Present-Biased Preferences

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Outline

Present-Biased Preferences

- Preference Reversals
- Quasi-Hyperbolic Discounting
- Stationarity, Time Invariance, and Dynamic (In)Consistency

Consumption-Savings Dynamics with Present Bias

- Three-Period Example
- Pathologies: Violations of Monotonicity and Continuity
- Infinite-Horizon Model with Uncertainty
- Illiquid Assets as Commitment Devices

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Preference Reversals

Example (Money now or later)

Problem 1: Decide now between

A: \$100 now B: \$110 in 4 weeks

Problem 2: Decide now between

C: \$100 in 26 weeks D: \$110 in 30 weeks

There are some issues with this example: saving/borrowing, risk of nonpayment, other income and consumption.

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Timing of Costs and Benefits

- Another type of question sometimes used in experiments involves choices between contemporaneous consumption alternatives, but where the associated rewards and costs might be incurred in (different) future periods.

Example (Healthy or unhealthy snack)

PROBLEM 1: Apple or Chocolate Bar (choice eaten next week).

PROBLEM 2: Apple or Chocolate Bar (choice eaten now).

People typically choose the healthy snack when choosing for the future, but are more likely to choose the unhealthy snack when they consume their choice immediately.

- A similar problem might be the choice between a documentary and another type of movie (comedy, action, etc.).
  (Q: How many unwatched documentaries do you have in your Netflix queue?)

Incompatibility with Exponential Discounting?

To understand why these examples are puzzling, think of them in the context of the standard exponential discounting model:

$$U = u_0 + \delta u_1 + \delta^2 u_2 + \delta^3 u_3 + \dots$$

where u_t is understood to be the instantaneous utility associated with some underlying consumption, work, etc. (for example, $u_t = u(x_t)$).

Incompatibility with Exponential Discounting?

Example (Money now or later (continued))

Normalizing $u(0) = 0$,

$$\begin{aligned}
&\$100 \text{ now} \succ \$110 \text{ in 4 weeks} \\
\iff\ & u(100) > \delta^4 u(110) \\
\iff\ & \delta^{26} u(100) > \delta^{30} u(110) \\
\iff\ & \$100 \text{ in 26 weeks} \succ \$110 \text{ in 30 weeks.}
\end{aligned}$$

- Thus, the typical pattern from this example is incompatible with exponential discounting.

Assumptions in this argument: no borrowing or lending; trust that future payments will be made; and no background consumption (so marginal utility of $\$x$ is the same in any period). These assumptions are certainly quite strong! It is always good to think critically about experimental or empirical evidence and question results where appropriate. Nonetheless, the intuition behind this example is very compelling, and other experiments (such as those involving dated costs and rewards) deliver the same conclusion under less stringent assumptions.

Incompatibility with Exponential Discounting?

Example (Work now or later (continued))

Let $u(h)$ denote the utility value for hours worked, and again normalize $u(0) = 0$, so that $u(8) < u(7) < u(0) = 0$. Then,

$$\begin{aligned} & \text{7 hours today} \prec \text{8 hours in 1 week} \\ \iff & u(7) < \delta u(8) \\ \iff & \delta^{10} u(7) < \delta^{11} u(8) \\ \iff & \text{7 hours in 10 weeks} \prec \text{8 hours in 11 weeks.} \end{aligned}$$

- Again, the typical pattern from this example is incompatible with exponential discounting.

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What type of discounting is consistent with these patterns?

To understand what these examples imply about preferences, abstract away from exponential discounting for a moment. Suppose the individual has a discount function $D(t)$:

- An instantaneous utility of u_t utils in period t is equivalent to $D(t)u_t$ utils in period 0 (in particular, $D(0) = 1$).
- Preferences are therefore represented by

$$U = u_0 + D(1)u_1 + D(2)u_2 + D(3)u_3 + \dots$$

- Exponential discounting is the special case where $D(t) = \delta^t$.

What type of discounting is consistent with these patterns?

