theorem 2.3.2. *There exist $\rho_0 \in C^\infty(\mathbb{T}^2)$, $\rho_0 \geqslant 0$, such that the corresponding solution $\rho(x,t)$ of equation (2.2) set on $\mathbb{T}^2$ blows up in finite time.*

Without loss of generality, we will assume that the spatial period of initial data and solution is equal to one, so $\mathbb{T}^2 = [-1/2, 1.2]^2$. Let us first state a lemma that will allow us to conveniently estimate the chemotactic term in the equation.

Lemma 2.3.3. *Assume $\mathbb{T}^d = [-1/2, 1/2]^d$, $d \geqslant 2$. For every $f(x) \in C^\infty(\mathbb{T}^d)$, we have*

$$\nabla(-\Delta)^{-1}(f(x) - \overline{f}) = -\frac{1}{c_d} \lim_{\gamma \to 0+} \int_{\mathbb{R}^d} \frac{(x-y)}{|x-y|^d}(f(y) - \overline{f}) e^{-\gamma |y|^2} dy. \tag{2.20}$$

Here on the right hand side $f(y)$ is extended periodically to all $\mathbb{R}^d$, $\overline{f}$ denotes the mean value of f, and c_d is the area of unit sphere in d dimensions.

The expression (2.20) is of course valid for a broader class of f, but the stated result is sufficient for our purpose.

Proof. Without loss of generality, let us assume that f is mean zero. By definition and properties of Fourier transformation, we have

$$\nabla(-\Delta)^{-1} f(x) = -\sum_{k \in \mathbb{Z}^d, k \neq 0} e^{2\pi i k x} \frac{ik}{2\pi |k|^2} \hat{f}(k).$$

To link this expression with (2.20), observe first that for a smooth f, a straightforward computation shows

$$-\sum_{k \in \mathbb{Z}^d, k \neq 0} e^{2\pi i k x} \frac{ik}{2\pi |k|^2} \hat{f}(k) = -\lim_{\gamma \to 0+} \int_{\mathbb{R}^d} e^{2\pi i p x} \frac{ip}{2\pi |p|^2} \int_{\mathbb{R}^d} e^{-2\pi i p y - \gamma |y|^2} f(y) dy dp, \tag{2.21}$$

where the function $f(y)$ is extended periodically to the whole plane. Indeed, all we need to do is plug in Fourier expansion $f(y) = \sum_{k \in \mathbb{Z}^d} e^{2\pi i k y} \hat{f}(k)$, integrate in y, and observe that $(\pi/\gamma)^{d/2} \exp(-\pi^2 |k-p|/\gamma)$ is an approximation of identity.

On the other hand, recall that the inverse Laplacian $(-\Delta)^{-1}$ of a sufficiently regular and rapidly decaying function g is given by

$$\int_{\mathbb{R}^d} e^{2\pi i p x} \frac{1}{(2\pi |p|)^2} \int_{\mathbb{R}^d} e^{-2\pi i p y} g(y) dy dp = \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^d} \log|x-y| g(y) dy & d = 2; \\ \frac{1}{c_d} \int_{\mathbb{R}^d} |x-y|^{2-d} g(y) dy & d \geqslant 3. \end{cases} \tag{2.22}$$

The expression on the right hand side of (2.21), with help of (2.22), can be written as

$$\text{Right hand side of (2.21)} = \lim_{\gamma\to 0+} \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| \nabla\left(f(y)e^{-\gamma|y|^2}\right) dy & d=2; \\ -\frac{1}{c_d}\int_{\mathbb{R}^d} |x-y|^{2-d} \nabla\left(f(y)e^{-\gamma|y|^2}\right) dy & d\geqslant 3. \end{cases}$$

Integrating by parts, we obtain (2.20). □

Suppose that the initial data ρ_0 is concentrated in a small ball B_a of radius a centered at the origin, so that $\int_{B_a}\rho_0 dx = \int_{\mathbb{T}^2}\rho_0 dx \equiv M$. Suppose that $\frac{1}{4} > b > 2a$, $M > 1$, and let ϕ be a cut-off function on scale b. Namely, assume that $\phi \in C^\infty(\mathbb{T}^2)$, $1 \geqslant \phi(x) \geqslant 0$, $\phi = 1$ on B_b, and $\phi = 0$ on B_{2b}^c. The function ϕ can be chosen so that for any multi-index $\alpha \in \mathbb{Z}^2$, $|D^\alpha \phi| \leqslant Cb^{-|\alpha|}$. The parameters a, M and b will be chosen below. The local existence of smooth solution $\rho(x,t)$ can be proved by standard method, see e.g. [54] for a closely related argument in $\mathbb{R}^2$ setting. It is straightforward to check using parabolic comparison principles that if $\rho_0 \geqslant 0$, then $\rho(x,t) \geqslant 0$ for all $t \geqslant 0$. Also, we have $\int_{\mathbb{T}^2}\rho(x,t)dx = M$ while $\rho(x,t)$ remains smooth.

The first quantity we would like to consider is $\int_{\mathbb{T}^2}\rho(x,t)\phi(2x)dx$. We need an estimate showing that the mass cannot leave B_b too quickly.

Lemma 2.3.4. *Suppose that a, b, ϕ, M and ρ_0 are as described above. Assume that the local solution $\rho(x,t)$ exists and remains regular in the time interval $[0,\tau]$. Then we have*

$$\int_{\mathbb{T}^2}\rho(x,t)\phi(2x)dx \geqslant M - C_1 M^2 b^{-2} t \tag{2.23}$$

for every $t \in [0,\tau]$.

Naturally, the bound (2.23) is only interesting if t is sufficiently small.

Proof. We have

$$\partial_t \int_D \rho(x)\phi(2x)dx = \int_D \Delta\rho(x)\phi(2x)dx - \int_D \phi(2x)\nabla\cdot(\rho(x)\nabla(-\Delta)^{-1}(\rho(x)-\overline{\rho}))dx.$$

First, using the periodic boundary conditions and integrating by parts, we find that

$$\left|\int_D \Delta\rho(x)\phi(2x)dx\right| = 4\left|\rho(x)\Delta\phi(2x)dx\right| \leqslant CMb^{-2}. \tag{2.24}$$

Next, let $\psi \in C_0^\infty(\mathbb{R}^2)$ be a cutoff function, $\psi(x)=1$ if $|x|\leqslant 1/2$, $\psi(x)=0$ if $|x|\geqslant 1$, $0\leqslant\psi(x)\leqslant 1$, $|\nabla\psi(x)|\leqslant C$. Using Lemma 2.3.3, we have

$$\begin{aligned}
&\left|\int_{\mathbb{T}^2} \phi(2x)\nabla\cdot(\rho(x)\nabla(-\Delta)^{-1}(\rho(x)-\overline{\rho}))dx\right| \\
&= \frac{1}{\pi}\left|\int_{\mathbb{T}^2} \nabla\phi(2x)\rho(x,t)\lim_{\gamma\to 0+}\int_{\mathbb{R}^2}\frac{(x-y)}{|x-y|^2}(\rho(y,t)-\overline{\rho})e^{-\gamma|y|}dydx\right| \\
&\leqslant C\left|\int_{\mathbb{T}^2} \nabla\phi(2x)\rho(x)\int_{\mathbb{T}^2}\frac{(x-y)}{|x-y|^2}(\rho(y)-\overline{\rho})\psi(y)dydx\right| \\
&+ C\left|\int_{\mathbb{T}^2} \nabla\phi(2x)\rho(x)\lim_{\gamma\to 0+}\int_{\mathbb{R}^2}\frac{(x-y)}{|x-y|^2}(\rho(y)-\overline{\rho})e^{-\gamma|y|^2}(1-\psi(y))dydx\right| \\
&= C(I)+C(II).
\end{aligned}$$

We passed the limit $\gamma\to 0+$ in the first integral since the integral of the limit converges absolutely. Using symmetrization, we can estimate

$$\begin{aligned}
(I) &\leqslant \overline{\rho}\left|\int_{\mathbb{R}^2}(\nabla\phi)(2x)\rho(x,t)\int_{\mathbb{R}^2}\frac{(x-y)}{|x-y|^2}\psi(y)dydx\right| \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x)\rho(y)\left|\left[\frac{(x-y)\cdot(\nabla\phi(2x)\psi(y)-\nabla\phi(2y)\psi(x))}{|x-y|^2}\right]\right|dxdy \\
&\quad \leqslant CM^2b^{-1} + \int_{B_1}\int_{B_1}\rho(x,t)\rho(y,t)(\|\nabla^2\phi\|_{L^\infty}+\|\nabla\phi\|_{L^\infty}\|\nabla\psi\|_{L^\infty})dxdy \\
&\quad \leqslant CM^2b^{-2}.
\end{aligned}$$

We use the fact that $\mathrm{supp}\phi \subset \mathrm{supp}\psi \subset B_1$ in the second step.

Next let us estimate (II). Note that in this case the kernel is not singular, since $\mathrm{supp}\phi \subset B_{2b}$ while $\mathrm{supp}(1-\psi) \subset B_1^c$. However, there is an issue of convergence of y integral over the infinite region. Suppose that $x \in B_{2b}$. Define mean zero function g via $\rho(y) - \overline{\rho} = \Delta g$. In fact, we have $\hat{g} = -\hat{\rho}(k)(2\pi|k|^2)^{-1}$ for $k \neq 0$. By working on the Fourier side, it is easy to show that $\|g\|_{L^1(\mathbb{T}^2)} \leqslant \|g\|_{L^2(\mathbb{T}^2)} \leqslant C\|\rho\|_{L^1(\mathbb{T}^2)}$. Now we can estimate

$$\begin{aligned}
&\lim_{\gamma\to 0+}\int_{\mathbb{R}^2}\frac{(x-y)}{|x-y|^2}(\rho(y,t)-\overline{\rho})e^{-\gamma|y|^2}(1-\psi(y))dy\\
&=\lim_{\gamma\to 0+}\int_{\mathbb{R}^2}\frac{(x-y)}{|x-y|^2}\Delta g(y,t)e^{-\gamma|y|^2}(1-\psi(y))dy\\
&=\lim_{\gamma\to 0+}\int_{\mathbb{R}^2}g(y,t)\Bigg(\Delta\left(\frac{(x-y)}{|x-y|^2}\right)e^{-\gamma|y|^2}(1-\psi(y))\\
&+2\nabla\left(\frac{(x-y)}{|x-y|^2}\right)\nabla\left(e^{-\gamma|y|^2}(1-\psi(y))\right)+\frac{(x-y)}{|x-y|^2}\Delta\left(e^{-\gamma|y|^2}(1-\psi(y))\right)\Bigg)dy.
\end{aligned} \tag{2.25}$$

Here $g(y,t)$ is extended periodically to the whole $\mathbb{R}^2$. Note that in the last integral of the first summand in (2.25) we can pass to the limit as $\gamma \to 0$ since $\Delta\left(\frac{(x-y)}{|x-y|^2}\right)$ decays sufficiently fast. For every $x \in B_b$, we obtain an integral which is bounded by $C\|g\|_{L^1}\sum_{n\in\mathbb{Z}^2,|n|>0}|n|^{-3} \leqslant CM$. It is straightforward to estimate that the last two summands in (2.25) are bounded by

$$C\int_{B^c_{1/2}}|g(y,t)|(\gamma|y|^{-1}+\gamma^2|y|)e^{-\gamma|y|^2}(1-\psi(y))dy \leqslant$$
$$C\|g\|_{L^1(\mathbb{T}^2)}\sum_{n\in\mathbb{Z}^2,|n|>0}(\gamma|n|^{-1}+\gamma^2|n|)e^{-\gamma|n|^2} \leqslant C\|g\|_{L^1(\mathbb{T}^2)}\gamma^{1/2} \xrightarrow{\gamma\to 0} 0.$$

Combining these estimates, we see that

$$(II) \leqslant CM\int_{\mathbb{R}^2}(\nabla\phi)(2x)\rho(x,t)dx \leqslant CM^2b^{-1}.$$

Therefore, for all times where smooth solution is still defined, and under our assumptions on values of parameters, we have

$$\left|\partial_t \int_{\mathbb{T}^2} \rho(x,t)\phi(2x)dx\right| \leqslant CM^2 b^{-2}.$$

This implies 2.23 and finishes the proof of the lemma. □

Let us now consider the second moment $\int_{\mathbb{T}^2} |x|^2\rho(x,t)\phi(x)dx$. Closely related quantities are well-known tools to establish finite time blow up in Keller-Segel equation; see e.g. [77], [74]. We have the following lemma.

Lemma 2.3.5. *Suppose $1/4 \geqslant b > 0$ and ϕ is a cutoff function on scale b as described above. Let $\rho_0 \in C^\infty(\mathbb{T}^2)$, and assume that a unique local smooth solution $\rho(x,t)$ to (2.2) set on $\mathbb{T}^2$ is defined on $[0,T]$. Then for every $t \in [0,T]$ we have*

$$\partial_t \int_{\mathbb{T}^2} |x|^2\rho(x,t)\phi(x)dx \leqslant -\frac{1}{2\pi}\left(\int_D \rho(x)\phi(x)dx\right)^2 + C_2 M\|\rho\|_{L^1(\mathbb{T}^2\setminus B_b)} + C_3 bM^2 + C_4 M. \tag{2.26}$$

Proof. In the estimate below, we will use the formula (2.20) with γ set to be zero. All the estimates can be done completely rigorously similar to the proof of Lemma 2.3.4; we will proceed with the formal computation to reduce repetitive technicalities.

We have

$$\begin{aligned}
\partial_t \int_{\mathbb{T}^2} |x|^2\rho(x,t)\phi(x)dx &= \int_{\mathbb{T}^2} |x|^2\Delta\rho(x)\phi(x)dx + \int_D |x|^2\phi\nabla\cdot(\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))dx \\
&= 4\int_{\mathbb{T}^2}\phi\rho dx + \int_{\mathbb{T}^2}|x|^2\Delta\phi\rho dx + 4\int_{\mathbb{T}^2}(x\cdot\nabla\phi)\rho dx \\
&- \frac{1}{\pi}\int_{\mathbb{T}^2}\phi(x)\rho(x)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}(\rho(y)-\overline{\rho})dydx \\
&- \frac{1}{2\pi}\int_{\mathbb{T}^2}|x|^2\rho(x)\int_{\mathbb{R}^2}\frac{\nabla\phi(x)\cdot(x-y)}{|x-y|^2}(\rho(y)-\overline{\rho})dxdy. \\
&\equiv (i)+(ii)+(iii).
\end{aligned}$$

Here (i) denotes the first three terms. By our choice of ϕ, (i) does not exceed C_4M for some constant C_4. Next, let us write

$$\begin{aligned}(ii) = &-\frac{1}{\pi}\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}(\rho(y,t)-\overline{\rho})\psi(y)dydx \\ &-\frac{1}{\pi}\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}(\rho(y,t)-\overline{\rho})(1-\psi(y))dydx,\end{aligned} \tag{2.27}$$

where ψ is a cutoff function as in Lemma 2.3.4. The absolute value of the integral

$$\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}(\rho(y,t)-\overline{\rho})(1-\psi(y))dydx$$

can be controlled similarly to the estimates applied in bounding the term (II) in the proof of Lemma 2.3.4, leading to an upper bound by CM^2b. Next, we can estimate

$$\left|\overline{\rho}\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}\psi(y)dxdy\right| \leqslant CM^2b$$

as well. Split the remaining part of the first integral in (2.27) into two parts:

$$\begin{aligned}&-\frac{1}{\pi}\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}\rho(y,t)\phi(y)dxdy \\ &-\frac{1}{\pi}\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}\rho(y,t)(1-\phi(y))\psi(y)dxdy\end{aligned}$$

Using symmetrization, we obtain

$$\begin{aligned}&-\frac{1}{\pi}\int_{\mathbb{T}^2}\phi(x)\rho(x)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}\rho(y)\phi(y)dydx \\ &= -\frac{1}{2\pi}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\phi(x)\rho(x)\phi(y)\rho(y)dxdy = -\frac{1}{2\pi}\left(\int_{\mathbb{T}^2}\rho(x)\phi(x)dx\right)^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{T}^2}\frac{x(x-y)}{|x-y|^2}\rho(y,t)(1-\phi(y))\psi(y)dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\rho(x,t)\rho(y,t)}{|x-y|^2}(x-y)\cdot[x\phi(x)(1-\phi(y))\psi(y)-y\phi(y)(1-\phi(x))\psi(x)]dxdy.\end{aligned}$$

Let us define $F(x,y) = x\phi(x)(1-\phi(y))\psi(y) - y\phi(y)(1-\phi(x))\psi(x)$. Observe that $F(x,y)=0$ on $B_b \times B_b$, $F(x,x)=0$ and $|\nabla F(x,y)| \leqslant C$ for all x,y. This means

$$\begin{aligned} |F(x,y)| = |F(x,y)-F(x,x)| &\leqslant \|\nabla F\|_{L^\infty}|x-y|\chi_{B_1\times B_1\setminus B_b\times B_b}(x,y) \\ &\leqslant C|x-y|\chi_{B_1\times B_1\setminus B_b\times B_b}(x,y), \end{aligned}$$

where $\chi_S(x,y)$ denotes the characteristic function of a set $S\subset \mathbb{R}^2\times\mathbb{R}^2$. Therefore,

$$\begin{aligned} &\int_{\mathbb{T}^2}\phi(x)\rho(x,t)\int_{\mathbb{R}^2}\frac{x(x-y)}{|x-y|^2}\rho(y)(1-\phi(y))\psi(y)dxdy \\ &\leqslant C\int\int_{B_1\times B_1\setminus B_b\times B_b}\rho(x,t)\rho(y,t)dxdy \\ &\leqslant CM\|\rho\|_{L^1(D\setminus B_b(0))}. \end{aligned}$$

To summarize, (ii) can be bounded above by

$$-\frac{1}{2\pi}(\int_{\mathbb{T}^2}\rho(x,t)\phi(x)dx)^2 + CM\|\rho(\cdot,t)\|_{L^1(\mathbb{T}^2\setminus B_b)} + CbM^2.$$

Finally, let us estimate (iii). Similarly to the previous part, we have

$$\int_{\mathbb{T}^2}|x|^2\rho(x,t)\int_{\mathbb{R}^2}\frac{\nabla\phi(x)\cdot(x-y)}{|x-y|^2}(\rho(y,t)-\overline{\rho})(1-\psi(y))dxdy \leqslant CbM^2.$$

Also,

$$\overline{\rho}\int_{\mathbb{T}^2}|x|^2\rho(x,t)\int_{\mathbb{R}^2}\frac{\nabla\phi(x)\cdot(x-y)}{|x-y|^2}(1-\psi(y))dxdy \leqslant CbM^2$$

as well. The remaining part of (iii) we can estimate by using symmetrization:

$$\begin{aligned} &\int_{\mathbb{R}^2}|x|^2\rho(x,t)\int_{\mathbb{R}^2}\frac{\nabla\phi(x)\cdot(x-y)}{|x-y|^2}\rho(y,t)\psi(y)dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x,t)\rho(y,t)\left(\frac{(x-y)\cdot(\nabla\phi(x)\psi(y)|x|^2-\nabla\phi(y)\psi(x)|y|^2)}{|x-y|^2}\right)dxdy. \end{aligned}$$

Observe that

$$||x|^2\nabla\phi(x)\psi(y) - |y|^2\nabla\phi(y)\psi(x)| \leqslant C\chi_{B_{2b}\times B_{2b}\setminus B_b\times B_b}|x-y|.$$

Therefore

$$(iii) \leqslant CbM^2 + CM\|\rho\|_{L^1(\mathbb{T}^2\setminus B_b)}.$$

Combine the estimate of (i), (ii) and (iii) yields 2.26, proving the lemma. □

We are now ready to complete the proof of Theorem 2.3.2.

Proof of Theorem 2.3.2. Let us recall that we assume $1/4 \geqslant b \geqslant 2a$, and the initial data ρ_0 is supported inside B_a. Assume that the unique solution $\rho(x,t)$ of (2.2) set on $\mathbb{T}^2$ remains smooth for all t. Then by Lemma 2.3.4, and conservation of mass, for all $t \geqslant 0$ we have

$$\|\rho(\cdot,t)\|_{L^1(\mathbb{T}^2\setminus B_b)} \leqslant M - \int_{\mathbb{T}^2} \rho(x,t)\phi(2x)dx \leqslant C_1 M^2 b^{-2} t.$$

Also,

$$\int_{\mathbb{T}^2} \rho(x,t)\phi(x)dx \geqslant \int_{\mathbb{T}^2} \rho(x,t)\phi(2x)dx \geqslant M - C_1 M^2 b^{-2} t.$$

Therefore, by Lemma 2.3.5, we have that

$$\partial_t \int_{\mathbb{T}^2} |x|^2 \rho(x,t)\phi(x)dx \leqslant -\frac{1}{2\pi}(M - C_1 M^2 b^{-2} t)^2 + C_2 M^3 b^{-2} t + C_3 M^2 b + C_4 M$$

for all $0 \leqslant t \leqslant \frac{b^2}{C_1 M}$. We will now make the choice of all our parameters.

