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Co-evolution of Density and Topology in a Model of City Formation

Classical models in urban economics usually describe transportation costs in a very simplified fashion. Transportation networks are usually absent and transportation of goods and individuals is usually interpreted as displacement taking place in continuous space. Because these classical models usually assume that the processes shaping the city and road changes take place in a relative long period of time, they often consider the evolution of a city an equilibrium process where the city's road system expansion is negligible in the economical processes at place.

An article written by Barthélemy and Flammini uses a model of city formation to study the interplay between the local density of population and the structure of the road network. This simple model neglects the potentially relevant factors such as the social, economical and geographical conditions in any particular historical period of time that causes the changes of the density and topology of the city. Here, the overall growth of population is considered an exogenous variable and the evolution of the city is considered inherently an "out-of-equalibrium" process. Various empirical studies (Batty 2005; Lammer et al. 2006; Makse et al. 1998, 2002) have shown that roads networks, despite the particular geographical, historical, social-economical processes that have shaped the urban districts in different ways, share significant quantitative similarities. Therefore, a model that demonstrates the general and simple mechanisms of the interplay between density and topology is possible. This model assumes that manmade transportation networks maximize efficiency and minimize costs and dissipated energy (Doyle and Snell 1989). The principle of optimality of road building is at work only locally.

1. The Model

The main assumption is that every road network constantly grows by connecting to a set of center points, whether they are homes, offices, or businesses, in an efficient and economic way. When a new center appears, it will connect to the nearest point of existing road network in the most economic and efficient way. When we grow a single new portion of the road from M to M' in order to grant the maximum reduction of the cumulative distance of two new centers A and B from the network, we have δd=d(M, A)+d(M, B)-[d(M', A)+d(M',B)]. δd is the optimal fixed length distance dx between M and M' which minimizes distance between A and B. This shows that $dMM \propto uA + uB$ where uA and uB are the unitary vectors from M to A and M to B. When the formula is extended to the centers with non-uniform weight η, the formula is modified as $dMM \propto \eta A\ uA + \eta B\ uB$, where ηA and ηB are two center weights that indicate "heavy" or "light" centers. The constant inter distance between nodes or intersections is $1{\sim}1/\sqrt{p}$, and because the number of points grows in time, the typical road length can be expressed $l1{\sim}1/\sqrt{t}$.

The empirical distribution of centers in real cities decreases exponentially from the center (Makse et al. 1998, 2002). The exponential expression $P(r)=\exp(-|r|/r_c)$ shows the central spatial location and measure the areas formed by the resulting network.

The betweeness centrality g(e) of an edge e is the fraction of shortest paths between any pair of nodes in the network that goes through e, where σ_{st} is the number of the shortest paths going from s to t and passing through e. We get

$$g(e) = \sum_{s \neq t} \frac{\sigma st\ (e)}{\sigma st} \tag{6}$$

This quantity g(e) is broadly distributed and varies over more than 6 orders of magnitude. Barthélemy and Flammini arbitrarily grouped the edges in three classes $[1, 10^4]$, $[10^4, 10^5]$, $[10^5, \infty]$ and plot them with different thicknesses (Fig. 7). The thickest lines represent the edges with the largest centrality, which form a tree of large arteries.

The housing rent price C_R is an increasing function of the local density $\rho(i)=N(i)/S$, where N(i) is the number of centers in the sector (i) . The rent price C_R is directly proportional to the local density $\rho(i)$

$$CR(i) = A\rho(i) \tag{7}$$

where A is some positive prefactor corresponding to the price per density.

Parallel to edge betweenness g(e), node betweeness centrality g(v) of a mode v is defined as the fraction of the shortest paths between any pair of points in the network that goes through v, where σ_{st} is the number of shortest paths going from s to t and $\sigma_{st}(v)$ is the number of shortest paths going from s to t and passing through v. The formula is

$$g(v) = \frac{1}{N(N-1)} \sum_{s \neq 1} \frac{\sigma st(v)}{\sigma st} \tag{8}$$

Betweenness centrality in planar graphs is strictly correlated to other measure of centrality. Fig 9 shows in three different ways that if shortest paths are chosen to move from and to arbitrary paths, central nodes are the most frequently visited.

Average commuting cost or average accessibility $\bar{g}$ is considered proportional to the distance to the center. For each sector S_i, the average betweeness centrality is computed as $\bar{g}(i)=\frac{1}{N(i)}\sum_{v\in Si} g(v)$. The transportation costs $C_T(i)$ for a center in sector (i) is given by

$$C_{T\,(i)} = B(g_m - \bar{g}(i)) \tag{10}$$

where B and g_m are positive constants. This formula shows that commuting cost is negatively correlated with average accessibility.