What does our first example require of the discount function?

$$\$100 \text{ now} \succ \$110 \text{ in 4 weeks}$$

$$\iff u(100) > D(4)u(110) \iff \frac{D(0)}{D(4)} > \frac{u(110)}{u(100)}$$

and

$$\$100 \text{ in 26 weeks} \prec \$110 \text{ in 30 weeks}$$

$$\iff D(26)u(100) < D(30)u(110) \iff \frac{D(26)}{D(30)} < \frac{u(110)}{u(100)}.$$

Together, these require:

$$\frac{D(0)}{D(4)} > \frac{D(26)}{D(30)}$$

so discounting between period 0 and 4 must be greater than discounting between period 26 and 30.

Present-Biased Preferences

- The second example has a similar implication for discount rates.
- Both of these examples require what we refer to as present-biased preferences.

Definition

An individual has present-biased preferences if their discount function satisfies

$$\frac{D(0)}{D(\tau)} > \frac{D(t)}{D(t+\tau)} \qquad \forall t, \tau > 0.$$

- As we already saw, exponential discounting precludes present bias since $\frac{D(0)}{D(\tau)} = \frac{D(t)}{D(t+\tau)} = \frac{1}{\delta^\tau}$.

Present-Biased Preferences

- One of the simplest extensions of the exponential discounting model that permits a present bias is the quasi-hyperbolic discounting model of Phelps and Pollak (1968) and Laibson (1997).

Definition

Given $\beta, \delta \in (0,1)$, the quasi-hyperbolic discounting function takes the form:

$$D(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta\delta^{\tau} & \text{if } t > 0. \end{cases}$$

- QH Discounting Preferences, also called (β, δ) preferences, are given by

$$\begin{aligned} U &= u_0 + \beta\delta u_1 + \beta\delta^2 u_2 + \beta\delta^3 u_3 + \dots \\ &= u_0 + \beta(\delta u_1 + \delta^2 u_2 + \delta^3 u_3 + \dots) \end{aligned}$$

- QH $\Longrightarrow$ Present Bias: $\frac{D(0)}{D(\tau)} = \frac{1}{\beta\delta^{\tau}} > \frac{1}{\delta^{\tau}} = \frac{D(t)}{D(t+\tau)}$ for $t, \tau > 0$.

Revisiting the Preference Reversal Examples

Can quasi-hyperbolic discounting rationalize the preference reversals from our previous examples?

Example (Money now or later (continued))

To illustrate the additional flexibility of the (β, δ) model simply, suppose $\beta = \frac{1}{2}$ and $\delta = 1$. Suppose also that $u(x) = x$. Then:

$$u(100) = 100 > 55 = \beta\delta^4 u(110)$$
$$\implies \$100 \text{ now} \succ \$110 \text{ in 4 weeks}$$

and

$$\beta\delta^{26} u(100) = 50 < 55 = \beta\delta^{30} u(110)$$
$$\implies \$100 \text{ in 26 weeks} \prec \$110 \text{ in 30 weeks.}$$

- Try applying QH discounting to the work example or the snack example.

Beyond Present Bias

The quasi-hyperbolic discounting model has the benefit of parsimoniously incorporating present bias. However, there is also evidence to suggest a bias for the near future, not just the present:

- For example, someone may prefer \$100 in 1 week to \$110 in 5 weeks, yet prefer \$110 in 30 weeks to \$100 in 26 weeks.
- Such preferences can be explained with neither exponential discounting nor quasi-hyperbolic discounting.

This suggests that present bias could be strengthened to strongly diminishing impatience: $\frac{D(t)}{D(t+\tau)}$ is strictly decreasing in t.

- Hyperbolic Discounting is one example with strongly diminishing impatience: $D(\tau) = (1 + \alpha\tau)^{-\gamma/\alpha}$ for some parameters $\alpha, \gamma > 0$.
- Quasi-hyperbolic discounting is a simplification that captures one of the most prominent features of hyperbolic discounting: present bias.