1. Choose b so that $C_3 b \leqslant 0.001$.
2. Choose M so that $M \geqslant 1000 C_4$.
3. Choose a so that the following three inequalities hold:

$$a \leqslant b/2, \;\; a \leqslant \frac{b}{10\sqrt{2C_1}}, \text{ and } a \leqslant \frac{b}{100\sqrt{C_2}}.$$

4. Choose the time $\tau = \frac{100a^2}{M}$.

With such choice of parameters, it is straightforward to check that

$$\partial_t \int_{\mathbb{T}^2} |x|^2 \rho(x,t)\phi(x)dx \leqslant -\frac{M^2}{50} \tag{2.28}$$

for every $t \in [0, \tau]$. But by assumption, $\text{supp}\rho_0 \subset B_a$, and so

$$\int_{\mathbb{T}^2} |x|^2 \rho_0(x)\phi(x)dx \leqslant a^2 M. \tag{2.29}$$

Together, (2.28), (2.29) and our choice of τ imply that $\int_{\mathbb{T}^2} |x|^2\rho(x,\tau)\phi(x)dx$ must be negative. This is a contradiction with the assumption that $\rho(x,t)$ stays smooth throughout $[0, \tau]$. □

2.3.3 Global existence: the L^2 criterion

In this section, we will show that to get the global regularity of (2.19), we only need to have certain control of spatial L^2 norm. The following theorem is a direct analog of Theorem 3.1 in [54], where it was proved in the $\mathbb{R}^2$ setting. We will provide a sketch of proof for the sake of completeness. Throughout the section, we will use notation H^s for the homogeneous Sobolev space in spatial coordinates, that is we set

$$\|f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{2s}|\hat{f}(k)|^2.$$

Theorem 2.3.6. *Suppose that $\rho_0 \in C^\infty(\mathbb{T}^d)$, $\rho_0 \geqslant 0$, and suppose that $u \in C^\infty$ is divergence free, $d = 2$ or $d = 3$. Assume $[0, T]$ is the maximal interval of existence of unique smooth solution $\rho(x,t)$ of equation (2.19). Then we must have*

$$\int_0^t \|\rho(\cdot,\tau) - \overline{\rho}\|_{L^2(\mathbb{T}^d)}^{\frac{4}{4-d}} d\tau \xrightarrow{t\to T} \infty. \tag{2.30}$$

In other words, the smooth solution can be continued as far as integral in time of appropriate power of the L^2 norm in space stays finite. Note that the mean value of ρ is conserved by evolution, so $\overline{\rho}(\cdot,t) \equiv \overline{\rho}_0$. We will denote it $\overline{\rho}$ throughout the rest of the section. One may or may not include $\overline{\rho}$ into (2.30), these criteria are equivalent. One can verify that the scaling of (2.30) is sharp in the sense that it is a critical quantity for (2.19).

Proof. The existence and uniqueness of smooth local solution can be proved by standard methods, so we will focus on global regularity. Let $s \geqslant 2$ be integer. Multiply (2.19) by $(-\Delta)^s\rho$ and integrate. We get

$$\frac{1}{2}\partial_t\|\rho\|^2_{H^s} \leqslant |\int_{\mathbb{T}^d}(\nabla\rho)\cdot(\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))(-\Delta)^s\rho dx| \\ + |\int_{\mathbb{T}^d}\rho(\rho-\overline{\rho})(-\Delta)^s\rho dx| + C\|u\|_{C^s}\|\rho\|^2_{H^s} - \|\rho\|^2_{H^{s+1}}.$$

Here we integrated by parts s times and used incompressibility of u to obtain

$$\left|\int_{\mathbb{T}^d}(u\cdot\nabla)\rho(-\Delta)^s\rho dx\right| \leqslant C\|u\|_{C^s}\|\rho\|^2_{H^s}.$$

Consider the expression

$$\int_{\mathbb{T}^d}\rho(\rho-\overline{\rho})(-\Delta)^s\rho dx.$$

Integrating by parts, we can represent this integral as a sum of terms of the form

$$\int_{\mathbb{T}^d}D^l\rho D^{s-l}(\rho-\overline{\rho})D^s\rho dx,$$

where $l = 0, 1, ..., s$ and D denotes any partial derivative. By Hölder inequality, we have

$$\int_{\mathbb{T}^d}D^l\rho D^{s-l}(\rho-\overline{\rho})D^s\rho dx \leqslant \|D^l\rho\|_{L^{p_l}}\|D^{s-l}(\rho-\overline{\rho})\|_{L^{q_l}}\|\rho\|_{H^s},$$

with any $2 \leqslant p_l, q_l \leqslant \infty$ satisfying $p_l^{-1} + q_l^{-1} = 1/2$. For any integer $0 \leqslant m \leqslant n$, and mean zero $f \in C^\infty(\mathbb{T}^d)$, we have Gagliardo-Nirenberg inequality

$$\|D^m f\|_{L^p} \leqslant C\|f\|^{1-a}_{L^2}\|f\|^a_{H^n}, \quad a = \frac{m - \frac{d}{p} + \frac{d}{2}}{n}, \tag{2.31}$$

which holds for $2 \leqslant p \leqslant \infty$ unless $a = 1$, and if $a = 1$ for $2 \leqslant p < \infty$. We will sketch a short proof of (2.31) in the appendix C to make the thesis more self-contained.

Take $p_l = \frac{2s}{l}$, $q_l = \frac{2s}{s-l}$. Then for all $l > 0$, applying (2.31), we get

$$\|D^l \rho\|_{L^{p_l}} \|D^{s-l}(\rho - \overline{\rho})\|_{L^{q_l}} \leqslant C\|\rho - \overline{\rho}\|_{L^2}^{\frac{2s-d+4}{2(s+1)}} \|\rho\|_{H^{s+1}}^{\frac{2s+d}{2(s+1)}}.$$

In the $l = 0$ case, we use

$$\|\rho\|_{L^\infty} \leqslant \|\rho - \overline{\rho}\|_{L^\infty} + \overline{\rho} \leqslant C\|\rho - \overline{\rho}\|_{L^2}^{1-\frac{d}{2(s+1)}} \|\rho\|_{H^{s+1}}^{\frac{d}{2(s+1)}} + \overline{\rho}.$$

Therefore

$$\left| \int_{\mathbb{T}^d} \rho(\rho - \overline{\rho})(-\Delta)\rho dx \right| \leqslant C \left(\|\rho - \overline{\rho}\|_{L^2}^{1-\frac{d}{2(s+1)}} \|\rho\|_{H^{s+1}}^{\frac{d}{2(s+1)}} + \overline{\rho} \right) \|\rho\|_{H^{s+1}}^{\frac{s}{s+1}} \|\rho - \overline{\rho}\|_{L^2}^{\frac{1}{s+1}} \|\rho\|_{H^s}.$$

Next, consider

$$\int_{\mathbb{T}^d} (\nabla \rho) \cdot (\nabla(-\Delta)^{-1}(\rho - \overline{\rho}))(-\Delta)^s \rho dx.$$

Integrating by parts s times, we get terms that can be estimated similarly to the previous case, using the fact that the double Riesz transform $\partial_{ij}(-\Delta)^{-1}$ is bounded on L^p, $1 < p < \infty$. The only exceptional terms that appear which have different structure are

$$\int_{\mathbb{T}^d} (\partial_{i_1} ... \partial_{i_s} \nabla \rho) \cdot (\nabla(-\Delta)^{-1}(\rho - \overline{\rho}))\partial_{i_1} ... \partial_{i_s} \rho dx$$

but these can be reduced to

$$\int_{\mathbb{T}^d} (\partial_{i_1} ... \partial_{i_s} \rho)^2 (\rho - \overline{\rho}) dx$$

by another integration by parts, and estimated as before. Altogether, we get

$$\begin{aligned} \frac{1}{2}\partial_t \|\rho\|_{H^s}^2 \leqslant C &\left(\|\rho - \overline{\rho}\|_{L^2}^{1-\frac{d}{2(s+1)}} \|\rho\|_{H^{s+1}}^{\frac{d}{2(s+1)}} + \overline{\rho} \right) \|\rho\|_{H^{s+1}}^{\frac{s}{s+1}} \|\rho - \overline{\rho}\|_{L^2}^{\frac{1}{s+1}} \|\rho\|_{H^s} \\ &+ C\|u\|_{C^s} \|\rho\|_{H^s}^2 - \|\rho\|_{H^{s+1}}^2. \end{aligned} \tag{2.32}$$

Observe that

$$\|\rho\|_{H^s} \leqslant \|\rho - \overline{\rho}\|_{L^2}^{\frac{1}{s+1}} \|\rho\|_{H^{s+1}}^{\frac{s}{s+1}}. \tag{2.33}$$

Split the first term on the right hand side of (2.32) into two parts, and estimate them as follows. First,

$$\begin{aligned} C\|\rho-\overline{\rho}\|_{L^2}^{\frac{2s-d+4}{2(s+1)}}\|\rho\|_{H^{s+1}}^{\frac{2s+d}{2(s+1)}}\|\rho\|_{H^s} &\leqslant \|\rho-\overline{\rho}\|_{L^2}\|\rho\|_{H^{s+1}}^{\frac{d}{2}}\|\rho\|_{H^s}^{2-\frac{d}{2}} \\ &\leqslant \frac{1}{4}\|\rho\|_{H^{s+1}}^2 + C\|\rho-\overline{\rho}\|_{L^2}^{\frac{4}{4-d}}\|\rho\|_{H^s}^2. \end{aligned}$$

Second,

$$\begin{aligned} \overline{\rho}\|\rho\|_{H^{s+1}}^{\frac{s}{s+1}}\|\rho-\overline{\rho}\|_{L^2}^{\frac{1}{s+1}}\|\rho\|_{H^s} &\leqslant \overline{\rho}\|\rho\|_{H^s}\|\rho\|_{H^{s+1}} \\ &\leqslant \frac{1}{4}\|\rho\|_{H^{s+1}}^2 + C\overline{\rho}^2\|\rho\|_{H^s}^2. \end{aligned}$$

We used Poincare inequality and (2.33) in the first step. Recall the following Nash-type inequality

$$\|\rho\|_{H^s} \leqslant C\|\rho\|_{H^{s+1}}^{\frac{2s+d}{2s+2+d}}\|\rho\|_{L^1}^{\frac{2}{2s+2+d}}, \tag{2.34}$$

the proof of which will be sketched in the appendix C. Since $\rho(x,t) \geqslant 0$ and hence $\|\rho(\cdot,t)\|_{L^1} = \overline{\rho} > 0$ is conserved in time, putting all estimates into (2.32) we get

$$\frac{1}{2}\partial_t\|\rho\|_{H^s}^2 \leqslant C\left(\|\rho-\overline{\rho}\|_{L^2}^{\frac{4}{4-d}} + \overline{\rho}^2 + \|u\|_{C^s}\right)\|\rho\|_{H^s}^2 - c\overline{\rho}^{-\frac{4}{2s+d}}\|\rho\|_{H^s}^{2+\frac{4}{2s+d}}. \tag{2.35}$$

From this differential inequality and integrability of $\|\rho(\cdot,t)-\overline{\rho}\|_{L^2(\mathbb{T}^d)}^{\frac{4}{4-d}}$ in time, a finite upper bound for $\|\rho(\cdot,t)\|_{H^s}$ follows for all times. In fact, due to the last term on the right hand side of (2.35), it is not hard to show there is a global, not growing in time, upper bound for any H^s norm of ρ.

□

2.3.4 An H^1 condition for an absorbing set in L^2

Due to Theorem 2.3.6, to show global regularity of solution $\rho(x,t)$ to (2.19), it suffices to control its L^2 norm in spatial variables. In this section, we prove a simple criterion that says that if the H^1 norm of a solution is sufficiently large compared to its L^2 norm, then in fact the L^2 norm is decaying. Our overall strategy will be then to show that mixing can increasing and sustain H^1 norm of solution. This will block the L^2 norm from ever growing too much, leading to global regularity.

Proposition 2.3.7. *Let $\rho(x,t) \geqslant 0$ be smooth local solution to (2.19) set on $\mathbb{T}^d$, $d = 2$ or 3. Suppose that $\|\rho(\cdot,t) - \overline{\rho}\|_{L^2} \equiv B > 0$ for some $t \geqslant 0$. Then there exists a universal constant C_1 such that*

$$\|\rho(\cdot, t+\tau) - \overline{\rho}\|_{L^2} \leqslant 2B \text{ for every } 0 \leqslant \tau \leqslant C_1 \min(1, \overline{\rho}^{-1}, B^{-\frac{4}{4-d}}). \tag{2.36}$$

Moreover, there exists a universal constant C_0 such that if in addition

$$\|\rho(\cdot,t)\|_{H^1}^2 \geqslant B_1^2 \equiv C_0 B^{\frac{12-2d}{4-d}} + 2\overline{\rho}B^2, \tag{2.37}$$

then $\partial_t \|\rho(\cdot,t)\|_{L^2} < 0$.

Remark 2.3.8. *The constant C_1, and other constants C_k employed later in this section, are not related to the constants C_k of section 2.3.2.*

Remark 2.3.9. *In particular, due to Theorem 2.3.6, it follows that if $\|\rho(\cdot,t) - \overline{\rho}\|_{L^2} \leqslant B$, then the local smooth solution persists at least till $t + C_1 \min\left(1, \overline{\rho}^{-1}, B^{-\frac{4}{4-d}}\right)$.*

Proof. Let us multiply both sides of (2.19) by $\rho - \overline{\rho}$ and integrate. Then

$$\frac{1}{2}\partial_t \|\rho - \overline{\rho}\|_{L^2}^2 = -\|\rho\|_{H^1}^2 + \int_{\mathbb{T}^d} \nabla \cdot (\rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho}))(\rho - \overline{\rho})dx. \tag{2.38}$$

Observe that

$$\int_{\mathbb{T}^d} \rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho}) \nabla \rho dx = \int_{\mathbb{T}^d} \rho^2 (\rho - \overline{\rho}) dx - \int_{\mathbb{T}^d} \nabla \rho \cdot \nabla (-\Delta)^{-1}(\rho - \overline{\rho}) dx.$$

Therefore,

$$\int_{\mathbb{T}^d} \rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\nabla\rho dx = \frac{1}{2}\int_{\mathbb{T}^d}\rho^2(\rho-\overline{\rho})dx,$$

and then the second integral on the right hand side of (2.38) is equal to

$$-\int_{\mathbb{T}^d}\nabla\cdot(\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho})(\rho-\overline{\rho}))dx = \frac{1}{2}\int_{\mathbb{T}^d}\rho^2(\rho-\overline{\rho})dx,$$

Next, notice that

$$\int_{\mathbb{T}^d}\rho^2(\rho-\overline{\rho})dx = \int_{\mathbb{T}^d}(\rho-\overline{\rho})^3dx + 2\overline{\rho}\int_{\mathbb{T}^d}(\rho-\overline{\rho})^2dx - 2\overline{\rho}^2.$$

By a Gagliardo-Nirenberg inequality (see e.g. [71] or [54] for a simple proof), we have

$$\|\rho-\overline{\rho}\|_{L^3}^3 \leqslant C\|\rho-\overline{\rho}\|_{L^2}^{3-\frac{d}{2}}\|\rho\|_{H^1}^{\frac{d}{2}} \leqslant \|\rho\|_{H^1}^2 + C_1\|\rho-\overline{\rho}\|_{L^2}^{\frac{12-2d}{4-d}},$$

where in the second step we applied Young's inequality. Applying all these estimates to (2.38) yields

$$\partial_t\|\rho-\overline{\rho}\|_{L^2}^2 \leqslant -\|\rho\|_{H^1}^2 + C_0\|\rho-\overline{\rho}\|_{L^2}^{\frac{12-2d}{4-d}} + 2\overline{\rho}\|\rho-\overline{\rho}\|_{L^2}^2. \tag{2.39}$$

Solving the differential equation

$$f'(\tau) = Cf^{\frac{6-d}{4-d}}(\tau) + 2\overline{\rho}f(\tau), \quad f(0) = B^2,$$

leads to the solution

$$f(\tau) = \frac{B^2\exp(2\overline{\rho}\tau)}{\left(1 - C\overline{\rho}^{-1}B^{\frac{4}{4-d}}(\exp(\frac{4\overline{\rho}\tau}{4-d}) - 1)\right)^{\frac{4-d}{2}}}. \tag{2.40}$$

A standard comparison argument can be used to show that $\|\rho(\cdot,t+\tau)-\overline{\rho}\|_{L^2}^2 \leqslant f(\tau)$. On the other hand, a straightforward estimate using (2.40) gives existence of a constant C_1 such that if $\tau \leqslant C_1\min(1,\overline{\rho}^{-1},B^{-\frac{4}{4-d}})$, then $f(\tau) \leqslant 4B^2$.

The second statement of the lemma follows directly from (2.39) and an assumption $\rho_0 \geqslant 0$.

□

2.3.5 The approximation lemma

We can now outline our general strategy of the proof of chemotactic blow up suppression in more detail. We know that the control of L^2 norm in spatial coordinates is sufficient for global regularity. We also see that if H^1 norm of the solution is large, then L^2 norm is not growing. On the other hand, flows with strong mixing properties tend to increase H^1 norm of solution. Hence our plan will be to deploy such flows, at a sufficiently strong intensity, to make sure that the H^1 norm of the solution stays high, at least whenever the L^2 norm is not small. The first hurdle we face, however, is to show that the mixing property of flow persists in the full nonlinear Keller-Segel.

In this section we prove a key result on approximation of solutions to Keller-Segel equation with advection (2.19) by solutions of pure advection equation. We will be looking at the intense advection regime, and consider small, relative to all parameters except the strength of advection, time intervals. It is natural to assume that in this case most of the dynamics we observe is due to advection, though the exact statement of the result requires care since both diffusion and chemotactic terms are not trivial perturbations.

Let us consider the equation (2.19)

$$\partial_t \rho + (u \cdot \nabla)\rho - \Delta\rho + \nabla \cdot (\rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho})) = 0, \quad \rho(x,0) = \rho_0(x),$$

$x \in \mathbb{T}^d$, with $d = 2, 3$. We will assume that the vector field u is divergence free and Lipschitz in spatial variables. It may be stationary or time dependent. Let us denote $\eta(x,t)$ the unique smooth solution of the equation

$$\partial_t \eta + (u \cdot \nabla)\eta = 0, \quad \eta(x,0) = \rho_0(x). \tag{2.41}$$

If we define the trajectories map by

$$\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x), t), \quad \Phi_0(x) = x, \tag{2.42}$$

then $\eta(x,t) = \rho_0(\Phi_t^{-1}(x))$.

We start with the following simple observation. Denote the Lipschitz semi-norm

$$\|f\|_{Lip} = \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|}.$$

Lemma 2.3.10. *Suppose that the vector field u is incompressible and Lipschitz in spatial variable for each $t \geqslant 0$, $\|u(\cdot,t)\|_{Lip(\mathbb{T}^d)} \leqslant D(t)$, $D(t) \in L^1_{loc}[0,\infty)$. Let $\eta(x,t)$ be the solution of (2.41). Then for every $t \geqslant 0$, and for every $\rho_0 \in H^1$, we have*

$$\|\eta(\cdot,t)\|_{H^1} \leqslant F(t)\|\rho_0\|_{H^1}, \quad \text{where } F(t) = \exp\left(C\int_0^t D(s)ds\right). \tag{2.43}$$

Proof. If u is incompressible and Lipschitz in spatial variable for each time, then the trajectories map $\Phi_t(x)$ is area preserving, Lipschitz in x and invertible for each t, and the inverse map $\Phi_t^{-1}(x)$ is also Lipschitz. Moreover, $\|\Phi_t^{-1}\|_{Lip} \leqslant \exp(C\int_0^t D(s)ds)$ (see e.g. [69]). The evolution $\eta(x,t) = \rho_0(\Phi_t^{-1})$ is a Lipschitz coordinate change of an H^1 function ρ_0. The bound (2.43) follows from well known properties of H^1 functions under Lipschitz transformations of coordinates [91]. □

We are now ready to prove the approximation lemma.

Lemma 2.3.11. *Suppose that the vector field $u(x,t)$ is incompressible and Lipschitz in x, and is such that (2.43) is satisfied with $F(t) \in L^\infty_{loc}[0,\infty)$. Let $\rho(x,t)$, $\eta(x,t)$ be solution of (2.19), (2.41) respectively with $\rho_0 \geqslant 0 \in H^1$. Suppose that the unique local smooth solution $\rho(x,t)$ exists for $t \in [0,T]$. Then for every $t \in [0,T]$ we have*

$$\frac{d}{dt}\|\rho-\eta\|_{L^2}^2 \leqslant -\|\rho\|_{H^1}^2 + 4\|\rho_0\|_{H^1}^2 F(t)^2 + C\|\rho-\bar{\rho}\|_{L^2}^2\left(\|\rho-\bar{\rho}\|_{L^2}^{\frac{12}{6-d}} + \bar{\rho}^2\right). \tag{2.44}$$

Of course a direct analog of lemma holds without assumption $\rho_0 \geqslant 0$; we only need to replace $\overline{\rho}_0$ in (2.44) with $\|\rho\|_{L^1}$.