Net income of a new center c in a sector (i) is denoted as K(i) and it is calculated by subtracting rent price $C_R(i)$ and transportation cost $C_T(i)$. Formally,

$$K(i)=Y- C_R(i) - C_T(i) \tag{11}$$

Because the location is usually chosen by minimizing costs, the probability that a new center will choose the sector (i) as its new location is given by

$$P(i) = \frac{e^{\beta K(i)}}{\sum_j e^{\beta K(j)}} \tag{12}$$

The parameter β describes in an effective way all the factors that have not been taken into account. For numerical simulations, the local density is normalized by the global density $\rho=N/L^2$, so we can make the correlation between P(i) and N(i). Here, P is the possibility of choosing the location and N is the number of nodes-the intersections. If β=0, then cost is irrelevant (net income K is negatively affected by costs) and new centers will appear uniformly distributed across the different sectors, as $P(i)\sim\frac{1}{N(i)}$. But if β ~ ∞, then the location with the minimal cost must be chosen determined by the number of P(i). If P(i)=1, then $e^{\beta K(i)}$is approximately $\sum_j e^{\beta K(j)}$, and β ~ ∞, then it must be the case net income K(i) is minimum or close to 0 so the cost must be large or relevant. If P(i) = 0, then $e^{\beta K(i)}$is negligible compared to $\sum_j e^{\beta K(j)}$, so net income K(i) would not be a negligible or minimum number, so the cost must be small or irrelevant. The parameter β can be used to determine the importance of the cost relative to other factors not explicitly included in the model.

If we assume that the centers are located in a one-dimensional segment [-L, L], the generic location x can be given by a function related to density at x which is denoted as ρ (x, t) and the number of nodes N. The function writes

$$g(x) = \int_{-L}^{x} \rho(y,t)dy[N - \int_{-L}^{x} \rho(y,t)dy] \tag{16}$$

The equation for the density therefore reads

$$\partial_t \rho(x,t) = e^{\beta[\lambda \frac{\int^{x}{}_{-L}\rho(y,t)dy}{N} * \frac{(N-\int^{x}{}_{-L}\rho(y,t)dy)}{N} - \frac{\rho(x,t)}{N}]} \tag{17}$$

Equation (17) is essential for understanding this city model because it expresses the co-evolution of the road network and the density. In this function, ρ (x, t) is the density, λ is the relative weight between centrality and density and β denotes all other factors not explicitly included in the model. $N = \int_{-L}^{L} \rho(y,t)dy$. Therefore, we can conclude that after a transient regime, the process locks in a pattern of growth in which the total population grows at a constant rate.

$$N = \int_{-L}^{L} \rho(y,t)dy \propto t \tag{18}$$

The two-dimensional case is similar to the one-dimensional case in which noise parameter β describes the overall importance of the cost-factors with respect to other factors. If β is very large, then the location which maximizes the cost is chosen. When λ is small density plays the dominant role in determining the location of the new centers. Specifically, the income $Y_2 = \sum_i [\frac{N(i)}{N}]^2$. In the uniform case, all N(i), the number of nodes or intersections, is approximately equal, therefore $Y_2 \sim 1/N_s$, which is usually small. A fraction of dominating sectors is calculated as $\sigma = 1/Y_2N_S$. σ is observed to decrease very fast when λ increases, meaning that a phenomenon of localization sets in as soon as transportation costs are involved.

2. Discussions

This basic city model describes the interplay between the co-evolution of the population density and the topology of the road network. Specifically the rent cost and the demand for accessibility transforms the population spatial density. If transportation costs are higher, a densely populated area emerges around which the density decays exponentially. The demand for accessibility increases when rent costs are high.

A very important element, how individuals or companies choose a location, is very important for modeling the evolution of a city. It is affected by two main factors, transportation costs and rent price. If β is very large, then the location which maximizes the cost is chosen. When λ is small density plays the dominant role in determining the location of the new centers.

References

Barthélemy, Marc & Flammini, Alessandro. (2009). Co-evolution of Density and Topology in a Simple Model of City Formation. *Networks and Spatial Economics*. **9**, 401-425.

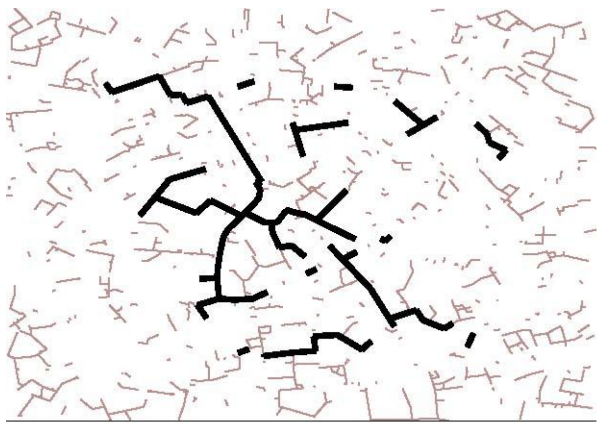

Fig. 7 Traffic map of the network. The edge centrality is computed and divided in three different groups and the thickness of the edge is plotted according to the group (from thin to thick for increasing edge centrality)