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Stationarity

- Dated rewards: (x, t) gives outcome x at time t
    - x could denote a level of money, costly effort, or any other outcome.
- Stationarity: $(x, t) \succsim_0 (y, s) \iff (x, t + \tau) \succsim_0 (y, s + \tau)$
    - Note that agents could have different preferences in different time periods, so we write $\succsim_0$ to emphasize that this is the "period 0" preference.
    - The previous examples of preference reversals are **violations** of stationarity. In the money example:
        - Problem 1: $(100, 0) \succ_0 (110, 4)$
        - Problem 2: $(100, 26) \prec_0 (110, 30)$

Decisions in Different Time Periods

Example (Money now or later)

PROBLEM 1: Decide now between

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Some Properties of Preferences Across Periods

- Time Invariance: $(x, t) \succsim_0 (y, s) \iff (x, t+\tau) \succsim_\tau (y, s+\tau)$.
  - In other words, pushing everything out τ periods into the future—both the time of the decision and the time of the rewards—does not change preferences.
  - The two previous examples **satisfy** time invariance. In the money example:
    - Problem 1: $(100, 0) \succ_0 (110, 4)$
    - Problem 2: $(100, 26) \prec_0 (110, 30)$
    - Problem 3: $(100, 26) \succ_{26} (110, 30)$

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    - The two previous examples **satisfy** time invariance. In the money example:
        - Problem 1: $(100, 0) \succ_0 (110, 4)$
        - Problem 2: $(100, 26) \prec_0 (110, 30)$
        - Problem 3: $(100, 26) \succ_{26} (110, 30)$
- Dynamic Consistency: $(x, t) \succsim_0 (y, s) \iff (x, t) \succsim_\tau (y, s)$ for all $\tau \leq t, s$.
    - In other words, the plans made today will not be reversed at some future period. This is also called time consistency.
    - The examples **violate** dynamic consistency. In the money example:
        - Problem 1: $(100, 0) \succ_0 (110, 4)$
        - Problem 2: $(100, 26) \prec_0 (110, 30)$
        - Problem 3: $(100, 26) \succ_{26} (110, 30)$

Connections Between Conditions

- Stationarity: $(x,t) \succsim_0 (y,s) \iff (x,t+\tau) \succsim_0 (y,s+\tau)$
- Time Invariance: $(x,t) \succsim_0 (y,s) \iff (x,t+\tau) \succsim_\tau (y,s+\tau)$
- Dynamic Consistency: $(x,t) \succsim_0 (y,s) \iff (x,t) \succsim_\tau (y,s)$

Observation

If preferences over dated rewards satisfy Time Invariance and Dynamic Consistency, then they satisfy Stationarity.

Proof

$$
\begin{aligned}
(x,t) \succsim_0 (y,s) &\iff (x,t+\tau) \succsim_\tau (y,s+\tau) && \text{(by TI)} \\
&\iff (x,t+\tau) \succsim_0 (y,s+\tau) && \text{(by DC)}
\end{aligned}
$$

Connections Between Conditions

- Thus, to accommodate preference reversals (violations of stationarity) as in these examples, we must give up either time invariance or dynamic consistency.
- If we view time invariance as a natural assumption (which was implicit in our discussion of Problem 3 in the previous examples), then we are forced to give up dynamic consistency!
- Present bias is therefore one natural source of dynamic inconsistency.

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Three-Period Example

- Setting:
  - Wealth: w_1, w_2, w_3
  - Consumption: c_1, c_2, c_3
  - Budget constraints: $w_3 = R(w_2 - c_2)$, $w_2 = R(w_1 - c_1)$
  - Note that in contrast to the discussion and examples in the previous section, we are now permitting consumption in all three periods.
- Preferences and Beliefs:
  - Quasi-hyperbolic (β-δ) discounting and instantaneous utility $u(c)$.
  - Sophistication about future present bias and dynamic inconsistency: agent correctly predicts her (time-inconsistent) future behavior. Therefore, optimal choices can be solved using backward induction.
  - An important development in the literature on time-inconsistent preferences is the introduction of naivete in the form of underestimation of future present bias. We'll discuss naivete in a future lecture; for now, assume sophistication.

Consumption-Savings Under Full Commitment

- If the agent can fully commit in period 1 to a consumption stream, then she chooses (c_1, c_2, c_3) to maximize

$$u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3)$$

subject to the constraints

$$c_3 = w_3 = R(w_2 - c_2) \qquad \text{and} \qquad w_2 = R(w_1 - c_1)$$

- Solving this problem yields the Euler equations

$$u'(c_1) = \beta\delta R u'(c_2) \qquad \text{and} \qquad u'(c_2) = \delta R u'(c_3)$$