Proof. A direct computation using divergence free property of u shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho-\eta\|_{L^2}^2 &= \int_{\mathbb{T}^d}\Delta\rho(\rho-\eta)dx - \int_{\mathbb{T}^d}\nabla\cdot(\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho}))(\rho-\eta)dx \\ &\leqslant -\|\rho\|_{H^1}^2 + \|\rho\|_{H^1}\|\eta\|_{H^1} + \|\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\|_{L^2}\|\rho\|_{H^1} \\ &+ \|\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\|_{L^2}\|\eta\|_{H^1}.\end{aligned} \tag{2.45}$$

Applying Hölder and Gagliardo-Nirenberg inequalities, we can estimate

$$\begin{aligned}\|\rho\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\|_{L^2} &\leqslant \|\rho\|_{L^3}\|\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\|_{L^6} \\ &\leqslant C\left(\|\rho\|_{H^1}^{\frac{d}{6}}\|\rho-\overline{\rho}\|_{L^2}^{1-\frac{d}{6}}+\overline{\rho}\right)\|\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\|_{H^1}^{\frac{d}{3}}\|\nabla(-\Delta)^{-1}(\rho-\overline{\rho})\|_{L^2}^{1-\frac{d}{3}} \\ &\leqslant C\left(\|\rho\|_{H^1}^{\frac{d}{6}}\|\rho-\overline{\rho}\|_{L^2}^{1-\frac{d}{6}}+\overline{\rho}\right)\|\rho-\overline{\rho}\|_{L^2}.\end{aligned}$$

Here the last step follows from simple estimates on Fourier side. Given these estimates, several applications of Young's inequality show that the right hand side of (2.45) can be bounded above by

$$\begin{aligned}&-\|\rho\|_{H^1}^2 + \frac{1}{4}\|\rho\|_{H^1}^2 + \|\eta\|_{H^1}^2 + \frac{1}{8}\|\rho\|_{H^1}^2 + C\|\rho-\overline{\rho}\|_{L^2}^{2+\frac{12}{6-d}} \\ &+ \frac{1}{8}\|\rho\|_{H^1}^2 + C\|\rho-\overline{\rho}\|_{L^2}^2\overline{\rho}^2 + \|\eta\|_{H^1}^2.\end{aligned}$$

With help of Lemma 2.3.10 the estimate (2.44) quickly follows. □

2.3.6 Proof of the main theorem: the relaxation enhancing flows

Our first example of flows that can stop chemotactic explosion will be relaxation enhancing flows of [17]. These stationary in time flows have been shown to be very efficient in speeding up convergence to the mean of the solution of diffusion-advection equation. A particular class of RE flows are weakly mixing flows, a well known class in dynamical systems theory which is intermediate in mixing properties between mixing and ergodic [22]. Let us briefly review the relevant definitions.

Given an incompressible Lipschitz in spatial variables vector field $u(x)$, recall the definition (2.42) for the trajectories map $\Phi_t(x)$. Then define a unitary operator $U^t f(x) = f(\Phi_t^{-1}(x))$ acting on $L^2(\mathbb{T}^d)$.

Definition 2.3.12. *The flow $u(x)$ is called weakly mixing if the spectrum of the operator $U \equiv U^1$ is purely continuous.*

The flow $u(x)$ is called relaxation enhancing (RE) if the operator U (or properly defined $(u \cdot \nabla)$) has no eigenfunctions in H^1 other than a constant function.

Remark. *The fact that we talk about the spectrum of U rather than $(u \cdot \nabla)$ is a minor technical point. The skew symmetric operator $(u \cdot \nabla)$ is unbounded on L^2, and sometimes need to be extended to its natural domain (rather than just H^1) to become self-adjoint and to be a generator for U. To avoid these technicalities, it is convenient to make the definition in terms of U, which can be defined on smooth functions and then extended to the entire L^2 by continuity.*

Examples of weakly mixing flows on $\mathbb{T}^d$ are classical and go back to von Neumann [75] (just continuous $u(x)$) and Kolmogorov [59] (smooth $u(x)$). The Kolmogorov construction is based on variable irrational rotation on the torus with appropriately selected invariant measure. Lack of eigenfunctions is established by analysis of a small denominator problem on Fourier side bearing some similarity to the core of the KAM theory. The original examples are not incompressible with respect to the Lebesgue measure on $\mathbb{T}^d$, but *a smooth change of coordinates can be applied to obtain incompressible flows with the same properties. Weakly mixing flows are also RE, but there are smooth RE flows which are not weakly mixing: these do have eigenfunctions but rough ones, lying in L^2 but not in H^1 (see [17] for more details and examples).*

We now state our first main theorem. Consider the equation

$$\partial_t \rho^A + A(u \cdot \nabla)\rho^A - \Delta \rho^A + \nabla \cdot (\rho^A \nabla (-\Delta)^{-1}(\rho - \overline{\rho})) = 0, \quad \rho^A(x,0) = \rho_0(x). \quad (2.46)$$

Here A is the coupling constant regulating the strength of the fluid flow that we will assume to be large. We note that dividing the equation by A and changing time, we can instead think of all the results below as applicable in the regime of weak diffusion and chemotaxis on long time scales.

Theorem 2.3.13. *Suppose that u is smooth and incompressible vector field on $\mathbb{T}^d$, $d = 2, 3$, which is also relaxation enhancing. Suppose that $\rho \geqslant 0 \in C^\infty(\mathbb{T}^d)$. Then there exists an amplitude A_0 which depends only on ρ_0 and u such that for every $A \geqslant A_0$ the solution $\rho^A(x,t)$ of the equation (2.46) is globally regular.*

We will only prove Theorem 2.3.13 in the case of weakly mixing flows. This serves our main purpose of providing an example of chemotactic blow up-arresting flow. In general case, the proof is a fairly straightforward extension of an argument for the weakly mixing case and the point spectrum estimates in [17]. (Lemma 3.3 and part of the proof of Theorem 1.4 dealing with point spectrum).

Before starting the proof, we need one auxiliary result from [17]. Let P_N be the projection operator on the subspace formed by Fourier modes $|k| \leqslant N$:

$$P_N f(x) = \sum_{|k| \leqslant N} e^{2\pi i k x} \hat{f}(k).$$

Lemma 2.3.14. *Let U be an unitary operator with purely continuous spectrum defined on $L^2(\mathbb{T}^d)$. Let $S = \{\phi \in L^2 : \|\phi\|_{L^2} = 1\}$, and let $K \subset S$ be a compact set. Then for every N and every $\sigma > 0$, there exists $T_c(N, \sigma, K, U)$ such that for all $T \geqslant T_c(N, \sigma, K, U)$ and every $\phi \in K$, we have*

$$\frac{1}{T}\int_0^T \|P_N U^t \phi\|^2 dt \leqslant \sigma. \tag{2.47}$$

This lemma connects the issues we are studying with one of the themes in quantum mechanics, namely the propagation rate of wave packets corresponding to continuous spectrum. Lemma 2.3.14 is an extension of the well-known RAGE theorem (see e.g. [24]) which is a rigorous variant of a folklore quantum mechanics statement that quantum states corresponding to the continuous spectrum travel to infinity. In our case, travel to infinity happens not in physical space, but in the modes of the operator $-\Delta$ (that is, in Fourier modes). We refer to [17] for the proof of Lemma 2.3.14.

Now we are ready to give the proof of Theorem 2.3.13.

Proof of Theorem 2.3.13. Fix any $B > \|\rho_0 - \overline{\rho}_0\|_{L^2}$. If for all times we have that $\|\rho^A(\cdot, t) - \overline{\rho}\|_{L^2} < B$, then the solution stays globally regular by Theorem 2.3.6. Otherwise, let

$$t_0 = \inf\{t \big| \|\rho^A(\cdot, t) - \overline{\rho}\|_{L^2} = B\}.$$

Since the solution is smooth, we also have that $\|\rho^A(\cdot, t_0) - \overline{\rho}\|_{L^2} = B$; thus t_0 is the first time the L^2 norm of $\rho^A - \overline{\rho}$ reaches B. Note that by Proposition 2.3.7, we must also have $\|\rho^A(\cdot, t_0)\|_{H^1} < B_1$, where $B_1 = C_0 B^{\frac{12-2d}{4-d}} + 2\overline{\rho}B^2$.

We are going to show that if $A \geqslant A_0(B, \overline{\rho}, u)$ is sufficiently large, then after a small time interval of length τ that we will define shortly, we will have $\|\rho^A(\cdot, t_0+\tau) - \overline{\rho}\|_{L^2} < B$. Moreover, we will have $\|\rho^A(\cdot, t) - \overline{\rho}\|_{L^2} \leqslant 2B$ for every $t \in [t_0, t_0 + \tau]$. This will prove the theorem, as the argument can be applied repeatedly each time the L^2 norm reaches the level B, showing that $\|\rho^A(\cdot,t)-\overline{\rho}\|_{L^2}\leqslant 2B$ for all t.

Denote λ_n the eigenvalues of $-\Delta$ on $\mathbb{T}^d$ in an increasing order, $0=\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\ldots$ Choose N so that

$$\lambda_N\geqslant 16C_0(2B)^{\frac{4}{4-d}}+32\overline{\rho}, \tag{2.48}$$

where C_0 is the constant appearing in (2.39). Observe that $\lambda_N B^2>B_1^2$. Define the compact set $K\subset S$ by

$$K=\{\phi\in S\big|\|\phi\|_{H^1}^2\leqslant\lambda_N\}$$

(recall S is the unit sphere in L^2). Let U be the unitary operator associated with our weakly mixing flow u as above. Fix $\sigma=0.01$. Let $T_c(N,\sigma,K,U)$ be the time threshold provided by Lemma 2.3.14.

We proceed to impose the first condition on $A_0(\rho_0,u)$. We define τ as below and require that

$$\tau\equiv\frac{T_c(N,\sigma,K,U)}{A}\leqslant C_1\min\left(1,\overline{\rho}^{-1},B^{-\frac{4}{4-d}}\right) \tag{2.49}$$

for every $A\geqslant A_0$, where C_1 is the constant appearing in Proposition 2.3.7 in (2.36). It follows from Proposition 2.3.7 and Theorem 2.3.6 that $\|\rho^A(\cdot,t)-\overline{\rho})\|_{L^2}\leqslant 2B$ for $t\in[t_0,t_0+\tau]$ and so ρ^A remains smooth on the time interval.

Let us introduce a short-cut notation $\phi_0(x)=\rho^A(x,t_0)$. Let $\eta^A(x,t)$ be the solution of the equation

$$\partial_t\eta^A+A(u\cdot\nabla)\eta^A=0,\quad \eta^A(x,0)=\phi_0.$$

Then $\eta^A(x,t)=U^{At}\phi_0$, and we have

$$\frac{1}{\tau}\int_0^\tau\|P_N\eta^A(x,t)\|_{L^2}^2dt=\frac{1}{\tau}\int_0^\tau\|P_NU^{At}\phi_0\|_{L^2}^2dt=\frac{1}{A\tau}\int_0^{A\tau}\|P_NU^s\phi_0\|_{L^2}^2ds\leqslant\sigma B^2. \tag{2.50}$$

Here we applied Lemma 2.3.14 to the vector $\phi_0/\|\phi_0\|_{L^2}$. Indeed, $\phi_0/\|\phi_0\|_{L^2} \in K$ since by (2.48) we have

$$\|\phi_0\|_{H^1}^2 \leqslant B_1^2 < \lambda_N B^2 < \lambda_N \|\phi_0\|_{L^2}^2.$$

Note also that (2.49) ensures the applicability of Lemma 2.3.14 and the validity of the last bound in (2.50).

We now impose the second and last condition on A_0. It will be convenient for us now to denote by t time elapsed since t_0. By the approximation Lemma 2.3.11, and since we know $\|\rho^A(\cdot, t_0)\|_{H^1}^2 \leqslant B_1 < \lambda_N B^2$, as well as $\|\rho^A(\cdot, t_0+t) - \overline{\rho}\|_{L^2} \leqslant 2B$, we have

$$\frac{d}{dt}\|\rho^A(\cdot, t_0+t) - \eta^A(\cdot, t)\|_{L^2}^2 \leqslant 4\lambda_N B^2 F(At)^2 + CB^2(B^{\frac{6}{6-d}} + \overline{\rho}^2) \tag{2.51}$$

for all $t \in [0, \tau]$. Here $\tau = T_c/A$ as before. Choose A_0 so that

$$\frac{4\lambda_N}{A}\int_0^{T_0} F(s)^2 ds + C\tau(B^{\frac{6}{6-d}} + \overline{\rho}^2) \leqslant 0.01 \tag{2.52}$$

for every $A \geqslant A_0$. Note that since u is smooth, $F(t)$ is a locally bounded function.

We claim that if $A \geqslant A_0$, then $\|\rho^A(\cdot, t_0+\tau) - \overline{\rho}\|_{L^2} \leqslant B$. First, the condition (2.52) allows us to control $\|\rho^A(\cdot, t_0+t) - \overline{\rho}\|_{L^2}$ more tightly, which is convenient. Indeed, since $\|\eta^A(\cdot, t)\|_{L^2} = \|\phi_0\|_{L^2} = B$ for all $t \geqslant 0$, (2.51) and (2.52) imply that

$$0.9B \leqslant \|\rho^A(\cdot, t_0+t)\|_{L^2} \leqslant 1.1B \tag{2.53}$$

for $t \in [0, \tau]$. Furthermore, by (2.50), (2.51) and (2.52) we have

$$\begin{aligned}\frac{1}{\tau}\int_0^{\tau} \|P_N\rho^A(\cdot, t_0+t)\|_{L^2}^2 dt \leqslant \frac{2}{\tau}\int_0^{\tau} \|P_N\eta^A(\cdot, t)\|_{L^2}^2 dt \\ + \frac{2}{\tau}\int_0^{\tau} \|P_N(\rho^A(\cdot, t_0+t) - \eta^A(\cdot, t))\|_{L^2}^2 dt \leqslant \frac{B^2}{25}.\end{aligned}$$

Combine this estimate with 2.53 we obtain

$$\frac{1}{\tau}\int_0^{\tau} \|\rho^A(\cdot, t_0+t)\|_{H^1}^2 dt \geqslant \frac{1}{\tau}\int_0^{\tau} \lambda_N\|(I - P_N)\rho^A(\cdot, t_0+t)\|_{L^2}^2 dt \geqslant \frac{1}{2}\lambda_N B^2. \tag{2.54}$$

Now we come back to (2.39):

$$\partial_t \|\rho^A - \overline{\rho}\|_{L^2}^2 \leqslant -\|\rho^A\|_{H^1}^2 + C_0\|\rho^A - \overline{\rho}\|_{L^2}^{\frac{12-2d}{4-d}} + 2\overline{\rho}\|\rho^A - \overline{\rho}\|_{L^2}.$$

By estimate (2.54), (2.53) and (2.48), we have

$$\begin{aligned} \|\rho^A(\cdot, t_0+\tau) - \overline{\rho}\|_{L^2}^2 &\leqslant B^2 + \tau\left(-\frac{1}{2}\lambda_N B^2 + C_0(2B)^{\frac{12-2d}{4-d}} + 2\overline{\rho}(2B)^2\right) \\ &\leqslant \left(1 - \frac{1}{4}\tau\lambda_N\right) B^2 \leqslant B^2. \end{aligned} \tag{2.55}$$

This complete the proof. □

We see that the bound we obtained on the decay of the L^2 norm in (2.55) is stronger than what we needed. In fact, with slightly more effort we can obtain stronger results. We now present an extension of Theorem 2.3.13 that establishes a complete analog of "relaxation enhancement" established in [17] for the diffusion-advection equation for the case that also includes chemotaxis. Namely, we show that not only fluid flow can prevent finite time blow up, but in fact it can enforce convergence of the solution to its mean in the long time limit. Intense fluid flow can also act to create an arbitrary strong and fast drop of $\|\rho^A(\cdot,t) - \overline{\rho}\|_{L^2}$.

Theorem 2.3.15. *Suppose $0 \leqslant \rho_0 \in C^\infty(\mathbb{T}^d)$, and let $\rho^A(x,t)$ be the solution of the equation (2.46). Let u be smooth, incompressible, relaxation enhancing flow. If $A_0(\rho_0, u)$ is the threshold value of Theorem 2.3.13, then for every $A \geqslant A_0$, we have*

$$\|\rho^A(\cdot,t) - \overline{\rho}\|_{L^2} \to 0 \tag{2.56}$$

as $t \to \infty$. The convergence rate is exponential in time, and can be made arbitrary fast by increasing the value of A. Namely, for every $\delta > 0$ and $\kappa > 0$, there exists $A_1 = A_1(\rho_0, u, \kappa, \delta)$ such that if $A \geqslant A_1$, then

$$\|\rho^A(\cdot,t) - \overline{\rho}\|_{L^2} \leqslant \|\rho_0 - \overline{\rho}\|_{L^2} e^{-\kappa t} \tag{2.57}$$

for all $t \geqslant \delta$.

Proof. Let us proof (2.57), since (2.56) follows from similar (and easier) arguments. The proof largely follows the argument in the proof of Theorem 2.3.13, but let us outline the necessary adjustments. We set $B_0 = \|\rho_0 - \overline{\rho}\|_{L^2}$. Choose N so that

$$\lambda_N \geqslant \max\left(100\kappa, 16C_0(2B_0)^{\frac{4}{4-d}} + 32\overline{\rho}\right). \tag{2.58}$$

Define the set K, as before, by $\{\phi \in S \big| \|\phi\|_{H^1}^2 \leqslant \lambda_N\}$.

For all times t where $\rho^A(x,t)/\|\rho^A(\cdot,t)\|_{L^2} \notin K$, we have $\|\rho^A(\cdot,t)\|_{H^1} \geqslant \lambda_N \|\rho^A(\cdot,t) - \overline{\rho}\|_{L^2}$. It follows from (2.39) and (2.58) that at such times we have

$$\begin{aligned}\partial_t \|\rho - \overline{\rho}\|_{L^2}^2 &\leqslant -\|\rho^A\|_{H^1}^2 + C_0\|\rho^A - \overline{\rho}\|_{L^2}^{\frac{12-2d}{4-d}} + 2\overline{\rho}\|\rho^A - \overline{\rho}\|_{L^2} \\ &\leqslant \left(-\lambda_N + C_0(2B_0)^{\frac{4}{4-d}} + 2\overline{\rho}\right)\|\rho^A - \overline{\rho}\|_{L^2}^2 \leqslant -\frac{1}{2}\lambda_N\|\rho^A - \overline{\rho}\|_{L^2}^2.\end{aligned} \tag{2.59}$$

Here in the second inequality we used that $\|\rho^A(\cdot,t) - \overline{\rho}\|_{L^2} \leqslant 2B_0$ for all times, as we know from the proof of Theorem 2.3.13 we can ensure by making A sufficiently large; we note that this bound will also follow from our argument below. Thus on the time intervals where $\rho^A(x,t)/\|\rho^A(x,t)\|_{L^2} \notin K$ we have exponential decay of $\|\rho^A(\cdot,t) - \overline{\rho}\|_{L^2}$ at rate that would imply (2.57) if all times were like that.

Suppose now that t_0 is the smallest time such that $\rho^A(x,t_0) \in K$ (t_0 could equal 0). Let $T_c(N,\sigma,K,U)$ be the time threshold provided by Lemma 2.3.14 (we set $\sigma = 0.01$ as before). Repeat all the steps in the proof of Theorem 2.3.13 from defining the time step τ (2.49) to (2.55), with B replaced by $\|\rho^A(\cdot,t_0) - \overline{\rho}\|_{L^2}$. In addition, require that A is large enough so that

$$\tau = \frac{T_c(N,\sigma,K,U)}{A} \leqslant \delta/2. \tag{2.60}$$

We arrive at the estimate

$$\begin{aligned}\|\rho^A(\cdot,t_0+\tau)-\overline{\rho}\|_{L^2} &\leqslant \left(1-\frac{1}{4}\lambda_N\tau\right)\|\rho^A(\cdot,t_0)-\overline{\rho}\|_{L^2}^2\\ &\leqslant e^{-\frac{1}{4}\lambda_N\tau}\|\rho^A(\cdot,t_0)-\overline{\rho}\|_{L^2}^2.\end{aligned} \tag{2.61}$$

Note that even though we do not control the L^2 norm of the solution for all times t, from (2.39) and (2.58) it is clear that for every $t\in[t_0,t_0+\tau]\equiv I_0$, we have

$$\|\rho^A(\cdot,t)-\overline{\rho}\|_{L^2}^2 \leqslant e^{\frac{1}{8}\lambda_N(t-t_0)}\|\rho^A(\cdot,t_0)-\overline{\rho}\|_{L^2}^2. \tag{2.62}$$

We continue further in time in a similar fashion. If $\rho^A(x,t)/\|\rho^A(\cdot,t)\|_{L^2}\notin K$, we have (2.59). On the other hand, if

$$t_n=\inf\{t\big|t\geqslant t_{n-1}+\tau,\rho^A(x,t)/\|\rho^A(\cdot,t)\|_{L^2}\in K\},$$

we can apply Lemma 2.3.14 and Lemma 2.3.11 on $I_n\equiv[t_n,t_n+\tau]$ obtaining

$$\begin{aligned}&\|\rho^A(\cdot,t_n+\tau)-\overline{\rho}\|_{L^2}^2 \leqslant e^{-\frac{1}{4}\lambda_N\tau}\|\rho^A(\cdot,t_n)-\overline{\rho}\|_{L^2}^2,\\ &\|\rho^A(\cdot,t)-\overline{\rho}\|_{L^2}^2 \leqslant e^{\frac{1}{8}\lambda_N(t-t_n)}\|\rho^A(\cdot,t_n)-\overline{\rho}\|_{L^2}^2, \quad \text{for every } t\in[t_n,t_n+\tau].\end{aligned} \tag{2.63}$$

Now given any $t\geqslant\delta$, we can represent

$$[0,t]=W\cup(\cup_{l=0}^n I_l),$$

where W is the set of times in $[0,t]$ outside all I_l. Note that (2.59) holds for every $s\in W$. Combining (2.59) and (2.63), we infer that for every $t\geqslant\delta$, we have

$$\|\rho^A(\cdot,t)-\overline{\rho}\|_{L^2}^2 \leqslant e^{-\frac{1}{4}\lambda_N(t-\frac{\delta}{2})}e^{\frac{1}{8}\lambda_N\frac{\delta}{2}}\|\rho_0-\overline{\rho}\|_{L^2}^2 \leqslant e^{-\frac{1}{8}\lambda_N t}\|\rho_0-\overline{\rho}\|_{L^2}^2.$$

This proves (2.57). $\square$

2.3.7 Yao-Zlatoš flow

In this section, we describe another class of flows that are capable of suppressing the chemotactic explosion. These flows arise as examples of almost perfect mixers satisfying some sort of natural constraints. They have the advantage of being somewhat more explicitly defined than the RE flows which may be harder to picture. For this reason we will be able to get an explicit estimate, albeit rather weak, for the intensity of mixing necessary to arrest the blow up as a function of the L^2 norm of the initial data. For the RE flows, such estimate would be difficult to obtain, primarily due to the challenge of estimating the time T_c from Lemma 2.3.14. A quantitative estimate on T_c would require delicate spectral analysis of the operator $u \cdot \nabla$, something that for the moment is out of reach. On the other hand, in contrast to the RE flows, Yao-Zlatoš flows are time dependent and active - their construction depends on the density being mixed. We remark that a re lated cl ass of efficient mixer flows has been also considered in [2].

