Puzzles and Paradoxes for Expected Utility

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Outline

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Allais Common Consequence Effect

PROBLEM 1: Choose between

$$A : \begin{cases} 2,500 & \text{with probability } .33 \\ 2,400 & \text{with probability } .66 \\ 0 & \text{with probability } .01 \end{cases} \qquad B : \begin{cases} 2,400 & \text{with probability } 1 \end{cases}$$

PROBLEM 2: Choose between

$$C : \begin{cases} 2,500 & \text{with probability } .33 \\ 0 & \text{with probability } .67 \end{cases} \qquad D : \begin{cases} 2,400 & \text{with probability } .34 \\ 0 & \text{with probability } .66 \end{cases}$$

- Kahneman and Tversky (1979) refer to this pattern as the *certainty effect*, but it also occurs away from certainty.

Why is this pattern a paradox?

$$A : \begin{cases} 2,500 & \text{with probability } .33 \\ 2,400 & \text{with probability } .66 \\ 0 & \text{with probability } .01 \end{cases} \qquad B : \begin{cases} 2,400 & \text{with probability } 1 \end{cases}$$

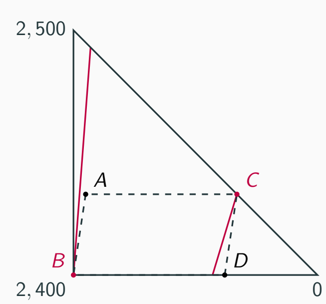
$$C : \begin{cases} 2,500 & \text{with probability } .33 \\ 0 & \text{with probability } .67 \end{cases} \qquad D : \begin{cases} 2,400 & \text{with probability } .34 \\ 0 & \text{with probability } .66 \end{cases}$$

Expected utility requires that $A \succsim B \iff C \succsim D$.

$$U(A) = u(2,500)(0.33) + u(2,400)(0.66) + u(0)(0.01)$$
$$\downarrow$$
$$U(C) = u(2,500)(0.33) + u(0)(0.66) + u(0)(0.01)$$

$$U(B) = u(2,400)(0.33) + u(2,400)(0.66) + u(2,400)(0.01)$$
$$\downarrow$$
$$U(D) = u(2,400)(0.33) + u(0)(0.66) + u(2,400)(0.01)$$

Graphical Illustration: Common Consequence Effect



$B \succ A$ and $C \succ D$ requires that indifference curve through B is steeper than indifference curve through C.

steeper indifference curves $\Longrightarrow$ greater risk aversion.

(Indifference curves need not be linear, but we will draw them that way for simplicity.)

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Paradoxes Related to Linearity of Risk Preferences

- Allais Common Consequence Effect
- Allais Common Ratio Effect
- Isolation Effect
- Probabilistic Insurance
- Generalizations of CCE and CRE

Attitudes toward Small and Large Risks

- Second-Order Risk Aversion
- Rabin Paradox
- Equity Premium Puzzle
- Summary

Allais Common Ratio Effect

Problem 1: Choose between

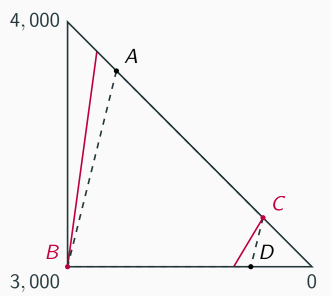
$$A: \begin{cases} 4,000 & \text{with probability } .8 \\ 0 & \text{with probability } .2 \end{cases} \qquad B: \begin{cases} 3,000 & \text{with probability } 1 \end{cases}$$

Problem 2: Choose between

$$C: \begin{cases} 4,000 & \text{with probability } .2 \\ 0 & \text{with probability } .8 \end{cases} \qquad D: \begin{cases} 3,000 & \text{with probability } .25 \\ 0 & \text{with probability } .75 \end{cases}$$

Since C is the $(1/4, 3/4)$ mixture of A with 0, and likewise for B and D, the independence axiom requires $A \succsim B \iff C \succsim D$.

Graphical Illustration: Common Ratio Effect



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Isolation Effect

Problem 10 in KT: Consider the following two-stage game. In the first stage, there is a probability of .75 to end the game without winning anything, and a probability of .25 to move into the second stage. If you reach the second stage you have a choice between

$$(4,000, .80) \qquad \text{and} \qquad (3,000).$$

Your choice must be made before the game starts, i.e., before the outcome of the first stage is known.