Consumption-Savings Without Commitment

- Consider now the case without commitment, so the agent is able to revise consumption plans in every period.
- Period 3 behavior: $c_3(w_3) = w_3$
- Period 2 preferences and behavior:
  - In period 2, chooses $c_2(w_2)$ (and hence w_3) to maximize

$$u(c_2) + \beta\delta u(c_3(w_3)) = u(c_2) + \beta\delta u(R(w_2 - c_2))$$

  - Note that this differs from the objective used to select $c_2(w_2)$ under full commitment.
  - The resulting Euler equation is:

$$u'(c_2) = \beta\delta R u'(R(w_2 - c_2)) = \beta\delta R u'(c_3)$$

Consumption-Savings Without Commitment

- Period 1 preferences and behavior:
    - Given the period 2 behavior $c_2(w_2)$ calculated above, utility from consumption c_1 and next period wealth w_2 is
    $$u(c_1) + \beta\delta V(w_2)$$
    where
    $$V(w_2) = u(c_2(w_2)) + \delta u(R(w_2 - c_2(w_2)))$$
    - Thus in period 1, chooses $c_1(w_1)$ to maximize
    $$u(c_1) + \beta\delta V(R(w_1 - c_1))$$
    - FOC:
    $$u'(c_1) = \beta\delta R V'(w_2).$$

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  - Thus in period 1, chooses $c_1(w_1)$ to maximize
    $$u(c_1) + \beta\delta V(R(w_1 - c_1))$$
  - FOC:
    $$u'(c_1) = \beta\delta R V'(w_2).$$
  - Also, using the period 2 Euler equation, note that (assume $c_2(w_2)$ is differentiable):
    $$\begin{aligned} V'(w_2) &= u'(c_2(w_2))c_2'(w_2) + \delta R u'(R(w_2 - c_2(w_2)))(1 - c_2'(w_2)) \\ &= u'(c_2(w_2))c_2'(w_2) + \frac{1}{\beta} u'(c_2(w_2))(1 - c_2'(w_2)) \end{aligned}$$

Consumption-Savings Without Commitment

$$u'(c_1) = \beta\delta R V'(w_2)$$
$$V'(w_2) = u'(c_2(w_2))c_2'(w_2) + \frac{1}{\beta}u'(c_2(w_2))(1 - c_2'(w_2))$$

- Period 1 preferences and behavior (continued):
  - Combining these two equations:

$$u'(c_1) = \beta\delta R u'(c_2(w_2))c_2'(w_2) + \beta\delta R\frac{1}{\beta}u'(c_2(w_2))(1 - c_2'(w_2))$$
$$= \underbrace{\left[c_2'(w_2)\beta\delta + (1 - c_2'(w_2))\delta\right]}_{\text{effective discount factor}} Ru'(c_2(w_2))$$

Consumption-Savings Without Commitment

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- Period 1 preferences and behavior (continued):
  - Combining these two equations:

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- Summing up, the Euler equations without commitment are:

$$u'(c_1) = \left[\frac{dc_2}{dw_2}\beta\delta + \left(1 - \frac{dc_2}{dw_2}\right)\delta\right] R u'(c_2)$$

$$u'(c_2) = \beta\delta R u'(c_3)$$

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Pathologies: Violations of Monotonicity and Continuity

$$u'(c_1) = \beta\delta R V'(w_2)$$
$$V'(w_2) = u'(c_2(w_2))c_2'(w_2) + \frac{1}{\beta}u'(c_2(w_2))(1 - c_2'(w_2))$$

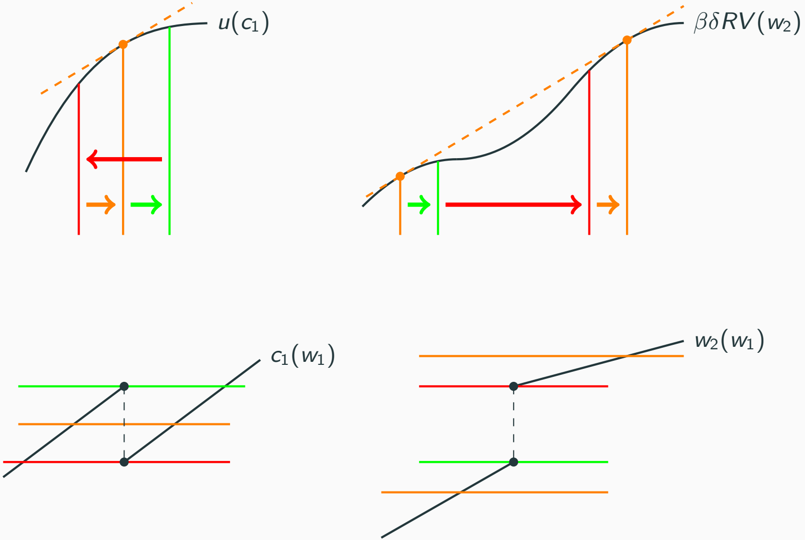
- Possible non-concavity of value function V (i.e., non-monotonicity of right side of the period 1 Euler equation) when $c''(w_2) < 0$:

  $$\uparrow w_2 \implies \uparrow c_2(w_2) \implies \downarrow u'(c_2(w_2)) \implies \downarrow V'(w_2)$$
  $$\uparrow w_2 \implies \downarrow c_2'(w_2) \text{ (if } c_2''(w_2) < 0) \implies \uparrow V'(w_2)$$

  If the second effect is stronger, V may be non-concave.

- This can result in non-monotonicity and discontinuity of $c_1(w_1)$.
  - Morris and Postlewaite (1997) provide a simple example where this happens, but it involves piecewise linear functions.
  - Harris and Laibson (2003) provide another example with CRRA utility, where income streams and borrowing constraints lead to discontinuous consumption.

Illustration: Impact of Non-Concave V



Pathologies: Violations of Monotonicity and Continuity

Why Does This Matter?

1. Breakdown of Euler equations:
   - Consider a consumption-savings problem with $T \geq 4$ periods.
   - Behavior in period $T-2$ is the same as in period 2 in our simple three-period model.
   - A discontinuous downward jump in $c_{T-2}(w_{T-2})$ results in a discontinuity in the period $T-3$ continuation value $V_{T-3}(w_{T-2})$.
   - In this case, solutions no longer characterized by simple Euler equations.
2. More importantly, Peleg and Yaari (1973) and Gul and Pesendorfer (2005) show that for more general consumption problems involving commitment devices, such discontinuities can lead to non-existence of forward-looking solutions—that is, we cannot formulate the model recursively using continuation values.

Pathologies: Violations of Monotonicity and Continuity

Solutions that smooth consumption:

1. Building on results from Carroll and Kimball (1996) about the shape of the consumption function, Morris (2002) provides sufficient conditions (e.g., CRRA utility) on utility functions to obtain monotonicity and continuity in a three-period problem.
   - Limitation: the aforementioned Harris and Laibson (2003) example shows that income streams and borrowing constraints can lead to discontinuities even for CRRA utility.
2. Harris and Laibson (2001) have related positive results for an infinite-horizon model with income uncertainty and borrowing constraints: By placing bounds on risk aversion and income uncertainty and taking β close to 1, they obtain continuous consumption functions.

Pathologies: Violations of Monotonicity and Continuity

Solutions that do NOT smooth consumption:

3. Equilibrium approach: Peleg and Yaari (1973) suggested treating dynamic decision problems as games and finding a SPNE.
    - Limitation: the equilibrium may not be "forward looking" (not Markov perfect) $\implies$ no recursive analysis.

4. Smoothing continuation values: Gul and Pesendorfer (2001, 2004) allow for discontinuous consumption and instead smooth the continuation value function by adding costly self-control.
    - Harris and Laibson (2013) provide a different approach to smoothing the value function by introducing the stochastic arrival of new "selves".

Outline

Infinite-Horizon Model with Uncertainty

- Based on Harris and Laibson (2001).
  - We derive the Euler equation assuming existence and differentiability of stationary continuation value $V(w)$ and solution $c(w)$.
  - Verifying these assumptions is however the main contribution of their paper.
- Setting:
  - Wealth: w_t
  - Consumption: c_t
  - Labor income (stochastic, iid): y_t
  - Budget constraint: $w_{t+1} = R(w_t - c_t) + y_{t+1}$

Recursive Quasi-Hyperbolic Discounting

- Suppose consumption c_t is given by the stationary consumption function $c(w_t)$.
- Thus $c(w_t)$ maximizes

$$u(c) + \beta\delta\mathbb{E}_t V\big(R(w_t - c) + y_{t+1}\big)$$

- Here $V(w_{t+1})$ is t's value for wealth in the subsequent period $t+1$:

$$V(w_{t+1}) = u(c(w_{t+1})) + \delta\mathbb{E}_{t+1} V\big(R(w_{t+1} - c(w_{t+1})) + y_{t+2}\big)$$

  - Notice that the period t self applies the present bias factor β only in the first equation, not in the recursive expression for $V(w_{t+1})$.
  - Hence $V(w_{t+1})$ is not the value that the period $t+1$ self would assign to wealth w_{t+1} (Harris and Laibson (2001) use $W(w_{t+1})$ to denote that value).