We refer to [88] for a detailed discussion of different notions of mixing and the general background, and for further references. For our purpose here, we need one particular result from [88], that we set about to explain. Let $\mathbb{T}^2 \equiv [-1/2, 1/2)^2$. Consider the dyadic partition of $\mathbb{T}^2$ with squares Q_{nij} given by

$$Q_{nij} = \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \quad i, j = -2^{n-1}, \dots, 2^{n-1} - 1. \tag{2.64}$$

Suppose $f_0 \in C^\infty(\mathbb{T}^2)$ and is mean zero, u is an incompressible flow with Lipschitz regularity. Let $f(x,t)$ denote the solution of transport equation

$$\partial_t f + (u \cdot \nabla) f = 0, \quad f(x,0) = f_0(x). \tag{2.65}$$

Theorem 2.3.16. *[Yao-Zlatoš] Given any $\kappa, \epsilon \in (0, 1/2]$, there exists an incompressible* *flow u such that following holds.*

$$1. \quad \|\nabla u(\cdot,t)\|_{L^\infty} \leqslant 1, \quad \textit{for every } t. \tag{2.66}$$

2. Let $n = [|\log_2(\kappa\epsilon)|] + 2$, where $[x]$ denotes the integer part of x. Then for some

$$\tau_{\kappa,\epsilon} \leqslant C\kappa^{-1/2}|\log(\kappa\epsilon)|^{3/2},$$

and every Q_{nij} as in (2.64) we have

$$\left| \frac{1}{|Q_{nij}|} \int_{Q_{nij}} f(x,\tau_{\kappa,\epsilon})\,dx \right| \leqslant \kappa \|f_0\|_{L^\infty}. \tag{2.67}$$

This theorem provides the flow u that satisfies uniform in time Lipschitz constraint (2.66) and mixes the initial density f_0 to scale ϵ with error κ in time $\tau_{\kappa,\epsilon}$. The construction in [88] employs a multi-scale cellular flow. On the nth stage of the construction, the goal is to make the mean of the function on each of Q_{nij} close to zero, starting with $n = 1$. This is achieved by cellular flows which are designed to have same rotation time period on streamlines away from a thin boundary layer. Such an arrangement makes evolution of density in each cell amenable to fairly precise control, and makes it possible to ensure that the "nearly mean zero" property of the solution propagates to smaller and smaller scales. We refer to [88] for the details. Below, we outline only some adjustments that are needed to obtain Theorem 2.3.16 from the arguments in [88], since it is not stated there in the precise form we need.

Proof. Theorem 2.3.16 is essentially Theorem 4.3 from [88] (or rather Theorem 5.1, which deals with the periodic boundary conditions instead of no flow - but Theorem 5.1 is a direct corollary of Theorem 4.3). We re-scaled the time compared to Theorem 4.3 from [88] to make (2.66) hold. We also replaced the conclusion of the ϵ-scale mixing (defined in [88]) with how it is actually proved: (2.67) follows directly from (4.8) and the next estimate in [88] as well as the choice of δ immediately below these two estimates. □

Here is the key corollary of Theorem 2.3.16 that we will use in our proof.

Corollary 2.3.17. *Let $f_0 \in C^\infty(\mathbb{T}^2)$ be a mean zero function. For every $\epsilon > 0$ there exists a Yao-Zlatoš flow $u(x,t)$ given by Theorem 2.3.16, such that $\|\nabla u\|_{L^\infty} \leqslant 1$ for every t and the solution $f(x,t)$ of the equation (2.65) satisfies*

$$\|f(\cdot,\tau)\|_{H^{-1}} \leqslant C_3 \|f_0\|_{L^\infty} \epsilon \tag{2.68}$$

for some

$$\tau \leqslant C_2 \epsilon^{-1/2} |\log \epsilon|^{3/2}. \tag{2.69}$$

Here $C_{2,3} \geqslant 1$ are universal constants.

Proof. To derive this corollary from Theorem 2.3.16, let us set $\kappa = \epsilon$. We need to address a couple of issues. The first one is the connection between (2.67) and H^{-1} norm of the solution.

Lemma 2.3.18. *Let $f \in C^\infty(\mathbb{T}^2)$ be mean zero. Fix $\epsilon > 0$ and suppose that*

$$\left| \frac{1}{|Q_{nij}|} \int_{Q_{nij}} f(x)dx \right| \leqslant \epsilon \|f\|_{L^\infty}$$

for some $n \geqslant [|\log_2 \epsilon|]$ and $i, j = -2^{n-1}, \ldots, 2^{n-1}-1$. Then

$$\|f\|_{H^{-1}} \leqslant C_3 \|f\|_{L^\infty} \epsilon. \tag{2.70}$$

Proof. The proof is by duality. Take any $g \in H^1$. Since f is mean zero, without loss of generality we can assume that g is also mean zero. Then, denoting $\overline{g}_{Q_{nij}}$ the average value of g over Q_{nij}, we have

$$\left|\int_{\mathbb{T}^2} fgdx\right| = \left|\sum_{i,j}\int_{Q_{nij}} fgdx\right| \leqslant \left|\sum_{i,j}\int_{Q_{nij}} f(x)(g(x)-\overline{g}_{Q_{nij}})dx\right| +$$
$$\left|\sum_{i,j}\overline{g}_{Q_{nij}}\int_{Q_{nij}} f(x)dx\right| \leqslant \sum_{i,j}\|f\|_{L^2(Q_{nij})}\|g-\overline{g}_{Q_{nij}}\|_{L^2(Q_{nij})}+$$
$$\epsilon\sum_{i,j}|\overline{g}_{Q_{nij}}||Q_{nij}|\|f\|_{L^\infty} \leqslant C2^{-n}\sum_{i,j}\|f\|_{L^2(Q_{nij})}\|\nabla g\|_{L^2(Q_{nij})} + \epsilon\|f\|_{L^\infty}\|g\|_{L^1}$$
$$\leqslant C\epsilon\|f\|_{L^2}\|g\|_{H^1} + \epsilon\|f\|_{L^\infty}\|g\|_{H^1} \leqslant C\epsilon\|f\|_{L^\infty}\|g\|_{H^1}.$$

Here we used Poincare inequality in the last and in the penultimate step, and Cauchy-Schwartz inequality in the last step. This proves the lemma. □

Another technical aspect we need to discuss is the smoothness of u. The construction in [88] does not explicitly control higher order derivatives of u beyond the Lipschitz condition $\|\nabla u\|_{L^\infty} \leqslant 1$. However, it is not difficult to see that a properly mollified velocity field will have the same mixing properties up to renormalization by some universal constant. Let ϕ be a mollifier, $\phi \geqslant 0$, $\phi \in C^\infty(\mathbb{T}^2)$, supp$\phi \subset B_{1/4}(0)$, $\int_{\mathbb{T}^2}\phi(x)dx = 1$. Denote $\phi_\delta(x) = \delta^{-d}\phi(x/\delta)$, and $u_\delta(x) = \phi_\delta * u(x)$.

Lemma 2.3.19. *Suppose that an incompressible vector field $u(x,t)$ satisfies $\|\nabla u\|_{L^\infty} \leqslant D$ for all x,t. Assume $f_0 \in C^\infty(\mathbb{T}^2)$, and denote $f(x,t)$ and $f_\delta(x,t)$ solutions of the transport equation (2.65) with velocity u and mollified velocity u_δ respectively. Fix any $T > 0$. Then as $\delta \to 0$, we have $\|f(x,t) - f_\delta(x,t)\|_{L^2} \to 0$ uniformly in $t \in [0,T]$.*

Proof. The proof of this lemma is standard and elementary. We provide a brief sketch. First note that $\|\nabla u_\delta\|_{L^\infty} \leqslant \|\nabla u\|_{L^\infty}$, so (2.66) also holds for u_δ. Consider $\Phi_t(x)$ and $\Phi_{t,\delta}(x)$, the trajectory maps corresponding to u and u_δ. A straightforward estimate based on Gronwall lemma gives

$$|\Phi_t(x) - \Phi_{t,\delta}(x)| \leqslant \delta e^{tD}.$$

Reversing time, we find that the same holds for the inverse maps:

$$|\Phi_t^{-1}(x) - \Phi_{t,\delta}^{-1}(x)| \leqslant \delta e^{tD}.$$

So we get

$$\int_{\mathbb{T}^2} |f(x,t) - f_\delta(x,t)|^2 dx = \int_{\mathbb{T}^2} |f_0(\Phi_t^{-1}(x)) - f_0(\Phi_{t,\delta}^{-1}(x))|^2 dx \leqslant \delta \|\nabla f_0\|_{L^\infty}^2 e^{tD}.$$

□

To complete the proof of the corollary, note now that we can choose $\delta = \delta(f_0)$ small enough so that

$$\|f_\delta(\cdot,\tau) - f(\cdot,\tau)\|_{H^{-1}} \leqslant \|f_\delta(\cdot,\tau) - f(\cdot,\tau)\|_{L^2} \leqslant C_3 \|f_0\|_{L^\infty} \epsilon.$$

Then if $u(x,t)$ is Yao-Zlatoš vector field yielding (2.68), we can take u_δ as our smooth flow and get that (2.68) holds with the renormalized constant $2C_3$. □

Before stating our main result on Yao-Zlatoš flows, we need one more auxiliary result. In our scheme, it is convenient to work with the L^2 norm of the solution. However Corollary 2.3.17 involves the L^∞ norm, so we need some control over it. We could get it from (2.35) and Sobolev embedding. However the bound in (2.35) involves norms of higher order derivatives of u and would result in weaker estimates for the flow intensity needed to suppress blow up. We prefer to estimate the L^∞ norm of the solution directly.

Let $\rho(x,t)$ be the solution of (2.19):

$$\partial_t \rho + (u \cdot \nabla)\rho - \Delta\rho + \nabla \cdot (\rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho})) = 0, \quad \rho(x,0) = \rho_0(x), \quad x \in \mathbb{T}^d.$$

where $u(x,t)$ is smooth and incompressible.

Proposition 2.3.20. *Let $0 \leqslant \rho \in C^\infty(\mathbb{T}^2)$. Suppose that $\|\rho(\cdot,t) - \overline{\rho}\|_{L^2} \leqslant 2B$ for all $t \in [0,T]$ and some $B \geqslant 1$. Then we also have $\|\rho(\cdot,t) - \overline{\rho}\|_{L^\infty} \leqslant C_4 B \max(B, \overline{\rho}^{1/2})$ for some universal constant C_4 and all time $t \in [0,T]$.*

We postpone the proof of this proposition to the appendix C.

We are now ready to state the main theorem of this section.

Theorem 2.3.21. *Let $\rho_0 \geqslant 0 \in C^\infty(\mathbb{T}^2)$, and suppose $\|\rho_0 - \overline{\rho}\|_{L^2} < B$ for some $B > 1$. Then there exists smooth incompressible flow $u(x,t)$ with $\|\nabla u(\cdot,t)\|_{L^\infty} \leqslant A(B,\overline{\rho})$ such that the solution $\rho(x,t)$ of the equation (2.19) is globally regular. Here we can choose*

$$A = C \exp\Big(C(1 + B + \overline{\rho}^{1/2})\big(\log(1 + B + \overline{\rho}^{1/2})\big)^{3/2}\Big) \tag{2.71}$$

for some universal constant C.

The flow $u(x,t)$ can be represented as

$$u(x,t) = \sum_j A u_j(x,t)\chi_{I_j}(t), \tag{2.72}$$

where I_j are disjoint time intervals, and u_j are Yao-Zlatoš flows given by Corollary 2.3.17 with a certain $\epsilon = \epsilon(B,\overline{\rho}) > 0$ and certain initial data.

Remark 2.3.22. *We have to deploy different Yao-Zlatoš flows in (2.72) due to the fact that these flows are designed to mix a specific initial data, and one can envision nonlinear dynamics attempting to break the L^2 barrier in different ways.*

Remark 2.3.23. *Similar to Theorem 2.3.15 for the RE flows, one can use combinations of Yao-Zlatoš flows to achieve stronger results, such as convergence of the solution to the mean, and at increasingly fast rate if the flow amplitude is allowed to grow. We will not pursue these results here.*

Proof. The scheme of the proof is similar to the RE flows case, but Corollary 2.3.17 replaces Lemma 2.3.14 with some necessary adjustments.

We start with $u = 0$ on some initial time interval. If $\|\rho(\cdot,t) - \overline{\rho}\|_{L^2} \leqslant B$ for all t, global regularity follows. Suppose this is not the case, and let t_0 be the first time when $\|\rho(\cdot,t_0) - \overline{\rho}\|_{L^2} = B$. Similarly to the RE case, we also know that $\|\rho(\cdot,t_0)\|_{H^1} \leqslant B_1$. In addition, Proposition 2.3.20 ensures that $\|\rho(\cdot,t_0) - \overline{\rho}\|_{L^\infty} \leqslant C_4 B \max(B, \overline{\rho}^{1/2})$.

Fix ϵ given by

$$\epsilon = \frac{1}{8C_3\sqrt{4C_0B^2 + 2\overline{\rho} + 1}(1 + C_4 \max(B, \overline{\rho}^{1/2}))}. \tag{2.73}$$

Take a flow u guaranteed by Corollary 2.3.17 corresponding to this value of ϵ and the initial density $\rho(x,t_0)$, and let τ be the time in (2.69). Denote $\eta^A(x,t)$ the solution of the equation

$$\partial_t \eta^A + A(u \cdot \nabla)\eta^A = 0, \quad \eta^A(x,0) = \eta_0 \equiv \rho(x,t_0). \tag{2.74}$$

Then by Corollary 2.3.17 and Proposition 2.3.20 we have

$$\|\eta^A(\cdot,\tau/A) - \overline{\rho}\|_{H^{-1}} \leqslant C_3\|\eta_0 - \overline{\rho}\|_{L^\infty}\epsilon \leqslant \frac{B}{8\sqrt{4C_0B^2 + 2\overline{\rho} + 1}}. \tag{2.75}$$

We will set $u(x,t) = 0$ in (2.74) for $t \geqslant \tau/A$.

Now we are going to turn on the same flow $u(x,t)$ in the equation for ρ^A at time t_0, for the duration τ/A. Let us denote $\rho^A(x,t_0+t)$ the solution of the equation

$$\partial_t \rho^A + A(u \cdot \nabla)\rho^A - \Delta\rho^A + \nabla \cdot (\rho^A \nabla(-\Delta)^{-1}(\rho^A - \overline{\rho})) = 0, \quad \rho^A(x,0) = \rho(x,t_0).$$

The first condition that we are going to impose on A is that

$$\frac{2\tau}{A} \leqslant C_1 \min\left(1, \overline{\rho}^{-1}, B^{-2}\right). \tag{2.76}$$

Given (2.69) and (2.73), it is easy to check that (2.76) holds if (2.71) is satisfied. Recall that given (2.76), Proposition 2.3.7 ensures

$$\|\rho^A(x, t_0 + t) - \overline{\rho}\|_{L^2} \leqslant 2B \tag{2.77}$$

for $t \in [0, 2\tau/A]$ and the solution stays smooth on this time interval. Next, by the approximation Lemma 2.3.11, we have

$$\begin{aligned}\frac{d}{dt}\|\rho^A(\cdot, t_0 + t) - \eta^A(\cdot, t)\|_{L^2}^2 &\leqslant -\|\rho^A(\cdot, t_0 + t)\|_{H^1}^2 + 4\|\rho(\cdot, t_0)\|_{H^1}^2 \exp\left(2C \int_0^{At} \|\nabla u\|_{L^\infty} ds\right) \\ &\quad + C\|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{L^2}^2 \left(\|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{L^2}^3 + \overline{\rho}^2\right) \\ &\leqslant 4B_1^2 \exp(2CAt) + 4CB^2 \left((2B)^3 + \overline{\rho}^2\right)\end{aligned} \tag{2.78}$$

for every $t \in [0, 2\tau/A]$. Here we used (2.77) in the last step. Let us now impose the second condition on A which says that it should be large enough so that

$$\frac{2B_1^2}{CA} \exp(4C\tau) + \frac{2\tau}{A} 4CB^2 \left((2B)^3 + \overline{\rho}^2\right) \leqslant \frac{B^2}{64(4C_0B^2 + 2\overline{\rho} + 1)}. \tag{2.79}$$

Note that in particular (2.79) implies that

$$0.8B \leqslant \|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{L^2} \leqslant 1.2B \tag{2.80}$$

for $t \in [0, 2\tau/A]$. Also, (2.79), (2.75) and (2.78) can be used to estimate that for every $t \in [\tau/A, 2\tau/A]$ we have

$$\|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{H^{-1}} \leqslant \|\eta^A(\cdot, t) - \overline{\rho}\|_{H^{-1}} + \|\rho^A(\cdot, t_0 + t) - \eta^A(\cdot, t)\|_{L^2} \leqslant \frac{B}{8\sqrt{4C_0B^2 + 2\overline{\rho} + 1}} + \frac{B}{8\sqrt{4C_0B^2 + 2\overline{\rho} + 1}} = \frac{B}{4\sqrt{4C_0B^2 + 2\overline{\rho} + 1}}.$$

Therefore, using (2.80), we obtain

$$\|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{H^1} \geqslant \frac{\|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{L^2}^2}{\|\rho^A(\cdot, t_0 + t) - \overline{\rho}\|_{H^{-1}}} \geqslant 2B\sqrt{4C_0B^2 + 2\overline{\rho} + 1} \tag{2.81}$$

for all $t \in [\tau/A, 2\tau/A]$. The proof of global regularity is completed similarly to the proof of Theorem 2.3.13 using (2.39), (2.80) and (2.81).

Finally, the sufficiency of the condition (2.71) follows from straightforward analysis of the bounds (2.73), (2.69), and (2.79). The approximation lemma constraint (2.79) is what truly determines the exponential form of (2.71). □

Remark 2.3.24. *One can check that by using Theorem 5.4 from [88] and similar arguments, we can change the periodic setting to the finite domain with no-slip or no-flow boundary condition for $u(x,t)$, and obtain analogous results but with a weaker estimate for the flow intensity. We leave the details to the interested reader.*

2.3.8 Discussion and generalization

The scheme developed in the previous sections of this section should be flexible enough to be applied in different situations. Here we briefly and informally discuss the main features that appear to be necessary to apply our analysis. On the most general informal level, one can say that the idea of the scheme is that the fluid flow, if sufficiently intense and with strong mixing properties, can make a supercritical equation into subcritical one for a given initial data.

It appears that for the scheme to be applicable to a nonlinearity $N(\rho)$ we need that for the solutions of the equation

$$\partial_t \rho + (u \cdot \nabla)\rho - \Delta\rho + N(\rho) = 0,$$

either the mean or some norm of ρ does not grow or at least obeys global finite (even if growing) bound in time. Without such assumption, it is difficult to rule our finite time blow up of the mean value of the solution, which large H^1 norm has no way to arrest.

The second condition that is needed concerns the bound on the nonlinear term in the spirit of

$$\left| \int N(\rho)\rho dx \right| \leqslant Cf(\|\rho\|_{L^2})\|\rho\|_{H^1}^a, \tag{2.82}$$

with $a < 2$. This would allow control of the L^2 norm growth by diffusion when H^1 norm is large. The third condition is that some analog of the approximation lemma holds. This seems to require bounds similar to (2.82).