Recall the second Common Ratio Effect problem:

Problem 4 in KT: Choose between

$$C: \begin{cases} 4,000 & \text{with probability } .2 \\ 0 & \text{with probability } .8 \end{cases} \qquad D: \begin{cases} 3,000 & \text{with probability } .25 \\ 0 & \text{with probability } .75 \end{cases}$$

Decision Trees for the Isolation Effect

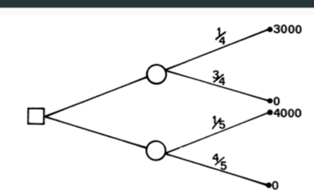


Figure 1.—The representation of Problem 4 as a decision tree (standard formulation).

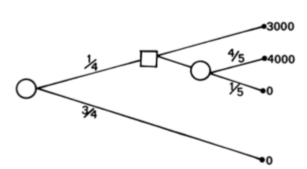


Figure 2.—The representation of Problem 10 as a decision tree (sequential formulation).

(Fig 2 implies first gamble realized before decision, contrary to phrasing of Prob 10.)

Outline

Probabilistic Insurance

PROBLEM 9 IN KT: Suppose you consider the possibility of insuring some property against damage, e.g., fire or theft. After examining the risks and the premium you find that you have no clear preference between the options of purchasing insurance or leaving the property uninsured.

It is then called to your attention that the insurance company offers a new program called *probabilistic insurance*. In this program you pay half of the regular premium. In case of damage, there is a 50 percent chance that you pay the other half of the premium and the insurance company covers all the losses; and there is a 50 percent chance that you get back your insurance payment and suffer all the losses.

Recall that the premium for full coverage is such that you find this insurance barely worth its cost.

Under these circumstances, would you purchase probabilistic insurance?

Another Description of Probabilistic Insurance

(Notation: w wealth, L potential loss, π prob of loss, y insurance premium.)

Indifferent between accepting or rejecting initial policy:

$$\begin{pmatrix} w-L & \pi \\ w & 1-\pi \end{pmatrix} \sim \begin{pmatrix} w-y \end{pmatrix}$$

Probabilistic insurance generates the following lottery:

$$\begin{pmatrix} w-L & \pi/2 \\ w-y & \pi/2 \\ w-y/2 & 1-\pi \end{pmatrix}$$

Would you choose the probabilistic insurance policy?

Majority (80%) of subjects answered NO.

Risk-Averse EU Agents Prefer Probabilistic Insurance

(Notation: w wealth, L potential loss, π prob of loss, y insurance premium.)

Indifferent between accepting or rejecting initial policy:

$$\begin{pmatrix} w-L & \pi \\ w & 1-\pi \end{pmatrix} \sim \begin{pmatrix} w-y \end{pmatrix}$$

Mixing the two, independence implies

$$\begin{pmatrix} w-L & \pi/2 \\ w-y & \pi/2 \\ w & (1-\pi)/2 \\ w-y & (1-\pi)/2 \end{pmatrix} \sim \begin{pmatrix} w-y \end{pmatrix}$$

Risk aversion (SOSD monotonicity) then implies

$$\begin{pmatrix} w-L & \pi/2 \\ w-y & \pi/2 \\ w-y/2 & 1-\pi \end{pmatrix} \succsim \begin{pmatrix} w-y \end{pmatrix}$$

Probabilistic Insurance versus Risk of Nonpayment

Risk of nonpayment: α chance of nonpayment with premium y':

$$\begin{pmatrix} w-L-y' & \alpha\pi \\ w-y' & (1-\alpha)\pi \\ w-y' & 1-\pi \end{pmatrix}$$

- Evidence: People tend to require a (much) more than proportional decrease in premium to tolerate risk of nonpayment: $y' < (1-\alpha)y$.
- Probabilistic insurance is similar to risk of nonpayment with $\alpha = 1/2$ and $y' = y/2$ (but the two are not quite the same!):

$$\begin{pmatrix} w-L & \pi/2 \\ w-y & \pi/2 \\ w-y/2 & 1-\pi \end{pmatrix} \quad \text{versus} \quad \begin{pmatrix} w-L-y/2 & \pi/2 \\ w-y/2 & \pi/2 \\ w-y/2 & 1-\pi \end{pmatrix}$$

- A disproportional change in willingness to pay in response to nonpayment risk is difficult to match with EU, but does not yield a direct contradiction of EU like probabilistic insurance.