Deriving the Euler Equation

$$c(w_t) \in \operatorname{argmax}\left[u(c) + \beta\delta\mathbb{E}_t V\big(R(w_t - c) + y_{t+1}\big)\right]$$

$$V(w_{t+1}) = u(c(w_{t+1})) + \delta\mathbb{E}_{t+1} V\big(R(w_{t+1} - c(w_{t+1})) + y_{t+2}\big)$$

- FOC:

$$u'(c(w_t)) = \beta\delta R\mathbb{E}_t V'\big(R(w_t - c(w_t)) + y_{t+1}\big) = \beta\delta R\mathbb{E}_t V'\big(w_{t+1}\big)$$

- Also note that:

$$\begin{aligned} V'(w_t) &= u'(c(w_t))c'(w_t) + \delta R\mathbb{E}_t V'\big(R(w_t - c(w_t)) + y_{t+1}\big)\big(1 - c'(w_t)\big) \\ &= u'(c(w_t))c'(w_t) + \frac{1}{\beta}u'(c(w_t))\big(1 - c'(w_t)\big) \qquad \text{(by the FOC)} \end{aligned}$$

- Combining these two equations:

$$\begin{aligned} u'(c(w_t)) &= \beta\delta R\mathbb{E}_t\left[u'(c(w_{t+1}))c'(w_{t+1}) + \frac{1}{\beta}u'(c(w_{t+1}))\big(1 - c'(w_{t+1})\big)\right] \\ &= \mathbb{E}_t \underbrace{\left[c'(w_{t+1})\beta\delta + (1 - c'(w_{t+1}))\delta\right]}_{\text{effective discount factor}} Ru'(c(w_{t+1})) \end{aligned}$$

Deriving the Euler Equation

$$u'(c(w_t)) = \mathbb{E}_t\left[c'(w_{t+1})\beta\delta + (1 - c'(w_{t+1}))\delta\right] Ru'(c(w_{t+1}))$$

- Referred to as the Strong Hyperbolic Euler Relation.
- Harris and Laibson (2001) assume a borrowing constraint, which gives $\geq$ (with equality for $c_t < w_t$).
- As previously noted, they obtain existence of the value function and continuous consumption function $c(w)$ by placing bounds on risk aversion and income uncertainty and taking β close to 1.

Outline

Environment and Preferences

Based on Laibson (1997).

- Households allocate between liquid and illiquid assets:
  - Motivation: 2/3 of household sector domestic assets are illiquid, e.g., retirement and pension plans, Social Security.
  - Illiquid assets can serve as a commitment device to constraint future consumption.
- Consumers have quasi-hyperbolic discounting (β-δ preferences):
  - Laibson (1997) discusses both hyperbolic and quasi-hyperbolic discounting in detail.
  - A key benefit of preferences with quasi-hyperbolic discounting is that they permit present-bias (like true hyperbolic discounting) but for commitments beyond the present they are time consistent (unlike hyperbolic). This property simplifies the analysis considerably.

Finite-Horizon Model with Liquid and Illiquid Assets

- T periods: $t = 1, \ldots, T$.
- Income: y_t (labor supply decision not modeled)
- Gross return on assets: $R = 1 + r$ (assume constant for simplicity)
- Assets chosen at $t-1$:
  - Liquid: x_{t-1}
  - Illiquid: z_{t-1}
- Initial endowments: $x_0, z_0 \geq 0$.
- Consumption constraint (no access to illiquid assets when choosing consumption):

$$c_t \leq y_t + Rx_{t-1}$$

- Asset allocation constraints:

$$c_t + x_t + z_t = y_t + R(z_{t-1} + x_{t-1})$$

$$x_t, z_t \geq 0$$

More on the Constraints

$$c_t \leq y_t + Rx_{t-1}$$

- Assumes borrowing against illiquid assets takes time.
- And no access non-collateralized short-term borrowing (e.g., credit cards)

$$c_t + x_t + z_t = y_t + R(z_{t-1} + x_{t-1})$$
$$x_t, z_t \geq 0$$

- The assumption that $x_t \geq 0$ is critical—without it future consumption could be perfectly controlled through forced savings contracts. Laibson argues that such contracts cannot be legally enforced.
- But restriction is only partially realistic: For example, a mortgage could be modeled as taking $x_t < 0$ (required payment next period) and $z_t > 0$ (equity in house). Laibson argues that mortgages do not have the necessary flexibility to control for consumption relative to variations in income and suggests the main lessons of this model will be robust.

Preferences and Solution Concept

- Quasi-hyperbolic discounting:

$$U_t = u(c_t) + \beta \sum_{\tau=t+1}^{T} \delta^{\tau-t} u(c_\tau).$$

- Marginal rates of substitution:

$$MRS^t_{t,t+1} = \beta\delta \frac{u'(c_{t+1})}{u'(c_t)} \qquad MRS^t_{t+1,t+2} = \delta \frac{u'(c_{t+2})}{u'(c_{t+1})}$$

- Equilibrium: T periods $\rightarrow$ T players. Use SPNE (backward induction).
    - Note that finite T matters. There are many SPNE in infinite-horizon game, just like with standard repeated games. Many papers with infinite-horizon models therefore use Markov-perfect equilibria.
    - In contrast, Bernheim, Ray, and Yeltekin (2015) allow for history dependence (non-Markov-perfect equilibria) in an interesting application of the idea of "personal rules" and self-reward and self-denial, and show that it can lead to poverty traps.

Preferences and Solution Concept

Assumption A1

$$u'(y_t) \geq \beta(\delta R)^{\tau - t} u'(y_\tau), \qquad \forall \tau > t.$$

- For example, for $\delta R = 1$ and $u(\cdot) = \log(\cdot)$ this implies $y_\tau \geq \beta y_t$.
- We will see that this implies $c_t \geq y_t \; \forall t$ in equilibrium.
- This is restrictive, but without this assumption we have the possibility of discontinuous payoffs which implies that marginal conditions cannot be used to characterize the equilibrium. This is a special case of the issues highlighted previously.
    - Intuition: Consider the marginal propensity to save of self $T-1$. If this is increasing at some wealth levels *and* self $T-1$ is saving part of y_{T-1}, then self $T-2$ may have a nonconvex utility from saving. This in turn implies a potential discontinuity of the savings of $T-2$ in wealth, which implies the utility from saving for self $T-3$ may be discontinuous.

Equilibrium Conditions

Theorem

Any consumption game satisfying A1 has a unique SPNE, and it is characterized by the following conditions for $1 \leq t \leq \tau \leq T$*:*

P1 $u'(c_t) \geq \max_{\tau > t} \beta(\delta R)^{\tau - t} u'(c_\tau)$

P2 $u'(c_t) > \max_{\tau > t} \beta(\delta R)^{\tau - t} u'(c_\tau) \implies c_t = y_t + Rx_{t-1}$

P3 $u'(c_{t+1}) < \max_{\tau > t+1} (\delta R)^{\tau - t - 1} u'(c_\tau) \implies x_t = 0$

P4 $u'(c_{t+1}) > \max_{\tau > t+1} (\delta R)^{\tau - t - 1} u'(c_\tau) \implies z_t = 0$

Resource exhausting: $z_T = x_T = 0$.

Lemma

A1+P1+P2 $\implies c_t \geq y_t \ \forall t$.

- Infinite-horizon approach: Apply the model simply by using conditions P1–P4 in the infinite horizon setting.