Of course, the scheme can also be adapted to the cases where diffusion is given by some dissipative operator other than Laplacian, for example a sufficiently strong fractional Laplacian, in which case the H^1 norm needs to be replaced by some other norm natural in the given context. It is also likely that diffusion term does not have to be linear, even though this may require subtler analysis.

As far as other possible classes of flows that may have the blow up arresting property, the main clearly sufficient requirement for our scheme to be applicable appears to be as follows. First, the flows should be sufficiently regular and in particular satisfy Lipschitz bound in space variables. Secondly, for every $\epsilon > 0$ it should be possible to find a flow u_ϵ from the given class, with uniform in time Lipschitz bound, such that for every $f_0 \in C^\infty$ the solution $f(x,t)$ of the transport equation

$$\partial_t f + (u \cdot \nabla) f = 0, \quad f(x,0) = f_0(x)$$

satisfies

$$\|f(\cdot, \tau_\epsilon)\|_{H^{-1}} \leqslant \epsilon C(\|f_0\|_{L^\infty}) \tag{2.83}$$

for some $\tau_\epsilon < \infty$. Observe that even though we did not frame the discussion of the mixing effect of the RE flows in terms of the H^{-1} norm, Lemma 2.3.14 clearly implies that (2.83) holds for the RE flows. There are other classes of flows that look likely to satisfy these properties, such as optimal mixing flows discussed in [62]. Generally, decay of the H^{-1} norm is one of the general measures of the mixing ability of the flow, hence we have a link between efficient mixing and suppression of blow up, which is quite natural. We refer to [62], [66], [48], [78] for further discussion of H^{-1} norm as a measure of mixing and some bounds on mixing rates for natural classes of flows. It also looks possible that some flows that do not in general lead to H^{-1} norm decay without diffusion can still be effective suppressors of blow up if diffusion is taken into account. A natural and common class to be investigated here are some families of stationary cellular flows. Furthermore, similarly to [17], our construction can be applied more generally to the case where the transport part of the equation is replaced by some other unitary evolution for which an analog of (2.83) holds. We plan to address some of these generalizations in future work.

Appendix A

Direct calculation of lemma 1.5.2

In this section, we will show a direct proof of the key lemma (lemma 1.5.2).

By proposition 1.3.8, we know we can write the Green function of Ω as following:

$$2\pi G_\Omega(x,y) = \chi_{B_r(0)}(y)(\log|x-y| - \log|x-y^*|) + (1-\chi_{B_r(0)}(y))(\log|x-y|) + C(x,y). \tag{A.1}$$

Here $\chi_{B_r}(y)$ is the smooth cut-off function. $C(x,y)$ is a function so that $\int_\Omega C(x,y)\omega(y)dy$ is $C^{2,\alpha}(B_\delta(0)\cap\Omega)$, for any small $\delta \leqslant \frac{r}{2}$, and $\omega(y)$ is a bounded function in $\overline{\Omega}$. Here y^* is the same as in proposition 1.3.8. And notice that in the sector D_1^γ, $\frac{|x|}{x_1} \leqslant C(\gamma)$, so the first order term of $\partial_{x_2}\int_\Omega C(x,y)\omega(y)dy$ can be written as $x_1 B_3(x,t) + M_1(\omega)$, for $B_3(x,t) \leqslant C(\gamma)||\omega||_{L^\infty}$ and $M_1(\omega) = \partial_{x_2}\int_\Omega C(x,y)\omega(y)dy|_{x=(0,0)}$. Then, for u_1, by direct computation we get

$$2\pi u_1 = x_1\int_{D^+}\frac{4y_1(x_2-y_2)}{|x-y|^2|\tilde{x}-y|^2}\omega(y)dy - x_1\int_{D^+}\frac{4y_1^*(x_2-y_2^*)}{|x-y^*|^2|\tilde{x}-y^*|^2}\chi_{B_r}(y)\omega(y)dy.$$

For the first term, we estimate it as follows.

1. For $0 \leqslant y_1 \leqslant 2x_1$, $-2 \leqslant y_2 \leqslant 2$, $y \in D^+$. The contribution from this region, or any of its subset, does not exceed a constant times $x_1\|\omega\|_{L^\infty}$ times

$$\begin{aligned}\int_0^{2x_1}\int_{-2}^{2}\frac{y_1|x_2-y_2|}{|x-y|^2|\tilde{x}-y|^2}dy_1dy_2 \leqslant & C\int_0^{x_1}\int_0^3\frac{x_1z_2}{(z_1^2+z_2^2)(x_1^2+z_2^2)}dz_1dz_2 \leqslant \\ & C\int_0^3\frac{x_1}{x_1^2+z_2^2}\arctan\frac{x_1}{z_2}dz_2 \leqslant C.\end{aligned} \tag{A.2}$$

2. For $2x_1 \leqslant y_1 \leqslant 2, \quad f(y_1) \leqslant y_2 \leqslant 2x_2$, where $\partial\Omega = (x_1, f(x_1))$ when x is small. If $f(y_1) \geqslant 0$ for $|y| \leqslant r$, then the contribution of this region does not exceed a constant times $x_1\|\omega_0\|$ times

$$\begin{aligned} &C\int_{2x_1}^{2}\int_{0}^{x_2}\frac{z_1z_2}{(z_1^2+z_2^2)^2}dz_1dz_2 \leqslant C\int_{0}^{x_2} z_2\left(\frac{1}{z_2^2+4x_1^2}-\frac{1}{z_2^2+4}\right)dz_2 = \\ &C\left(\log\left(\frac{x_2^2+4x_1^2}{x_2^2+4}\right)+\log\frac{1}{x_1^2}\right) \leqslant C\log(1+\frac{x_2}{x_1}) \leqslant C(\gamma), \end{aligned} \tag{A.3}$$

where the last step we used the fact that $x \in D_1^\gamma$.

Otherwise, for the part that $-2|x_2| \leqslant y_2 \leqslant 2|x_2|$, the proof is the same. For the region $2x_1 \leqslant y_1 \leqslant 2, \quad \min\{0, f(y_1)\} \leqslant y_2 \leqslant -2|x_2|$ we need to take integral in y_2 first. Notice that if $f(y_1) \leqslant 0$, $f(y_1) \geqslant -a|y_1|^2$ for some $a > 0$ since $f(0) = f'(0) = 0$. Actually, for r small enough, the contribution of this resion does not exceed a constant times $x_1\|\omega_0\|$ times

$$\begin{aligned} &C\int_{2x_1}^{2}\int_{\min\{f(y_1),0\}-x_2}^{-C|x_2|}\frac{z_1z_2}{(z_1^2+z_2^2)^2}dz_1dz_2 \\ &\leqslant C\int_{2x_1}^{2}\int_{-az_1^2-x_2}^{-x_2}\frac{z_1z_2}{(z_1^2+z_2^2)^2}dz_1dz_2 \\ &= C\int_{2x_1}^{2} z_1\left(\frac{1}{z_1^2+x_2^2}-\frac{1}{z_1^2+(az_1^2+x_2)^2}\right)dz_1 \\ &= C\int_{2x_1}^{2} z_1\left(\frac{z_1^2|2ax_2+a^2z_1^2|}{(z_1^2+x_2^2)(z_1^2+(az_1^2+x_2)^2)}\right)dz_1 \\ &= C\int_{4x_1^2}^{4}\left(\frac{t|2ax_2+a^2t|}{(t+x_2^2)(t+(at+x_2)^2)}\right)dt \\ &\leqslant C\int_{4x_1^2}^{4}\left(\frac{|2ax_2+a^2t|}{t+(at+x_2)^2}\right)dt \\ &\leqslant C\int_{4x_1^2}^{4}\frac{2a|x_2|}{t+(at+x_2)^2}dt + C\int_{4x_1^2}^{4}\frac{t}{t+(at+x_2)^2}dt \\ &\leqslant C|x_2|\int_{4x_1^2}^{4}\frac{1}{t}dt + C \\ &\leqslant C(\gamma). \end{aligned} \tag{A.4}$$

3. The region $Q(2x_1, 2x_2)$. Here the contribution we get is equal to, up to a constant,

$$x_1 \int_{Q(2x_1,2x_2)} \frac{y_1(x_2-y_2)}{|x-y|^2|\tilde{x}-y|^2}\omega(y)dy.$$

First, observe that

$$\begin{aligned}\int_{Q(2x_1,2x_2)} \frac{y_1x_2}{|x-y|^2|\tilde{x}-y|^2}\omega(y)dy \leqslant & Cx_2\|\omega\|_{L^\infty}\int_{2x_2}^2\int_{2x_1}^2 \frac{z_1}{(z_1^2+z_2^2)^2}dz_1dz_2 \leqslant \\ & Cx_2\|\omega_0\|_{L^\infty}\int_{2x_2}^2 \frac{1}{z_2^2+4x_1^2}dz_2 \leqslant C(\gamma).\end{aligned} \tag{A.5}$$

Then also

$$\frac{y_1y_2}{|x-y|^2|\tilde{x}-y|^2} - \frac{y_1y_2}{|y|^2|\tilde{x}-y|^2} = \frac{y_1y_2(2x_1y_1+2x_2y_2-x_1^2-x_2^2)}{|x-y|^2|\tilde{x}-y|^2|y|^2}. \tag{A.6}$$

If $x_2 \geqslant 0$, the contribution of this term does not exceed $Cx_1\|\omega_0\|_{L^\infty}$ times

$$\int_{Q(2x_1,2x_2)} \frac{y_1y_2(2x_1y_1-x_1^2)}{(y_1^2+y_2^2)^3}dy_1dy_2 \leqslant C\int_{2x_1}^2 \frac{y_1(2x_1y_1-x_1^2)}{(y_1^2+4x_2^2)^2}dy_1 \leqslant C\int_{2x_1}^2 \frac{2x_1y_1-x_1^2}{y_1^3}dy_1 \leqslant C.$$

And we can estimate the contribution coming from the difference

$$\frac{y_1y_2}{(y_1^2+y_2^2)^2} - \frac{y_1y_2}{|y|^2|\tilde{x}-y|^2}$$

in a similar way.

Now if $x_2 < 0$, based on the above estimate, we only need to estimate the following term

$$\int_{Q(2x_1,2x_2)} \frac{y_1y_2(2x_2y_2-x_2^2)}{|x-y|^2|\tilde{x}-y|^2|y|^2}\omega(y)dy_1dy_2.$$

We will estimate it step by step. First,

$$
\begin{aligned}
&\int_{2x_2}^{0}\int_{2x_1}^{2}\frac{y_1y_2(2x_2y_2-x_2^2)}{|x-y|^2|\tilde{x}-y|^2|y|^2}\omega(y)dy\\
&\leqslant C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\int_{2x_1}^{2}\frac{-y_1y_2|x_2^2-2x_2y_2|}{|x-y|^2|\tilde{x}-y|^2|y|^2}dy\\
&\leqslant C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\int_{2x_1}^{2}\frac{-y_1y_2|x_2^2-2x_2y_2|}{(y_1^2+(y_2-x_2)^2)^2(y_1^2+y_2^2)}dy_1dy_2\\
&\leqslant C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\int_{2x_1}^{2}\frac{-y_1y_2}{(y_1^2+(y_2-x_2)^2)^2}dy_1dy_2\\
&+C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\int_{2x_1}^{2}\frac{-y_1y_2}{(y_1^2+(y_2-x_2)^2)(y_1^2+y_2^2)}dy_1dy_2\\
&\leqslant C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\frac{-y_2}{4x_1^2+(y_2-x_2)^2}dy_2+C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\frac{-y_2}{x_2^2-2x_2y_2}\log\left(1+\frac{x_2^2-2x_2y_2}{4x_1^2+y_2^2}\right)dy_2\\
&\leqslant C\|\omega\|_{L^\infty}\int_{x_2}^{-x_2}\frac{a}{4x_1^2+a^2}dy_2+C\|\omega\|_{L^\infty}\int_{x_2}^{0}\frac{x_2}{4x_1^2+a^2}dy_2+C\|\omega\|_{L^\infty}\int_{2x_2}^{0}\frac{-y_2}{4x_1^2+y_2^2}dy_2\\
&\leqslant C(\gamma)\|\omega\|_{L^\infty}.
\end{aligned}
\tag{A.7}
$$

Second,

$$
\begin{aligned}
&\int_{0}^{2}\int_{2x_1}^{2}\frac{y_1y_2(2x_2y_2-x_2^2)}{|x-y|^2|\tilde{x}-y|^2|y|^2}\omega(y)dy\\
&\leqslant C\|\omega\|_{L^\infty}\int_{0}^{2}\int_{2x_1}^{2}\frac{y_1y_2(x_2^2-2x_2y_2)}{|x-y|^2|\tilde{x}-y|^2|y|^2}dy\\
&\leqslant C\|\omega\|_{L^\infty}\int_{0}^{2}\int_{2x_1}^{2}\frac{y_1y_2(x_2^2-2x_2y_2)}{(y_1^2+y_2^2)^3}dy_1dy_2\\
&\leqslant C\|\omega\|_{L^\infty}\int_{0}^{2}\frac{y_2(x_2^2-2x_2y_2)}{(x_1^2+y_2^2)}dy_2\\
&\leqslant C\|\omega\|_{L^\infty}\int_{0}^{4}\frac{x_2^2}{(x_1^2+a)^2}da+C\|\omega\|_{L^\infty}\int_{0}^{2}\frac{x_2y_2^2}{(x_1^2+y_2^2)^2}dy\\
&\leqslant C(\gamma)\|\omega\|_{L^\infty}.
\end{aligned}
\tag{A.8}
$$

The last step is due to the elementrary indefinite integral $\int \frac{x^2}{(x^2+a^2)^2}dx = \frac{1}{2a}\arctan(\frac{x}{a}) - \frac{x}{2(x^2+a^2)} + C$. And again we can estimate the contribution coming from the difference

$$\frac{y_1y_2}{(y_1^2+y_2^2)^2} - \frac{y_1y_2}{|y|^2|\tilde{x}-y|^2}$$

in a similar way.

Now we only need to focus on the second term of u_1. For this, we use the change of coordinate. The second term is equal to

$$\int_{y^{-*}(D^+)} \frac{4y_1(x_2-y_2)}{|x-y|^2|\tilde{x}-y|^2}\omega(y^{-*}(y))\chi_{B_r(0)}(y^{-*}(y))|\det(\nabla(y^*))(y^{-*}(y))|^{-1}dy.$$

Then, we write the domain $y^{-*}(D^+)$ as the union of $y^{-*}(D^+)\cap\{y : 0 \leqslant y_1 \leqslant 2x_1, -2 \leqslant y_2 \leqslant 2\}$, $y^{-*}(D^+)\cap\{y : 2x_1 \leqslant y_1 \leqslant 2, -2x_2 \leqslant y_2 \leqslant f(y_1)\}$ and $y^{-*}(D^+)\cap\{y : 2x_1 \leqslant y_1 \leqslant 2, -2 \leqslant y_2 \leqslant -2x_2\}$. For the first two domains, the estimates are the same as the first part. Now we concentrate on the third domain. By the same way as the first term we can write the integral as

$$\int_{y\in y^{-*}(D^+), 2x_1\leqslant y_1\leqslant 2, -2\leqslant y_2\leqslant -2x_2} -\frac{y_1y_2}{|y|^4}\omega(y^{-*}(y))\chi_{B_r(0)}(y^{-*}(y))|\det(\nabla(y^*))(y^{-*}(y))|^{-1}dy$$
$$+ \text{ something bounded.}$$

So we only need to estimate the following term:

$$\int_{y\in y^{-*}(D^+), 2x_1\leqslant y_1\leqslant 2, -2\leqslant y_2\leqslant -2x_2} -\frac{y_1y_2}{|y|^4}\omega(y^{-*}(y))\chi_{B_r(0)}(y^{-*}(y))|\det(\nabla(y^*))(y^{-*}(y))|^{-1}dy$$
$$- \int_{Q(2x_1,2x_2)} \frac{y_1y_2}{|y|^4}\omega(y_1,y_2)dy \tag{A.9}$$

We call $\tilde{Q}(2x_1, -2x_2) = \{2x_1 \leqslant y_1 \leqslant 2, -2 \leqslant y_2 \leqslant -2x_2\}$. And if we change the coordinate of the first term back to D^+, we can find that it is nothing but

$$\int_{y\in D^+, y\in y^*(\tilde{Q}(2x_1,-2x_2))} -\frac{y_1^* y_2^*}{|y^*|^4}\omega(y)\chi_{B_r(0)}(y^{-*}(y))dy - \int_{Q(2x_1,2x_2)} \frac{y_1y_2}{|y|^4}\omega(y_1,y_2)dy$$

So,

$$\begin{aligned}&\int_{y\in D^+, y\in y^*(\tilde{Q}(2x_1,-2x_2))} -\frac{y_1^* y_2^*}{|y^*|^4}\omega(y)\chi_{B_r}(y^{-*}(y))dy - \int_{Q(2x_1,2x_2)} \frac{y_1y_2}{|y|^4}\omega(y)dy \\ &\leqslant \int_{y\in D^+, y\in y^*(\tilde{Q}(2x_1,-2x_2))} |\left(-\frac{y_1^* y_2^*}{|y^*|^4} - \frac{y_1y_2}{|y|^4}\right)||\omega(y)|dy + \int_H |\frac{y_1y_2}{|y|^4}||\omega(y)|dy\end{aligned} \tag{A.10}$$

Where $H = Q(2x_1, 2x_2) \setminus y^*(\tilde{Q}(2x_1, -2x_2)) = \{y \in D^+, y_1 \geqslant 2x_1, y_2 \geqslant 2x_2, y_1^{-*}(y) \leqslant 2x_1, y_2^{-*}(y) \geqslant -2x_2\}$. We call $H' = H \cap \{y : y_1 \geqslant y_2\}$. Then in H', we have

$$y_1^{-*}(y) = y_1 + C(y)|y|^2 \geqslant y_1 - Cy_1^2,$$

for some $C > 0$. So $H' \subseteq \{y \in D^+ : 2x_1 \leqslant y_1, 2x_1 \geqslant y_1 - Cy_1^2, 2x_2 \leqslant y_2, y_2 \leqslant y_1\} \subseteq \{y \in D^+ : 2x_1 \leqslant y_1 \leqslant Cx_1, 2x_2 \leqslant y_2, y_2 \leqslant y_1\}$, for some $C > 2$, provided that $|x|$ is small. So

$$\begin{aligned}\int_{H'} |\frac{y_1y_2}{|y|^4}||\omega(y)|dy &\leqslant C\|\omega_0\|_{L^\infty} \int_{2x_1}^{Cx_1}\int_{2x_2}^{2} \frac{|y_1y_2|}{(y_1^2+y_2^2)^2}dy_1dy_2 \\ &\leqslant C\|\omega_0\|_{L^\infty} \int_{2x_1}^{Cx_1} \frac{y_1}{y_1^2+4} + \frac{y_1}{y_1^2+4x_2^2}dy_1 + C\|\omega_0\|_{L^\infty} \\ &\leqslant C\|\omega_0\|_{L^\infty}\left(\log\left(\frac{Cx_1^2+4}{4x_1^2+1}\right) + \log\left(\frac{Cx_1^2+4x_2^2}{4x_1^2+4x_2^2}\right) + 1\right) \\ &\leqslant C(\gamma)\|\omega_0\|_{L^\infty}.\end{aligned} \tag{A.11}$$

We have exactly the same estimate for $H \setminus H'$. So $\int_H \frac{y_1y_2}{|y|^4}\omega(y)dy$ is bounded as $x \to 0$, we only need to estimate the first term. Remember $y^* = (y_1, -y_2) + O(|y|^2)$ by lemma

1.3.3, we have

$$
\begin{aligned}
&\int_{y\in D^+, y\in y^*(\tilde{Q}(2x_1,-2x_2))} |\left(-\frac{y_1^* y_2^*}{|y^*|^4} - \frac{y_1 y_2}{|y|^4}\right)||\omega(y)|dy \\
&= \int_{y\in D^+, y\in y^*(\tilde{Q}(2x_1,-2x_2))} |\left(\frac{y_1 y_2 + O(|y|^3)}{|y|^4 + O(|y|^5)} - \frac{y_1 y_2}{|y|^4}\right)||\omega(y)|dy \\
&= \int_{y\in D^+, y\in y^*(\tilde{Q}(2x_1,-2x_2))} |\frac{O(|y|^3)}{|y|^4 + O(|y|^5)}||\omega(y)|dy \\
&\leqslant C||\omega||_{L^\infty}
\end{aligned}
\tag{A.12}
$$

Which means the second term also satisfies the expression in the lemma.