Outline

Machina's generalization of the common consequence effect

Machina (1987) describes a generalization of the CCE: Let δ_x be the Dirac measure that gives x with certainty, P involve outcomes both greater and less than x, and $Q \geq_{FOSD} R$.

Problem 1: Choose between

$$A:\ \alpha P + (1-\alpha)Q \qquad B:\ \alpha\delta_x + (1-\alpha)Q$$

Problem 2: Choose between

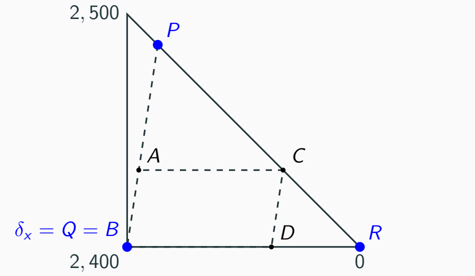
$$C:\ \alpha P + (1-\alpha)R \qquad D:\ \alpha\delta_x + (1-\alpha)R$$

- Independence requires $P \succ \delta_x \iff A \succ B \iff C \succ D$
- Machina (1987) suggests that violations of independence tend to be of the form $B \succ A$ and $C \succ D$.
- However, we will see that the evidence is actually mixed.

Illustration: Kahneman and Tversky (1979) example

$$A:\ \alpha P + (1-\alpha)Q \qquad B:\ \alpha\delta_x + (1-\alpha)Q$$
$$C:\ \alpha P + (1-\alpha)R \qquad D:\ \alpha\delta_x + (1-\alpha)R$$

Kahneman and Tversky (1979) example was $x = 2,400$, $Q = \delta_x$, $R = \delta_0$, $\alpha = .34$, $P = (33/34)\delta_{2,500} + (1/34)\delta_0$.



Machina's Hypothesis II

$$A:\ \alpha P + (1-\alpha)Q \qquad B:\ \alpha\delta_x + (1-\alpha)Q$$
$$C:\ \alpha P + (1-\alpha)R \qquad D:\ \alpha\delta_x + (1-\alpha)R$$

- Independence requires $A \succsim B \iff C \succsim D$
- Suppose we want to relax independence and describe a weaker restriction that permits reversals of the form $B \succ A$ and $C \succ D$, but not reversals of the form $A \succ B$ and $D \succ C$, what would such a condition look like?

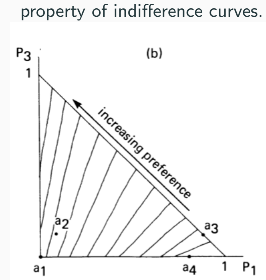
Machina (1982)'s Hypothesis II

If $Q \geq_{FOSD} R$, then for any P and x,

$$\alpha P + (1-\alpha)Q \succsim \alpha\delta_x + (1-\alpha)Q \implies \alpha P + (1-\alpha)R \succsim \alpha\delta_x + (1-\alpha)R$$

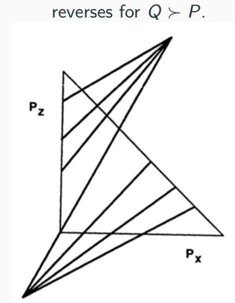
Hypothesis II and "fanning out" of indifference curves

Hypothesis II implies a "fanning out" property of indifference curves.



(Machina (1982))

Others have argued preference exhibit a "mixed fan": Hypothesis II holds for $P \succ Q$, but the pattern reverses for $Q \succ P$.



(Gul (1991))

"Fanning out" versus "mixed fan" versus something else?

Allais paradoxes are all about the lower-right section of the simplex. There is less evidence available to judge "fanning out" versus "mixed fan", which is about the upper-left section.