Illustration

P1 $u'(c_t) \geq \max_{\tau > t} \beta(\delta R)^{\tau - t} u'(c_\tau)$

P2 $u'(c_t) > \max_{\tau > t} \beta(\delta R)^{\tau - t} u'(c_\tau) \implies c_t = y_t + Rx_{t-1}$

P3 $u'(c_{t+1}) < \max_{\tau > t+1} (\delta R)^{\tau - t - 1} u'(c_\tau) \implies x_t = 0$

P4 $u'(c_{t+1}) > \max_{\tau > t+1} (\delta R)^{\tau - t - 1} u'(c_\tau) \implies z_t = 0$

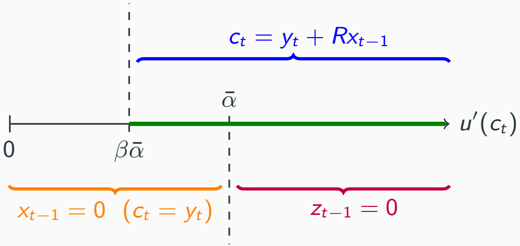


Illustration: $\bar{\alpha} \equiv \max_{\tau > t} (\delta R)^{\tau - t} u'(c_\tau)$

$(\uparrow u'(c_t) \implies \downarrow c_t)$

Infinite-Horizon Examples

Example (Constant Income)

Suppose $\delta R = 1$ and $y_t = \bar{y}$ for all t, and suppose initial wealth w_0 is divided between the assets according to

$$x_0 = \left(1 - \frac{1}{R}\right) w_0 \quad \text{and} \quad z_0 = \frac{1}{R} w_0.$$

Then, the solution is

$$\begin{aligned} c_t &= \bar{y} + R x_{t-1} = \bar{y} + (R-1) w_0 \\ x_t &= \left(1 - \frac{1}{R}\right) w_0 \\ z_t &= \frac{1}{R} w_0. \end{aligned}$$

Note that if wealth w_0 is not divided between the assets x_0 and z_0 as described above, then consumption could be higher or lower in the first period, but the subsequent pattern will follow the pattern above but with $w_1 = R z_0$ in the place of w_0 (assuming P1 is satisfied for $c_1 = \bar{y} + R x_0$, i.e., x_0 is not so large that the period 1 self voluntarily saves part of $R x_0$).

Example (Time-Varying Income)

Suppose $\delta R = 1$ and

$$y_t = \begin{cases} \bar{y} & \text{if } t \text{ odd} \\ \underline{y} & \text{if } t \text{ even,} \end{cases}$$

where $\underline{y} < \bar{y}$. Suppose also that $x_0 = 0$ and $w_0 = z_0 > 0$ satisfies

$$\underline{y} + z_0(R^2 - 1) \leq \bar{y}.$$

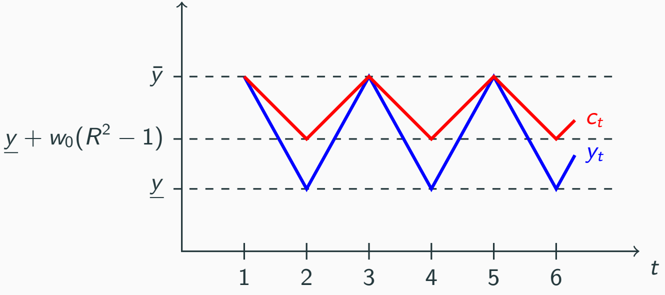
Then, the solution is

$$c_t = \begin{cases} \bar{y} & \text{if } t \text{ odd} \\ \underline{y} + w_0(R^2 - 1) & \text{if } t \text{ even} \end{cases}$$

$$x_t = \begin{cases} \frac{w_0}{R}(R^2 - 1) & \text{if } t \text{ odd} \\ 0 & \text{if } t \text{ even} \end{cases}$$

$$z_t = \begin{cases} \frac{w_0}{R} & \text{if } t \text{ odd} \\ w_0 & \text{if } t \text{ even} \end{cases}$$

Illustration



Asset Holdings

	0	1	2	3	4
z_t	w_0	$\frac{w_0}{R}$	w_0	$\frac{w_0}{R}$	...
x_t	0	$\frac{w_0}{R}(R^2 - 1)$	0	$\frac{w_0}{R}(R^2 - 1)$	...
Rz_{t-1}		Rw_0	w_0	Rw_0	...

Question: What happens if $\underline{y} + z_0(R^2 - 1) > \bar{y}$?

Discussion

- Asset specific MPC (Thaler (1990))
- Ricardian Equivalence fails (even when agents hold substantial wealth).
- Credit markets effect on growth and welfare.

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