Then we prove that u_2 also satisfies the expression. First of all, we show that for u_2, the integrals of types

$$
\int_{\{y\in\Omega:|y|\leqslant r, y_2\leqslant 0\}} \frac{x_i - y_i}{|x-y|^2}\omega(y)dy \quad , \quad \int_{\{y\in\Omega:|y|\leqslant r, y_2\leqslant 0\}} \frac{x_i - y_i^*}{|x-y^*|^2}\omega(y)dy \tag{A.13}
$$

can be written as $|x|\|\omega\|_{L^\infty}B_0(x) + M_0(\omega)$, here $B_0(x)$ is a bounded function, and M_0 is a constant. Recall near the oringin we can write $\partial\Omega$ as $(x_1, f(x_1))$, for some C^3 function f; If $B_r(0)\cap\Omega$ is in upper half plane, then (A.13) are all zeros; otherwise, we can choose a smaller r so that if $B_r(0)\cap\partial\Omega = \{A^-, O, A^+\}$, where A^+ is in D^+, then the slope of tangent line at A^+ is negative. This r is still only depend on Ω, we use this new r instead of the old one. Then the estimate of these "lower half plane terms" can be done in the following way:

First, for the terms without y^*. We call the set $\Omega^- = \{y \in \Omega : |y| \leqslant r, y_2 \leqslant 0\}$, then

$$
\begin{aligned}
\int_{\Omega^-} \frac{x_i - y_i}{|x-y|^2}\omega(y)dy &= \int_{\Omega^-} \left(\frac{x_i - y_i}{|x-y|^2} + \frac{y_i}{|y|}\right)\omega(y)dy + M_0(\omega) \\
&= \sum_{l,m,n} \int_{\Omega^-} \frac{(x_l - y_l)x_m y_n}{|x-y|^2|y|^2}\omega(y)dy + M_0(\omega)
\end{aligned}
\tag{A.14}
$$

For the first part, each term is less than or equal to

$$|x|\|\omega\|_{L^\infty}\int_{\Omega^-}\frac{|y_n|}{|x-y||y|^2}dy.$$

For $n=1$, recall in D_2^γ, $x_2\geqslant 0$, so we have that this term is less than or equal to $|x|\|\omega\|_{L^\infty}$ times:

$$\begin{aligned}
&\int_0^r\int_{\min\{f(y_1),0\}}^0\frac{y_1}{|x-y||y|^2}dy_2dy_1+\int_{-r}^0\int_{\min\{f(y_1),0\}}^0\frac{-y_1}{|x-y||y|^2}dy_2dy_1\\
&\leqslant\int_0^r\int_{\min\{f(y_1),0\}}^0\frac{y_1}{\sqrt{(x_1-y_1)^2+x_2^2}(y_1^2+y_2^2)}dy_2dy_1\\
&+\int_{-r}^0\int_{\min\{f(y_1),0\}}^0\frac{-y_1}{\sqrt{(x_1-y_1)^2+x_2^2}(y_1^2+y_2^2)}dy_2dy_1\\
&\leqslant\int_0^r\frac{|\arctan(\frac{f(y_1)}{y_1})|}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1+\int_{-r}^0\frac{|\arctan(\frac{f(y_1)}{y_1})|}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1
\end{aligned}\tag{A.15}$$

Now remember $\partial\Omega$ tangent to x_1-axis at the origin, so we have $f(y_1)=O(y_1^2)$, so $|\arctan(\frac{f(y_1)}{y_1})|\leqslant C|y_1|$, with some universal constant C. So we have the above expression is actually less or equal to some universal constant times

$$\begin{aligned}
&\int_0^2\frac{y_1}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1+\int_{-2}^0\frac{-y_1}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1\\
&\leqslant\int_0^2\frac{y_1-x_1+x_1}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1+\int_{-2}^0\frac{x_1-y_1-x_1}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1\\
&\leqslant 4+x_1\int_0^2\frac{1}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1-x_1\int_{-2}^0\frac{1}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1\\
&\leqslant 4+x_1\int_0^2\frac{1}{(x_1-y_1)^2+x_2^2}dy_1-x_1\int_{-2}^0\frac{1}{(x_1-y_1)^2+x_2^2}dy_1\\
&\leqslant C(\gamma)
\end{aligned}\tag{A.16}$$

For $n = 2$, again we have this term is less than or equal to $|x|\|\omega\|_{L^\infty}$ times

$$\begin{aligned} &\leqslant \int_0^2 \int_{\min\{f(y_1),0\}}^0 \frac{y_2}{\sqrt{(x_1-y_1)^2+x_2^2(y_1^2+y_2^2)}}dy_2dy_1 \\ &+ \int_{-2}^0 \int_{\min\{f(y_1),0\}}^0 \frac{-y_2}{\sqrt{(x_1-y_1)^2+x_2^2(y_1^2+y_2^2)}}dy_2dy_1 \\ &\leqslant \int_0^2 \frac{\log(1+\frac{f(y_1)^2}{y_1^2})}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1 + \int_{-2}^0 \frac{-\log(1+\frac{f(y_1)^2}{y_1^2})}{\sqrt{(x_1-y_1)^2+x_2^2}}dy_1 \end{aligned} \tag{A.17}$$

And similarly, $|\log(1+\frac{f(y_1)^2}{y_1^2})| = O(|y_1|^2) \leqslant |y_1|$, so the estimate we made still hold.

For the terms involve y^*, notice that by similar triangle, since the slopre of tangent line at $x \in \partial\Omega \cap (B_r(0) \setminus 0)$ is negative, the map $y \to y^*$ maps $(0,r) \times \{y : y_2 = 0\}$ into the curve $(t, \min\{2f(t), 0\})$ for $t \leqslant r$, so we have this map maps Ω^- to the similar domain bounded by y_1-axis, $B_r(0)$, $(t, \min\{2f(t), 0\})$. Also remember that the determinant of Hessian of map y^* is bounded since it is at least C^2, so we do the change of variables and run the above argument again by replacing nothing but $f(y_1)$ by $2f(y_1)$ and we will get the proof of terms involve y^*. So this proves that the contribution of Ω^- is neglectable. With a little abusement of notation, we use D^+ again instead of $D^+ \cap \{y : y_2 \geqslant 0\}$ in the rest of the proof.

Then we need to write u_2 term by term:

$$u_2 = -\frac{1}{2\pi}\int_{D^+}\left(\frac{x_1-y_1}{|x-y|^2} - \frac{x_1+y_1}{|\tilde{x}-y|^2} - \frac{x_1-y_1^*}{|x-y^*|^2} + \frac{x_1+y_1^*}{|\tilde{x}-y^*|^2}\right)\chi_{B_r(0)}(y)\omega(y)dy + x_2B_5(x,t) + M(\omega). \tag{A.18}$$

Here the last two terms $x_2B_5(x,t) + M(\omega)$ came from the derivative of the terms $(1 - \chi_{B_r}(y))(\log|x-y|)$ and $C(x,y)$ of expression (A.1). We only need to handle the terms in the integral. And since $|\chi_{B_r(0)}(y)| \leqslant 1$, in the rest of the proof we just let it be absorbed by $\omega(y)$. First notice if we use $\tilde{y} = (y_1, -y_2)$ instead of y^* in the expression, then by exactly the same way as we deal with u_1, we can get the desired expression. So what we need is to use some $x_2 B_6(x,t)$ to bound the first two terms of the following terms:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{D^+}\left(\frac{x_1 - y_1^*}{|x-y^*|^2} - \frac{x_1 - y_1}{|x-\tilde{y}|^2}\right)\omega(y)dy \\
&+ \frac{1}{2\pi}\int_{D^+}\left(\frac{-x_1 - y_1^*}{|\tilde{x}-y^*|^2} - \frac{-x_1 - y_1}{|\tilde{x}-\tilde{y}|^2}\right)\omega(y)dy + M(\omega) \\
&= I + II + M(\omega).
\end{aligned} \tag{A.19}$$

The idea to estimate these terms is, instead of y^* terms, we add and subtract the first order approximation of y^* terms as we did in lemma 1.3.3 and estimate the differences separately.

We first give an estimate of I. For this term, we set $(x_*, f(x_*)) = e(x)$, where $e(x)$ as before, is the nearest point to x in $\partial\Omega$. We can choose δ small so this definition is valid. Then, for given x, we set $\overline{y}_1 = x_* + \frac{1-f'(x_*)^2}{1+f'(x_*)^2}(y_1 - x_*) + \frac{2f'(x_*)}{1+f'(x_*)^2}(y_2 - f(x_*))$, $\overline{y}_2 = f(x_*) + \frac{2f'(x_*)}{1+f'(x_*)^2}(y_1 - x_*) + \frac{f'(x_*)^2-1}{1+f'(x_*)^2}(y_2 - f(x_*))$, and $\overline{y} = (\overline{y}_1, \overline{y}_2)$. Then, for I, we write it as following:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{D^+}\left(\frac{x_1 - y_1^*}{|x-y^*|^2} - \frac{x_1 - \overline{y}_1}{|x-\overline{y}|^2} + \frac{x_1 - \overline{y}_1}{|x-\overline{y}|^2} - \frac{x_1 - y_1}{|x-\tilde{y}|^2}\right)\omega(y)dy \\
&= \frac{1}{2\pi}\int_{D^+}\left(\frac{x_1 - y_1^*}{|x-y^*|^2} - \frac{x_1 - \overline{y}_1}{|x-\overline{y}|^2}\right)\omega(y)dy \\
&+ \frac{1}{2\pi}\int_{D^+}\left(\frac{x_1 - \overline{y}_1}{|x-\overline{y}|^2} - \frac{x_1 - y_1}{|x-\tilde{y}|^2}\right)\omega(y)dy \\
&= (i) + (ii).
\end{aligned} \tag{A.20}$$

For (i), we want to write it as $|x|B(x)$ for B bounded above by $\|\omega\|_{L^\infty}$, so we need first of all let it equal to 0 when $x = 0$, which is, we write it like this:

$$\frac{1}{2\pi}\int_{D^+}\left(\frac{x_1 - y_1^*}{|x-y^*|^2} - \frac{-y_1^*}{|y^*|^2}\right)\omega(y)dy - \frac{1}{2\pi}\int_{D^+}\left(\frac{x_1-\overline{y}_1}{|x-\overline{y}|^2} - \frac{-\overline{y}_1}{|\overline{y}|^2}\right)\omega(y)dy \\ - L. \tag{A.21}$$

Where $L = \frac{1}{2\pi}\int_{D^+}\left(\frac{-y_1^*}{|y^*|^2} - \frac{-\overline{y}_1}{|\overline{y}|^2}\right)\omega(y)dy$ is a constant. We go back to (i). So it can be written as $\frac{1}{2\pi}$ times:

$$\int_{D^+}\frac{y_1^*x_1(x_1-y_1^*) + x_1y_2^*(y_2^*-x_2) + x_2y_1^*(x_2-y_2^*) + x_2y_2^*(x_1-y_1^*)}{|x-y^*|^2|y^*|^2}\omega(y)dy \\ - \int_{D^+}\frac{\overline{y}_1x_1(x_1-\overline{y}_1) + x_1\overline{y}_2(\overline{y}_2-x_2) + x_2\overline{y}_1(x_2-\overline{y}_2) + x_2\overline{y}_2(x_1-\overline{y}_1)}{|x-\overline{y}|^2|\overline{y}|^2}\omega(y)dy \tag{A.22}$$

We can write it as the sum of terms like

$$\int_{D^+}\left(\frac{x_iy_j^*(x_k-y_k^*)}{|x-y^*|^2|y^*|^2} - \frac{x_i\overline{y}_j(x_k-\overline{y}_k)}{|x-\overline{y}|^2|\overline{y}|^2}\right)\omega(y)dy \tag{A.23}$$

For some $i, j, k \in \{1, 2\}$. Then, (A.23) can be written as x_i times the following term:

$$\int_{D^+}\left(\frac{y_j^*(x_k-y_k^*)}{|x-y^*|^2|y^*|^2} - \frac{y_j^*(x_k-\overline{y}_k)}{|x-\overline{y}|^2|y^*|^2}\right)\omega(y)dy \\ + \int_{D^+}\left(\frac{y_j^*(x_k-\overline{y}_k)}{|x-\overline{y}|^2|y^*|^2} - \frac{\overline{y}_j(x_k-\overline{y}_k)}{|x-\overline{y}|^2|\overline{y}|^2}\right)\omega(y)dy \\ = (a) + (b) \tag{A.24}$$

For (a), we can write it as:

$$\int_{D^+}\left(\frac{x_k-y_k^*}{|x-y^*|^2} - \frac{x_k-\overline{y}_k}{|x-\overline{y}|^2}\right)\frac{y_j^*}{|y^*|^2}\omega(y)dy \tag{A.25}$$

Then, we compute $|x-y^*|^2$.

$$\begin{aligned} |x-y^*|^2 &= |x-(x_*, f(x_*)) + (x_*, f(x_*)) - y^*|^2 \\ &= |x-(x_*, f(x_*)) + (x_*, f(x_*)) - \overline{y} + O(|(x_*, f(x_*)) - y)|^2)|^2 \\ &= |x-\overline{y}|^2 + O(|(x_*, f(x_*)) - y|^2|x-\overline{y}|) + O(|(x_*, f(x_*)) - y|^4) \\ &= |x-\overline{y}|^2 + O(|(x_*, f(x_*)) - \overline{y}|^2|x-\overline{y}|) + O(|(x_*, f(x_*)) - \overline{y}|^4) \end{aligned} \tag{A.26}$$

Here we use the fact $|(x_*, f(x_*)) - y| = |(x_*, f(x_*)) - \overline{y}|$ by definition. We claim that $|(x_*, f(x_*)) - \overline{y}| \lesssim |x - \overline{y}|$. First we can see that $|(x_*, f(x_*)) - y^*| \leqslant |x - y^*|$, this can be seen from the geometry. We call $e(y) = (x(y), f(x(y)))$, then since $y^* - x(y)$ is parallel to the normal vector $n(x(y))$, and $x - (x_*, f(x_*))$ is parallel to $n((x_*, f(x_*)))$, and we choose $r(\Omega)$ small so $x(y)$ is also close to $(x_*, f(x_*))$, we must have $((x_*, f(x_*)) - y) \cdot (x - (x_*, f(x_*)) < 0$. So we have $|(x_*, f(x_*)) - y^*| \leqslant |x - y^*|$ by sine law. And we know

$$|x - y^*|^2 \geqslant |(x_*, f(x_*)) - y^*|^2 = |(x_*, f(x_*)) - \overline{y}|^2 + O(|(x_*, f(x_*)) - \overline{y}|^3). \tag{A.27}$$

Notice that for $|(x_*, f(x_*)) - y|$ small, $O(|(x_*, f(x_*)) - \overline{y}|^4) < \frac{1}{2}(|(x_*, f(x_*)) - \overline{y}|^2 + O(|(x_*, f(x_*)) - \overline{y}|^3))$. Combine (A.26) and (A.27), by solving a quadratic inequality, we know that $|(x_*, f(x_*)) - \overline{y}| \lesssim |x - \overline{y}|$. Which means from (A.26), we get $|x - y^*|^2 = |x - \overline{y}|^2 + O(|x - \overline{y}|^3)$. Now we go back to (a), it can be written as

$$\begin{aligned}
&\int_{D^+} \left(\frac{x_k - y_k^*}{|x - \overline{y}|^2(1 + O(|x - \overline{y}|))} - \frac{x_k - \overline{y}_k}{|x - \overline{y}|^2} \right) \frac{y_j^*}{|y^*|^2} \omega(y) dy \\
&= \int_{D^+} \left(\frac{\overline{y}_k - (x_k)_* + (x_k)_* - y_k^*}{|x - \overline{y}^2|(1 + O(|x - \overline{y}|))} \right) \frac{y_j^*}{|y^*|^2} \omega(y) dy \\
&+ \int_{D^+} \left(\frac{O(|x - \overline{y}|^2)}{|x - \overline{y}^2|(1 + O(|x - \overline{y}|))} \right) \frac{y_j^*}{|y^*|^2} \omega(y) dy \\
&= \int_{D^+} \left(\frac{O(|x - \overline{y}|^2)}{|x - \overline{y}^2|(1 + O(|x - \overline{y}|))} \right) \frac{y_j^*}{|y^*|^2} \omega(y) dy + B_7(x) ||\omega||_{L^\infty}
\end{aligned} \tag{A.28}$$

Here $(x_k)_* = x_*$ if $k = 1$, $(x_k)_* = f(x_*)$ if $k = 2$. The last equality is by the definition of $\overline{y}$ and $|(x_*, f(x_*)) - \overline{y}| \lesssim |x - \overline{y}|$, $B_7(x)$ is a bounded function, so from the above computation, we know that (a) is a bounded function in x.

Now we estimate (b). Let's first compute $|y^*|^2$, by taking r small enough, we can compute it near $(0, 0)$. By lemma 1.3.3, we take $(s_0, f(s_0)) = (0, 0)$, we know that $y^* = (y_1, -y_2) + O(|y|^2)$, which means $|y^*|^2 = |y|^2 + C(y)|y|^3$, here $C(y)$ is a bounded function in y. We call $\tilde{y} = (-y_1, y_2) = (\tilde{y}_1, \tilde{y}_2)$,

$$\begin{aligned}
&\int_{D^+} \left(\frac{y_j^*}{|y^*|^2} - \frac{\overline{y}_j}{|\overline{y}|^2} \right) \frac{(x_k - \overline{y}_k)}{|x - \overline{y}|^2} \omega(y) dy \\
&= \int_{D^+} \left(\frac{y_j^*}{|y|^2 + C(y)|y|^3} - \frac{\tilde{y}_j}{|y|^2} \right) \frac{(x_k - \overline{y}_k)}{|x - \overline{y}|^2} \omega(y) dy + \int_{D^+} \left(\frac{\tilde{y}_j}{|y|^2} - \frac{\overline{y}_j}{|\overline{y}|^2} \right) \frac{(x_k - \overline{y}_k)}{|x - \overline{y}|^2} \omega(y) dy \\
&= \int_{D^+} \frac{O(|y|^2)}{|y|^2 + C(y)|y|^3} \frac{(x_k - \overline{y}_k)}{|x - \overline{y}|^2} \omega(y) dy + \int_{D^+} \left(\frac{\tilde{y}_j}{|y|^2} - \frac{\overline{y}_j}{|\overline{y}|^2} \right) \frac{(x_k - \overline{y}_k)}{|x - \overline{y}|^2} \omega(y) dy
\end{aligned} \tag{A.29}$$

Where the first term is bounded by a constant times $||\omega||_{L^\infty}$. So we only need to control the second term. To do this, we need a more precisely computation of $\overline{y}$.

For simplicity, we call $A = f'(x_*)$, $B = Ax_* - f(x_*)$. Then we know $A = O(x_*)$, $B = O(x_*^2)$. By definition,

$$\begin{aligned}
\overline{y}_1 &= \tilde{y}_1 + \frac{-2A^2}{1+A^2} y_1 + \frac{2A}{1+A^2} y_2 + \frac{2A^2}{1+A^2} x_* - \frac{2A}{1+A^2} f(x_*), \\
\overline{y}_2 &= \tilde{y}_2 + \frac{2A}{1+A^2} y_1 + \frac{2A^2}{1+A^2} y_2 + \frac{-2A}{1+A^2} x_* + \frac{2}{1+A^2} f(x_*).
\end{aligned} \tag{A.30}$$

We set $E_1(x) = \frac{2A}{1+A^2}(Ax_* - f(x_*)) = \frac{2}{1+A^2} AB$, $E_2(x) = \frac{-2A}{1+A^2} x_* + \frac{2}{1+A^2} f(x_*) = -\frac{2}{1+A^2} B$, and by the above definition we can write $\overline{y}$ as $\tilde{y} + AO(|y|) + BO(1)$.

First of all let's assume $A \geqslant 0$. Then, by the same trick we use in (A.23), we write the second term of (A.29) as a sum of the following terms:

$$\begin{aligned}
&\int_{D^+}\left(\frac{\tilde{y}_{i'}\overline{y}_{j'}(\tilde{y}_{k'}-\overline{y}_{k'})}{|y|^2|\overline{y}|^2}\right)\frac{(x_k-\overline{y}_k)}{|x-\overline{y}|^2}\omega(y)dy\\
&\leqslant C||\omega||_{L^\infty}\left(\int_{D^+}\frac{A}{|\overline{y}||x-\overline{y}|}dy+\int_{D^+}\frac{|B||\tilde{y}_{i'}||\overline{y}_{j'}|}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy\right)\\
&\leqslant C||\omega||_{L^\infty}\left(\frac{A}{|x_*|}\int_{D^+}\frac{|(x_*-\overline{y_1}+\overline{y_1})|}{|\overline{y}||x-\overline{y}|}dy+\int_{D^+}\frac{|B|y_{i'}(|y|+|B|+A|y|)}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy\right)\\
&\leqslant C||\omega||_{L^\infty}\left(\frac{A}{|x_*|}\int_{D^+}\frac{|x_*-\overline{y_1}|}{|\overline{y}||x-\overline{y}|}dy+\int_{D^+}\frac{|B||y|^2}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy+\int_{D^+}\frac{B^2y_{i'}}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy+1\right)\\
&\leqslant C||\omega||_{L^\infty}\left(\frac{A}{|x_*|}+|B|\int_{D^+}\frac{1}{|\overline{y}|^2|x-\overline{y}|}dy+B^2\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy\right)
\end{aligned}\tag{A.31}$$

Now we give an estimate of $B\int_{D^+}\frac{1}{|\overline{y}|^2|x-\overline{y}|}dy$ and $B^2\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy$. First, by use the same notation before, we write $\overline{y}$ as $U(x)y+E(x)$, where U is a orthogonal matrix given by

$$\begin{pmatrix}\frac{1-A^2}{1+A^2} & \frac{2A}{1+A^2}\\ \frac{2A}{1+A^2} & \frac{A^2-1}{1+A^2}\end{pmatrix}$$

As a consequence, $|\overline{y}|=|U(x)y+E(x)|=|y+U(x)^tE(x)|$. Similarly, $|\overline{y}-x|=|U(x)y+E(x)-x|=|y+U(x)^tE(x)-U(x)^tx|$. We call $A_1=-U(x)^tE(x)$, $A_2=-(U(x)^tE(x)-U(x)^tx)$. First of all,

$$\begin{aligned}
&\int_{D^+}\frac{1}{|\overline{y}|^2|x-\overline{y}|}dy\\
&\leqslant\frac{1}{|A_1-A_2|^2}\int_{D^+}\frac{|A_1-A_2|^2}{|y-A_1|^2|y-A_2|}dy\\
&\leqslant\frac{1}{|U(x)^tx|^2}\int_{D^+}\frac{|(y-A_1)\cdot(y-A_1)-(y-A_1)(y-A_2)-(A_1-A_2)(y-A_2)|}{|y-A_1|^2|y-A_2|}dy\\
&\leqslant\frac{1}{|x|^2}\left(\int_{D^+}\frac{1}{|y-A_2|}dy+\int_{D^+}\frac{1}{|y-A_1|}dy+|A_1-A_2|\int_{D^+}\frac{1}{|y-A_1|^2}dy\right)\\
&\leqslant C\frac{1}{|x|^2}\left(1+|x|\int_{D^+}\frac{1}{|y-A_1|^2}dy\right).
\end{aligned}\tag{A.32}$$

Now we need another lemma.

Lemma A.0.25. $|x| \approx |x - (x_*, f(x_*))|$*, for* x *small. Where the constant only depend on* γ*.*

This lemma is an easy observation based on the geometry. In Cartesian coordinate, we call x point A, origin point O and $(x_*, f(x_*))$ point B, then we know that the angle between OA and x-axis is greater or equal to γ; as x is close to O, by the smooth dependence in x of x_*, $|(x_*, f(x_*))|$ is also small. So, we can choose δ small enough such that the angle between OB and x_1-axis is less than $\frac{\gamma}{2}$, which means $\frac{\gamma}{2} \leqslant \angle AOB \leqslant \frac{\pi}{2}$. Meanwhile, x_* small means $\angle ABO$ is close to $\frac{\pi}{2}$, because AB is the normal vector of the curve near the origin. So we can choose δ small so that $\frac{\pi}{4} \leqslant \angle ABO \leqslant \frac{3\pi}{4}$, by sine law we know that $AB \approx AO$, with constant only depend on γ, which is what we want.

Now back to our estimate. First we assume $B < 0$. By lemma A.0.25, we know that $C(\gamma)|x| \geqslant |(x_*, f(x_*))|$, where $C(\gamma)$ is a constant only depend on γ. Also notice that for x small, $A_1 = -U(x)^t E(x) = (E_1(x), -E_2(x))^t + A^2 B O(1)$, which means for x small, if we call $A_1 = (A_{11}, A_{12})$, we have

$$\frac{1}{4}C_0 AB \leqslant \frac{1}{2}|E_1(x)| \leqslant A_{11} \leqslant 2|E_1(x)| \leqslant 8C_0 AB,$$

$$-8C_0 B \leqslant -2|E_2(x)| \leqslant A_{12} \leqslant -\frac{1}{2}|E_2(x)| \leqslant -\frac{1}{4}C_0 B.$$

For some $C_0 > 0$. For $B > 0$, we only need to switch the sign of these two estimates.

Without loss of generality for the rest of the proof we just assume $C_0 = 1$. This means

$$\begin{aligned}
&\int_{D^+} \frac{1}{|y-A_1|^2} dy \\
&\leqslant \int_{[0,2]\times[0,2]} \frac{1}{(y_1-A_{11})^2+(y_2-A_{12})^2} dy_1 dy_2 \\
&\leqslant \int_{-A_{12}}^{2-A_{12}} \int_{-A_{11}}^{2-A_{11}} \frac{1}{y_1^2+y_2^2} dy_1 dy_2 \\
&\leqslant \int_{\frac{1}{2}B}^{2+8B} \frac{1}{y_2} \left(|\arctan(\frac{1-A_{11}}{y_2}) - \arctan(\frac{-A_{11}}{y_2}|) \right) dy_2 \\
&\leqslant C \int_{\frac{1}{2}B}^{2+8B} \frac{1}{y_2} dy_2 \\
&\leqslant C(|\log(B)|+1)
\end{aligned} \tag{A.33}$$

Here the constant C only depend on the domain Ω. So the right hand side of (A.32) is less than

$$C\left(\frac{1}{|x|^2} + \frac{|\log(B)|}{|x|}\right)$$

Which means $B\int_{D^+} \frac{1}{|\overline{y}|^2|x-\overline{y}|} dy$ is bounded when x tends to zero. Because $B = O(x_*^2)$.

For $B^2\int_{D^+} \frac{y_{i'}}{|y|^2|\overline{y}|^2|x-\overline{y}|} dy$, it is a little bit tricker. First let's use the same method as we estimate the previous term:

$$
\begin{aligned}
&B^2\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy\\
&\leqslant \frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}(x_*-\overline{y}_1+\overline{y}_1)}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}||x-\overline{y}|}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}(x_*-\overline{y}_1+\overline{y}_1)}{|y|^2|\overline{y}||x-\overline{y}|}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}}{|y|^2|x-\overline{y}|}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}}{|y|^2|y-A_1|}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}}{|y|^2|y-A_2|}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}}{|y|^2|y-A_1|}dy+\frac{B^2}{x_*^2}\int_{D^+}\frac{y_{i'}}{|y|^2|y-A_2|}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2|A_1|}\int_{D^+}\frac{y_{i'}|A_1|}{|y|^2|y-A_1|}dy+\frac{B^2}{x_*^2|A_2|}\int_{D^+}\frac{y_{i'}|A_2|}{|y|^2|y-A_2|}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2|A_1|}\int_{D^+}\frac{y_{i'}}{|y|^2}dy+\frac{B^2}{x_*^2|A_1|}\int_{D^+}\frac{y_{i'}}{|y||y-A_1|}dy\right)\\
&+C(\gamma)\left(\frac{B^2}{x_*^2|A_2|}\int_{D^+}\frac{y_{i'}}{|y||y-A_2|}dy+\frac{B^2}{x_*^2|A_2|}\int_{D^+}\frac{y_{i'}}{|y|^2}dy\right)\\
&\leqslant C(\gamma)\left(\frac{B^2}{x_*}\int_{D^+}\frac{y_{i'}}{|y|^2|\overline{y}|^2}dy+\frac{B^2}{x_*^2|A_1|}+\frac{B^2}{x_*^2|A_2|}\right)
\end{aligned}
\tag{A.34}
$$

We know that $|A_1| = |E(x)| \geqslant B$, $|A_2| \geqslant ||x| - |E(x)|| \geqslant ||x| - 4|B|| \geqslant |x|$, so $\dfrac{B^2}{x_*^2|A_1|} + \frac{B^2}{x_*^2|A_2|} \leqslant C(\gamma)$ for some constant only depend on γ. Then we only need to control $\dfrac{B^2}{x_*}\int_{D^+}\dfrac{y_{i'}}{|y|^2|\overline{y}|^2}dy$.

First we assume $B < 0$. For different i', we need to discuss seperately. For $i' = 1$, since

$A_{12} \leqslant -\frac{1}{2}B$, we have

$$\begin{aligned}\frac{B^2}{x_*}\int_{D^+}\frac{y_1}{|y|^2|\overline{y}|^2}dy &\leqslant \frac{B^2}{x_*}\int_0^2\int_0^2\frac{y_1}{((y_1-A_{11})^2+(\frac{1}{2}B)^2)(y_1^2+y_2^2)}dy_2dy_1\\ &\leqslant C\frac{B^2}{x_*}\int_0^2\frac{1}{((y_1-A_{11})^2+(\frac{1}{2}B)^2)}dy_1\\ &\leqslant C(\frac{B^2}{x_*}\frac{1}{B}+1)\leqslant C.\end{aligned}\tag{A.35}$$

Similarly for $i'=2$, we have

$$\begin{aligned}\frac{B^2}{x_*}\int_{D^+}\frac{y_2}{|y|^2|\overline{y}|^2}dy &\leqslant \frac{B^2}{x_*}\int_0^2\int_0^2\frac{y_2}{((y_2-A_{12})^2)(y_1^2+y_2^2)}dy_1dy_2\\ &\leqslant C\frac{B^2}{x_*}\int_0^2\frac{1}{(y_2-A_{12})^2}dy_2\\ &\leqslant C(\frac{B^2}{x_*}\frac{1}{-A_{12}}+1)\leqslant C.\end{aligned}\tag{A.36}$$

For $B>0$, again we only need to change the indices 1 and 2 and also discuss separately. Here C is universal constant only depend on γ and Ω. So we finally finish the estimate of (b) and so that the (i) could be bounded by $B_9(x)$ for $B_9(x)\leqslant C(\gamma)||\omega||_{L^\infty}$.

For $A\leqslant 0$, we will estimate the second term of (A.29) directly. Again, that term can be written as a sum of the following terms:

$$\begin{aligned}
&\int_{D^+}(\frac{\tilde{y}_{i'}\overline{y}_{j'}(\tilde{y}_{k'}-\overline{y}_{k'})}{|y|^2|\overline{y}|^2})\frac{(x_k-\overline{y}_k)}{|x-\overline{y}|^2}\omega(y)dy \\
&\leqslant C||\omega||_{L^\infty}\left(\int_{D^+}\frac{A}{|\overline{y}||x-\overline{y}|}dy+\int_{D^+}\frac{|B||\tilde{y}_{i'}||\overline{y}_{j'}|}{|y|^2|\overline{y}|^2|x-\overline{y}|}dy\right) \\
&\leqslant C||\omega||_{L^\infty}\left(\frac{A}{|x_*|}+|B|\int_{D^+}\frac{1}{|y||\overline{y}||x-\overline{y}|}dy\right) \\
&\leqslant C||\omega||_{L^\infty}\left(1+\frac{|B|}{|A_1-A_2|}\int_{D^+}\frac{|A_1-A_2|}{|y||y-A_1||y-A_2|}dy\right) \\
&\leqslant C||\omega||_{L^\infty}\left(1+\frac{|B|}{|x|}\int_{D^+}\frac{1}{|y||y-A_2|}dy+\frac{|B|}{|x|}\int_{D^+}\frac{1}{|y||y-A_1|}dy\right) \\
&\leqslant C||\omega||_{L^\infty}\left(1+\frac{|B|}{|x||A_2|}\int_{D^+}\frac{|A_2|}{|y||y-A_2|}dy+\frac{|B|}{|x|}\int_{D^+}\frac{1}{|y||y-A_1|}dy\right) \\
&\leqslant C||\omega||_{L^\infty}\left(1+\frac{|B|}{|x|^2}+\frac{|B|}{|x|}\int_{D^+}\frac{1}{|y||y-A_1|}dy\right) \\
&\leqslant C||\omega||_{L^\infty}+C||\omega||_{L^\infty}\frac{|B|}{|x|}\int_{D^+}\frac{1}{|y||y-A_1|}dy
\end{aligned} \tag{A.37}$$

Let's focus on the second term of right hand side. First, if $B<0$, then the second term is less or equal to a constant times $||\omega||_{L^\infty}\frac{|B|}{|x|}$ times

$$\begin{aligned}
&\int_0^2\int_0^2\frac{1}{\sqrt{y_1^2+y_2^2}\sqrt{(y_1-A_{11})^2+(y_2-A_{12})^2}}dy_1dy_2 \\
&=\int_{-A_{12}}^{2-A_{12}}\int_{-A_{11}}^{2-A_{11}}\frac{1}{\sqrt{y_1^2+y_2^2}\sqrt{(y_1+A_{11})^2+(y_2+A_{12})^2}}dy_1dy_2 \\
&=\int_{D_1}+\int_{D_2}+\int_{D_3}+\int_{D_4}\frac{1}{\sqrt{y_1^2+y_2^2}\sqrt{(y_1+A_{11})^2+(y_2+A_{12})^2}}dy_1dy_2.
\end{aligned} \tag{A.38}$$

Where

$$\begin{aligned}
D_1&=\{y:-A_{11}\leqslant y_1\leqslant-\frac{1}{2}A_{11},\quad -A_{12}\leqslant y_2\leqslant-\frac{1}{2}A_{12}\}, \\
D_2&=\{y:-\frac{1}{2}A_{11}\leqslant y_1\leqslant 2-A_{11},\quad -\frac{1}{2}A_{12}\leqslant y_2\leqslant 2-A_{12}\}, \\
D_3&=\{y:-\frac{1}{2}A_{11}\leqslant y_1\leqslant 2-A_{11},\quad -A_{12}\leqslant y_2\leqslant-\frac{1}{2}A_{12}\} \\
D_4&=\{y:-A_{11}\leqslant y_1\leqslant-\frac{1}{2}A_{11},\quad -\frac{1}{2}A_{12}\leqslant y_2\leqslant 2-A_{12}\}.
\end{aligned} \tag{A.39}$$

For D_1, the integral in this region is less than or equal to

$$\begin{aligned}
&C\frac{1}{|A_1|}\int_{-A_{11}}^{-\frac{1}{2}A_{11}}\int_{-A_{12}}^{-\frac{1}{2}A_{12}}\frac{1}{\sqrt{(y_1+A_{11})^2+(y_2+A_{12})^2}}dy_1dy_2\\
&\leqslant C\frac{1}{|A_1|}\int_0^{\frac{1}{2}A_{11}}\int_0^{\frac{1}{2}A_{12}}\frac{1}{\sqrt{y_1^2+y_2^2}}dy_1dy_2\\
&\leqslant C\frac{1}{|A_1|}\int_0^{\frac{1}{2}A_{11}}|\log(\sqrt{y_1^2+\frac{A_{12}^2}{4}}+\frac{A_{12}}{2})|dy_1+C\frac{1}{|A_1|}\int_0^{\frac{1}{2}A_{11}}|\log(y_1)|dy_1\\
&\leqslant C\frac{|A_{11}|\log(|A_{12}|)}{|A_1|}+C\frac{|A_{11}|\log(|A_{11}|)}{|A_1|}\\
&\leqslant C\frac{|A||B|\log(|B|)+|A||B|\log(|AB|)}{|B|}
\end{aligned} \tag{A.40}$$

So $\frac{|B|}{|x|}$ times this term is bounded. Then for D_2, the integral in this region is less than or equal to

$$\begin{aligned}
&C\int_{-\frac{1}{2}A_{11}}^{2-A_{11}}\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{1}{\sqrt{y_1^2+y_2^2}}\frac{1}{(y_2+A_{12})}dy_1dy_2\\
&\leqslant C\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{\log(\sqrt{y_2^2+(2-A_{11})^2}+2-A_{11})-\log(\sqrt{y_2^2+\frac{A_{11}^2}{4}}-\frac{A_{11}}{2})}{y_2+A_{12}}dy_2\\
&\leqslant C\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{1-\log(y_2^2)+\log(\sqrt{y_2^2+\frac{A_{11}^2}{4}}+\frac{A_{11}}{2})}{y_2+A_{12}}dy_2\\
&\leqslant C\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{1+|\log(|y_2|)|-\log(y_2^2)}{y_2+A_{12}}dy_2\\
&\leqslant C\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{C+\frac{1}{\sqrt{|y_2|}}}{y_2+A_{12}}dy_2\\
&\leqslant C|\log(A_{12})|+C+\int_{-\frac{1}{2}A_{12}}^{0}\frac{1}{\sqrt{-y_2}(y_2+A_{12})}dy_2+\int_0^2\frac{1}{\sqrt{y_2}(y_2+A_{12})}dy_2\\
&\leqslant C|\log(A_{12})|+\frac{1}{\sqrt{A_{12}}}(C+\log(\frac{1+2\sqrt{A_{12}}}{1-2\sqrt{A_{12}}}))+C\frac{1}{\sqrt{A_{12}}}\\
&\leqslant C|\log(A_{12})|+C\frac{1}{\sqrt{A_{12}}}.
\end{aligned} \tag{A.41}$$

Remember $|A_{12}| \gtrsim B$, $B=O(x_*^2)$ for x small, so $\frac{|B|}{|x|}$ times this is bounded.

Then, the integral in D_3 is no more than

$$\begin{aligned}
&\int_{-\frac{1}{2}A_{11}}^{2-A_{11}}\int_{-A_{12}}^{-\frac{1}{2}A_{12}}\frac{1}{\sqrt{y_1^2+y_2^2}\sqrt{y_1^2+(y_2+A_{12})^2}}dy_2dy_1\\
&\leqslant C\int_0^2\int_{-A_{12}}^{-\frac{1}{2}A_{12}}\frac{1}{\sqrt{y_1^2+A_{12}^2}\sqrt{y_1^2+(y_2+A_{12})^2}}dy_2dy_1\\
&\leqslant C\int_0^2\int_0^{\frac{1}{2}A_{12}}\frac{1}{\sqrt{y_1^2+A_{12}^2}\sqrt{y_1^2+y_2^2}}dy_2dy_1\\
&\leqslant C\int_0^2\frac{\log(\sqrt{y_1^2+\frac{A_{12}^2}{4}}+\frac{A_{12}}{2})-\log(y_1)}{\sqrt{y_1^2+A_{12}^2}}dy_1\\
&\leqslant C\int_0^{\frac{2}{A_{12}}}\frac{\log(\sqrt{y_1^2+\frac{1}{4}}+\frac{1}{2})-\log y_1}{\sqrt{y_1^2+1}}dy_1\\
&\leqslant C\int_0^1\frac{\log(\sqrt{y_1^2+\frac{1}{4}}+\frac{1}{2})-\log y_1}{\sqrt{y_1^2+1}}dy_1+C\int_1^{\frac{2}{A_{12}}}\frac{\log(\sqrt{y_1^2+\frac{1}{4}}+\frac{1}{2})-\log y_1}{\sqrt{y_1^2+1}}dy_1\\
&\leqslant C\int_0^1-\log(y_1)dy_1+C\int_1^{\frac{2}{A_{12}}}\frac{1}{\sqrt{y_1^2+1}}dy_1\\
&\leqslant C+C|\log(|A_{12}|)|.
\end{aligned}\tag{A.42}$$

For x small, this term times $\frac{|B|}{|x|}$ is bounded.

Finally for D_4, we have the integral in it is no more than

$$\begin{aligned}
&\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\int_{-A_{11}}^{-\frac{1}{2}A_{11}}\frac{1}{\sqrt{y_1^2+y_2^2}(y_2+A_{12})}dy_1dy_2\\
&\leqslant C\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{\log(\sqrt{y_2^2+\frac{A_{11}^2}{4}}+\frac{A_{11}}{2})-\log(\sqrt{y_2^2+A_{11}^2}+A_{11})}{y_2+A_{12}}dy_2\\
&\leqslant C\int_{-\frac{1}{2}A_{12}}^{2-A_{12}}\frac{1}{y_2+A_{12}}dy_2\\
&\leqslant C\log(|A_{12}|).
\end{aligned}\tag{A.43}$$

Which means this term times $\frac{|B|}{|x|}$ is also bounded.

Now we assume $B>0$. Now we know $A_{11}\leqslant 0$, $A_{12}\leqslant 0$. Then, by the previous estimate, we only need to estimate $\frac{|B|}{|x|}$ times the following integral

$$\begin{aligned}
&\int_0^2\int_0^2 \frac{1}{\sqrt{y_1^2+y_2^2}\sqrt{(y_1-A_{11})^2+(y_2-A_{12})^2}}dy_1dy_2\\
&\leqslant \int_0^2\int_0^2 \frac{1}{\sqrt{y_1^2+y_2^2}(y_2-A_{12})}dy_1dy_2\\
&\leqslant \int_0^2 \frac{-\log(y_2)+\log(\sqrt{4+y_2^2}+2)}{y_2-A_{12}}\\
&\leqslant C+C\int_0^2\frac{1}{\sqrt{y_2}(y_2-A_{12})}dy_2\\
&\leqslant C+C\frac{1}{\sqrt{-A_{12}}}.
\end{aligned} \tag{A.44}$$

So we know $\frac{|B|}{|x|}$ times this term is bounded as $|A_{12}| \gtrsim |B|$ and $B = O(x_*^2)$.

Now we give an estimate of (ii). Once again up to a sign we can write it as the sum of terms like

$$\int_{D^+}\left(\frac{(x_l-\overline{y}_l)(x_m-\tilde{y}_m)(\overline{y}_n-\tilde{y}_n)}{|x-\overline{y}|^2|x-\tilde{y}|^2}\right)\omega(y)dy \tag{A.45}$$

And we have $\overline{y}_n-\tilde{y}_n = x_*(O(|\tilde{y}-x|)+O(|x|)+O(|x_*|))$, which means $|\overline{y}_n-\tilde{y}_n| \leqslant Cx_*(|\tilde{y}-x|+|x|)$, where C is a universal constant. In the right hand side we don't have x_* because $|x_*| \lesssim |x|$ by lemma 1.3.7. Next we call $U^t(x)x = (I_1(x), I_2(x)) = (x_1, -x_2) + O(|x||x_*|)$

and $J_1 = A_{11} + I_1(x) - x_1$, $J_2 = A_{12} + I_2(x) + x_2$. We have (ii) can be seen as

$$\begin{aligned}
&\leqslant C||\omega||_{L^\infty}\left(\int_{D^+}\frac{x_*}{|x-\overline{y}|}dy + \int_{D^+}\frac{x_*|x|}{|x-\overline{y}||x-\tilde{y}|}\right)\\
&\leqslant C||\omega||_{L^\infty}\left(x_* + |x|x_*\int_{x_2}^{2+x_2}\int_{-x_1}^{2-x_1}\frac{1}{\sqrt{(y_1-J_1)^2+(y_2-J_2)^2}\sqrt{y_1^2+y_2^2}}dy_1dy_2\right)\\
&\leqslant C||\omega||_{L^\infty}\left(x_* + |x|x_*\int_{x_2}^{2+x_2}\int_{-x_1}^{2-x_1}\frac{1}{|y_2-J_2|\sqrt{y_1^2+y_2^2}}dy_1dy_2\right)\\
&\leqslant C||\omega||_{L^\infty}\left(x_* + |x|x_*\int_{x_2}^{2+x_2}\frac{|\log(y_2)|+1+|\log(|x|)|}{|y_2-J_2|}dy_2\right)\\
&\leqslant C||\omega||_{L^\infty}\left(x_* + |x|x_*\int_{x_2}^{2+x_2}\frac{|\frac{1}{\sqrt{y_2}}|+1+|\log(|x|)|}{|y_2-J_2|}dy_2\right)\\
&\leqslant C||\omega||_{L^\infty}\left(x_* + |x|x_*\int_{x_2-J_2}^{2+x_2-J_2}\frac{|\frac{1}{\sqrt{y_2}}|+1+|\log(|x|)|}{|y_2|}dy_2\right)\\
&\leqslant C||\omega||_{L^\infty}(x_*+x_*)\\
&\leqslant C(\gamma)||\omega||_{L^\infty}x_2
\end{aligned} \tag{A.46}$$

For the last three inequalities, the reason for $\frac{1}{\sqrt{|y_2+J_2|}} \lesssim \frac{1}{\sqrt{y_2}}$ is, if $J_2 \geqslant 0$, this is trivial; if $J_2 \leqslant 0$, then if we can show that $x_2 \geqslant -2J_2$, then we have $y_2 \lesssim y_2 + J_2$, since $y_2 \geqslant x_2 - J_2 \geqslant -2J_2$. This is true since $|J_2| = |A_{12}+I_2+x_2| = |O(x_*^2)+O(|x|x_*)| \leqslant \frac{1}{2}x_2$ for $|x|$ small. And also we have $|J_2 - x_2| = |A_{12} + I_2| \geqslant C|x|$

So we finally complete term I. For term II, observe that we only need to change x_1 to $-x_1$ and $x_*(\tilde{x}) = -x_*$, the estimate of II is exactly the same as I. Then from the above estimate we know that $I + II$ can be written as the form $xB_8(x) + M'$, where $B_8(x) \leqslant C(\gamma)||\omega||_{L^\infty}$ and $M'(\omega)$ is a constant. Also notice that $u_2(0,0) = 0$, so $M(\omega) + M'(\omega) + M_0(\omega) = 0$. So we finish the proof of the key lemma.

Appendix B

Real line case of the stability result

One can also consider the model equation (1.8) and (1.9) with law (1.12) under the Dirichlet boundary condition in bounded domain of $\mathbb{R}$. Without loss of generality we assume the domain of this system is $[-1, 1]$, which means the system is defined on $[-1, 1]$ and satisfies the boundary condition $\omega(1) = \omega(-1) = \theta(1) = \theta(-1) = 0$. In this case, similar argument like in section 2 can show that the corresponding kernel will be

$$F(x, y, a) = \frac{y}{x}\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right) + \log\left(\frac{(x+y)^2 + a}{(x-y)^2 + a}\right)\right], \tag{B.1}$$

for $a > 0$.

The analogue of Lemma 1.5.2 will be the following:

Lemma B.0.26. **(a)** *For any $a \neq 0$, there is a constant $C(a) > 0$ such that for any $0 < x < y < 1$, $F(x, y, a) \leqslant -C(a)$.*

(b) *For any $0 < y < x < \infty$, $F(x, y, a)$ is increasing in x.*

(c) *For any $0 < x, y < \infty$, $\frac{1}{y}(\partial_x F)(x, y, a) + \frac{1}{x}(\partial_x F)(y, x, a)$ is positive.*

Proof. First $F(x, y, a)$ is easy to see that it is non-positive. For part (a), one can follow the similar but easier argument as in the proof of part (a) of lemma 1.5.2. Now let us prove part (b) and (c).

Proof of (b)

By direct computation

$$\begin{aligned}
\frac{1}{y}\partial_x F(x,y,a) &= -\frac{1}{x^2}\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right) + \log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&+ \frac{1}{x}\left[\frac{2(x-y)}{(x-y)^2} - \frac{2(x-y)}{(x-y)^2+a} - \frac{2(x+y)}{(x+y)^2} + \frac{2(x+y)}{(x+y)^2+a}\right] \\
&= -\frac{1}{x^2}\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right) + \log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&+ \frac{1}{x}\left[\frac{2a(x-y)}{(x-y)^2((x-y)^2+a)} - \frac{2a(x+y)}{(x+y)^2((x+y)^2+a)}\right] \\
&= I + II.
\end{aligned}$$

For I, by the same argument as in the proof of the first statement, it is positive. For II, we have

$$II = \frac{1}{x}(f(x-y) - f(x+y)),$$

where $f(t) = \frac{2a}{t(t^2+a)}$. It's easy to see that for $t > 0$, $f(t)$ is decreasing in t, which means $II \geqslant 0$ whenever $0 < y < x$.

Proof of (c)

First of all, let's call our target function $G(x,y,a)$, which means

$$\begin{aligned}
G(x,y,a) &= \frac{1}{y}(\partial_x F)(x,y,a) + \frac{1}{x}(\partial_x F)(y,x,a) \\
&= -\left(\frac{1}{x^2} + \frac{1}{y^2}\right)\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right) + \log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&+ \left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{2a(x-y)}{(x-y)^2((x-y)^2+a)}\right) - \left(\frac{1}{y} + \frac{1}{x}\right)\left(\frac{2a(x+y)}{(x+y)^2((x+y)^2+a)}\right) \\
&= -\left(\frac{1}{x^2} + \frac{1}{y^2}\right)\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right) + \log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&- \frac{2a}{xy((x-y)^2+a)} - \frac{2a}{xy((x+y)^2+a)}.
\end{aligned}$$

Now our aim is to prove the positivity of $G(x, y, a)$. Notice that when $a = 0$, $G(x, y, a) = 0$, as a consequence, to prove the positivity of $G(x, y, a)$, the only thing we need to show is this function is increasing in a for any x, y in the domain. On the other hand,

$$\begin{aligned}\partial_a G(x,y,a) = &-\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left(\frac{1}{(x+y)^2+a}-\frac{1}{(x-y)^2+a}\right)\\ &-\frac{2}{xy}\left[\frac{(x-y)^2}{((x-y)^2+a)^2}+\frac{(x+y)^2}{((x+y)^2+a)^2}\right].\end{aligned}$$

As a conclusion,

$$\begin{aligned}&((x-y)^2+a)^2((x+y)^2+a)^2\partial_a G(x,y,a)\\ &=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)((x+y)^2-(x-y)^2)((x+y)^2+a)((x-y)^2+a)\\ &-\frac{2}{xy}\left[(x-y)^2((x+y)^2+a)^2+(x+y)^2((x-y)^2+a)^2\right]\end{aligned}$$

It is easy to see this is a quadratic polynomial in a. Let's call the coefficient of the second order term A_2 , then

$$\begin{aligned}A_2 &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)((x+y)^2-(x-y)^2)-\frac{2}{xy}[(x-y)^2+(x+y)^2]\\ &=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\cdot 4xy-\frac{2}{xy}[2x^2+2y^2]\\ &=\frac{4}{x^2y^2}((x^2+y^2)xy-xy(x^2+y^2))\\ &=0.\end{aligned}$$

Similarly, for coefficient of the first order term A_1, we have

$$\begin{aligned}A_1 &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)(4xy)((x+y)^2+(x-y)^2)-\frac{2}{xy}[2(x-y)^2(x+y)^2+2(x+y)^2(x-y)^2]\\ &=\frac{1}{x^2y^2}[(x^2+y^2)^2\cdot 8xy-8xy(x^2-y^2)^2]\\ &\geqslant 0.\end{aligned}$$

Lastly, for the coefficient of the constant term A_0, we have

$$
\begin{aligned}
A_0 &= \left(\frac{1}{x^2} + \frac{1}{y^2}\right)(4xy)(x+y)^2(x-y)^2 - \frac{2}{xy}[(x-y)^2(x+y)^4 + (x+y)^2(x-y)^4] \\
&= \frac{(x+y)^2(x-y)^2}{x^2y^2}[(x^2+y^2)\cdot 4xy - 2xy((x+y)^2 + (x-y)^2)] \\
&= 0.
\end{aligned}
$$

In all, we have $\partial_a G(x,y,a) \geqslant 0$ for $x, y > 0$. □

From this lemma, one can do the same argument to get the blow up result, which is the following theorem:

Theorem B.0.27. *There exists initial data such that solutions to* (1.8) *and* (1.9)*, with velocity given by* (1.67)*, and* $F(x,y,a)$ *defined by (B.1), blow up in finite time.*

In fact, we can prove the following type of initial data will lead to blow up:

- θ_{0x}, ω_0 smooth odd and are supported in $[-1, 1]$.
- $\theta_{0x}, \omega_0 \geqslant 0$ on $[0, 1]$.
- $\theta_0(0) = 0$.
- $\|\theta_0\|_\infty \leqslant M$.

And similarly, for general pertubation (analogue of theorem 1.6.1), we also have the similar blow up result.

Assume the velocity u is given by the following choice of Biot-Savart Law

$$
u(x) = \frac{1}{\pi}\int_{-1}^{1} (\log|(x-y)]| + f(x,y))\,\omega(y)\,dy, \tag{B.2}
$$

(B.3)

where f is a smooth function whose precise properties we will specify later. We view f as a perturbation and we will show solutions to the system (1.8) and (1.9) can still blow-up in finite time.

Theorem B.0.28. *Let $f \in C^2$ be supported on $[-1,1]$, such that $f(0,y) = f(0,-y)$ for all y. Then there exists initial data ω_0, θ_0 such that solutions of (1.8) and (1.9), with velocity given by (B.2), blow up in finite time.*

Again we can prove the following type of initial data will form finite time singularity:

- θ_{0x}, ω_0 smooth odd and are supported in $[-1,1]$.
- $\theta_{0x}, \omega_0 \geqslant 0$ on $[0,1]$.
- $\theta_0(0) = 0$.
- $\operatorname{supp} \omega_0 \subset [0,\epsilon]$.
- $\|\theta_0\|_\infty \leqslant M$.

We leave the proofs of these theorems as exercises for interested reader.

Appendix C

Some inequalities in Keller-Segel equation

Here we first prove Proposition 2.3.20, and then sketch the proofs of the multiplicative inequalities (2.31) and (2.34).

Let $\rho(x,t)$ be a solution of (2.19)

$$\partial_t \rho + (u \cdot \nabla)\rho - \Delta\rho + \nabla \cdot (\rho \nabla (-\Delta)^{-1}(\rho - \overline{\rho})) = 0, \quad \rho(x,0) = \rho_0(x),$$

with some smooth incompressible vector field $u(x,t)$.

Proposition C.0.29. *Let $\rho \in C^\infty(\mathbb{T}^2)$. Suppose that $\|\rho(\cdot,t) - \overline{\rho}\|_{L^2} \leqslant 2B$ for all $t \in [0,T]$ and some $B \geqslant 1$. Then we also have $\|\rho(\cdot,t) - \overline{\rho}\|_{L^\infty} \leqslant C_4 B \max(B, \overline{\rho}^{1/2})$ for some universal constant C_4 and all $t \in [0,T]$.*

Proof. Observe first that by a direct computation, for every integer $p \geqslant 1$ we have

$$\begin{aligned} \partial_t \int_{\mathbb{T}^2} (\rho - \overline{\rho})^{2p} dx = -(4 - \frac{2}{p}) \int_{\mathbb{T}^2} |\nabla((\rho - \overline{\rho})^p)|^2 dx \\ +(2p-1) \int_{\mathbb{T}^2} (\rho - \overline{\rho})^{2p+1} dx + 2p\overline{\rho} \int_{\mathbb{T}^2} (\rho - \overline{\rho})^{2p} dx. \end{aligned} \tag{C.1}$$

To obtain (C.1), we need to multiply (2.19) by $2p(\rho - \overline{\rho})^{2p-1}$, integrate, and then simplify the obtained terms using integration by parts and the fact that u is divergence free.

Let us estimate $\|\rho-\overline{\rho}\|_{L^{2^n}}$ inductively. By assumption, we have $\|\rho(\cdot,t)-\overline{\rho}\|_{L^2}\leqslant 2B$ for all $t\in[0,T]$. Assume that for some $n\geqslant 1$, we have

$$\|\rho-\overline{\rho}\|_{L^{2^n}}\leqslant \Upsilon_n \quad \text{for all } t\in[0,T].$$

Let us derive an estimate for an upper bound Υ_{n+1} on $\|\rho-\overline{\rho}\|_{L^{2^{n+1}}}$ on $[0,T]$. For that purpose, let us set $p=2^n$ in (C.1) and let us define $f(x,t)=(\rho-\overline{\rho})^p\equiv(\rho-\overline{\rho})^{2^n}$. Then (C.1) implies

$$\partial_t\int_{\mathbb{T}^2}|f|^2dx\leqslant -2\int_{\mathbb{T}^2}|\nabla f|^2dx+2^{n+1}\int_{\mathbb{T}^2}|f|^{2+2^{-n}}dx+2^{n+1}\overline{\rho}\int_{\mathbb{T}^2}|f|^2. \tag{C.2}$$

Also, in terms of f, our induction assumption is that $\int_{\mathbb{T}^2}|f|dx\leqslant\Upsilon_n^{2^n}$.

We will now need the following Gagliardo-Nirenberg inequality.

Lemma C.0.30. *Suppose $v\in C^\infty(\mathbb{T}^d)$, $d\geqslant 2$, and the set where v vanishes is nonempty. Assume that $q,r>0$, $\infty>q>r$, and $\frac{1}{d}-\frac{1}{2}+\frac{1}{r}>0$. Then*

$$\|v\|_{L^q}\leqslant C(d,p)\|\nabla v\|_{L^2}^a\|v\|_{L^r}^{1-a},\quad a=\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{d}-\frac{1}{2}+\frac{1}{r}}. \tag{C.3}$$

The constant $C(d,p)$ for a fixed d is bounded uniformly when q varies in any compact set in $(0,\infty)$.

Proof. This inequality is well known in the case $v\in C_0^\infty(\mathbb{R}^d)$, see e.g. [71]. A simple proof is contained in [54]. Going through the proof in [54], it is not difficult to verify that the result still holds in the periodic case under the assumption that v vanish somewhere in $\mathbb{T}^d$ (which rules out increasing the mean value without increasing variance). One can similarly trace the claim regarding the constant $C(d,p)$. We refer to [54] for details. □

Applying Lemma C.0.30 with $d=2$, $r=2$, and $q=2+2^{-n}$ yields

$$\|f\|_{L^{2+2^{-n}}}^{2+2^{-n}}\leqslant C\|\nabla f\|_{L^2}^{2^{-n}}\|f\|_{L^2}^2\leqslant\frac{1}{2^{n+1}}\|\nabla f\|_{L^2}^2+C\|f\|_{L^2}^{\frac{2}{1-2^{-n-1}}}, \tag{C.4}$$

where we used Young's inequality in the last step. Moreover, we also have

$$\|f\|_{L^2} \leqslant C\|\nabla f\|_{L^2}^{1/2}\|f\|_{L^1}^{1/2}. \tag{C.5}$$

Applying (C.5) and (C.4) to (C.2), we obtain

$$\partial_t\|f\|_{L^2}^2 \leqslant -C_1\|f\|_{L^2}^4\|f\|_{L^1}^{-2} + C_2 2^{n+1}\|f\|_{L^2}^{\frac{2}{1-2^{-n-1}}} + 2^{n+1}\overline{\rho}\|f\|_{L^2}^2, \tag{C.6}$$

where $C_{1,2}$ are some fixed universal constants (not connected to C_1 and C_2 used earlier in section 1.3). Clearly, given the upper bound on $\|f\|_{L^1}$, the right hand side of (C.6) turns negative if $\|f\|_{L^2}$ becomes sufficiently large. Thus $\|f\|_{L^2}$ cannot cross this threshold. Assuming without loss of generality that $\Upsilon_n \geqslant 1$ for all n, a direct computation shows that if $\|\rho-\overline{\rho}\|_{L^{2^{n+1}}}$ reaches the value Υ_{n+1} which satisfies the following recursive equality, then the right hand side of (C.6) is negative:

$$\log \Upsilon_{n+1} = \max(\Gamma_n, \Theta_n)$$

where

$$\Gamma_n = \frac{2^{n+1}-1}{2^{n+1}-2}\log \Upsilon_n + \frac{1}{2^{n+1}}((n+1)\log 2 + \log C) \tag{C.7}$$

$$\Theta_n = \log \Upsilon_n + \frac{1}{2^{n+1}}((n+1)\log 2 + \log C + \max(\log \overline{\rho}, 0)). \tag{C.8}$$

Here $C \geqslant 1$ is some universal constant. Denote $q_j = \frac{2^{j+1}-1}{2^{j+1}-2}$ and observe that due to telescoping,

$$\prod_{j=1}^{n} q_j = \frac{2^{n+1}-1}{2^n} \xrightarrow{n\to\infty} 2.$$

An elementary inductive computation shows that if $B \gtrsim \overline{\rho}^{1/2}$, then the first recursive relation (C.7) determines the size of Υ_{n+1}, yielding the estimate $\Upsilon_{n+1} \leqslant CB^2$. If $B \lesssim \overline{\rho}^{1/2}$, then the second relation (C.8) dominates, yielding the estimate $\Upsilon_{n+1} \leqslant CB\overline{\rho}^{1/2}$.

Since

$$\|\rho - \overline{\rho}\|_{L^\infty} = \lim_{n\to\infty} \|\rho - \overline{\rho}\|_{L^{2^n}},$$

we obtain that

$$\|\rho - \overline{\rho}\|_{L^\infty} \leqslant CB\max(B, \overline{\rho}),$$

proving the proposition. □

Proposition C.0.31. *Suppose that $f \in C^\infty(\mathbb{T}^d)$ and mean zero. Then*

$$\|D^m f\|_{L^p} \leqslant C\|f\|_{L^2}^{1-a}\|f\|_{H^n}^a, \quad a = \frac{m - \frac{d}{p} + \frac{d}{2}}{n},$$

where D stands for any partial derivative, $2 \leqslant p \leqslant \infty$, and we assume $n > M + d/2$.

Note that the last assumption is not necessary. However it makes the proof simpler, and this is the only case we need in this section. Indeed, in the estimates of Section 3 we have $s+1 > l + d/2$ unless $l = s$ (recall $d \leqslant 3$). But when $l = s$, we are only estimating $\|D^s\rho\|_{L^2}$, which is straightforward.

Proof. Consider $p = 2$. Then

$$\|D^m f\|_{L^2} \leqslant \|f\|_{L^2}^{1-\frac{m}{n}}\|f\|_{H^n}^{\frac{m}{n}} \tag{C.9}$$

by Hölder inequality on Fourier side.

Next consider $p = \infty$. Then

$$\|D^m f\|_{L^\infty} \leqslant C\sum_{0<|k|<\Lambda}|k|^m|\hat{f}(k)| + C\sum_{|k|\geqslant\Lambda}|k|^m|\hat{f}(k)| \equiv (I) + (II).$$

Now

$$(I) \leqslant C\Lambda^{m+\frac{d}{2}}\left(\sum_{0<|k|<\Lambda}|\hat{f}(k)|^2\right)^{1/2}$$

by Cauchy-Schwartz. On the other hand,

$$(II) \leqslant C \left(\sum_{|k| \geqslant \Lambda} |k|^{2n} |\hat{f}(k)|^2 \right)^{1/2} \left(\sum_{|k| \geqslant \Lambda} |k|^{2(m-n)} \right)^{1/2} \leqslant C \|f\|_{H^n} \Lambda^{(m-n)+\frac{d}{2}},$$

provided that $n > m + \frac{d}{2}$. Choose Λ so that

$$\|f\|_{L^2} \Lambda^{m+\frac{d}{2}} = \|f\|_{H^n} \Lambda^{m-n+\frac{d}{2}}.$$

Such choice leads to the bound

$$\|D^m f\|_{L^\infty} \leqslant C \|f\|_{L^2}^{\frac{n-m+d/2}{n}} \|f\|_{H^n}^{\frac{m+d/2}{n}}. \tag{C.10}$$

The general case $2 < p < \infty$ follows immediately from (C.9) and (C.10). □

Proposition C.0.32. *Suppose that* $f \in C^\infty(\mathbb{T}^d)$*, and* $m > 0$*. Then*

$$\|\rho\|_{H^s} \leqslant C \|\rho\|_{H^{s+1}}^{\frac{2s+d}{2s+2+d}} \|\rho\|_{L^1}^{\frac{2}{2s+2+d}}.$$

Proof. The proof of this proposition can be done similarly to the previous one. One needs to use that $\|\hat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$. We leave details to the interested reader. □

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