An example of the kind of lottery needed to test (E and F are new):

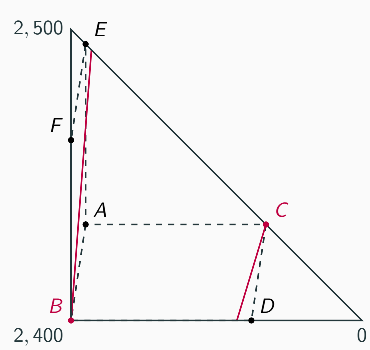
$$A: \begin{cases} 2,500 & \text{with probability } .33 \\ 2,400 & \text{with probability } .66 \\ 0 & \text{with probability } .01 \end{cases} \qquad B: \begin{cases} 2,400 & \text{with probability } 1 \end{cases}$$

$$C: \begin{cases} 2,500 & \text{with probability } .33 \\ 0 & \text{with probability } .67 \end{cases} \qquad D: \begin{cases} 2,400 & \text{with probability } .34 \\ 0 & \text{with probability } .66 \end{cases}$$

$$E: \begin{cases} 2,500 & \text{with probability } .99 \\ 0 & \text{with probability } .01 \end{cases} \qquad F: \begin{cases} 2,500 & \text{with probability } .66 \\ 2,400 & \text{with probability } .34 \end{cases}$$

- The pattern BCF is consistent with fanning out and mixed fan.
- The pattern BCE is consistent with mixed fan but not fanning out.

Illustration



(only $F \succ E$ is consistent with fanning out; either ranking consistent with mixed fan)

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Second-Order Risk Aversion

An expected-utility maximizer (with differentiable u) is willing to bear at least a small amount of actuarially favorable risk. This is difficult to reconcile with:

- Avoiding some actuarially favorable investments.
- Purchasing full insurance coverage at actuarially unfair premiums.

These (inaccurate) predictions of the expected utility are closely related to one of its properties, known as second-order risk aversion: The risk premium of a zero-expectation gamble is proportional to its variance σ^2.

The patterns described above can be explained by (non-expected-utility) models that generate first-order risk aversion: The risk premium of a zero-expectation gamble is proportional to its standard deviation σ.

Arrow-Pratt Approximation

To illustrate second-order risk aversion and how it relates to expected utility, we will develop the Arrow-Pratt approximation.

- initial wealth w
- risk $\tilde{x}$ with mean μ and variance σ^2
- investment level (scale of risk) t

Let $f(t)$ denote the certainty equivalent of the random wealth $w + t\tilde{x}$ from investment level t:

$$u(f(t)) = \mathbb{E}[u(w + t\tilde{x})]$$

We can approximate $f(t)$ by taking a second-order Taylor expansion of f around $t = 0$. This is called the Arrow-Pratt approximation of the certainty equivalent.

Arrow-Pratt Approximation

First, note that

$$\begin{aligned} &u(f(t)) = \mathbb{E}[u(w + t\tilde{x})] \\ &\implies f(0) = w \end{aligned}$$

Next, differentiating with respect to t gives:

$$\begin{aligned} &u'(f(t))f'(t) = \mathbb{E}[u'(w + t\tilde{x})\tilde{x}] \\ &\implies u'(w)f'(0) = u'(w)\mathbb{E}[\tilde{x}] \qquad (t = 0) \\ &\implies f'(0) = \mu \end{aligned}$$

Differentiating with respect to t again gives:

$$\begin{aligned} &u''(f(t))(f'(t))^2 + u'(f(t))f''(t) = \mathbb{E}[u''(w + t\tilde{x})\tilde{x}^2] \\ &\implies u''(w)\mu^2 + u'(w)f''(0) = u''(w)\mathbb{E}[\tilde{x}^2] \qquad (t = 0) \\ &\implies f''(0) = \frac{u''(w)}{u'(w)}\left(\mathbb{E}[\tilde{x}^2] - \mu^2\right) = -A(w)\sigma^2 \end{aligned}$$

where $A(w) = -\frac{u''(w)}{u'(w)}$ is the Arrow-Pratt coefficient of risk aversion.

Arrow-Pratt Approximation and Second-Order Risk Aversion

Arrow-Pratt approximation: The second-order Taylor expansion of $f(t)$ around $t = 0$ is therefore given by

$$\begin{aligned} f(t) &\approx f(0) + tf'(0) + \frac{1}{2}t^2 f''(0) \\ &= w + t\mu - \frac{1}{2}t^2\sigma^2 A(w) \end{aligned} \tag{1}$$

- $t\mu$ is the expected value of $t\tilde{x}$
- $t\sigma$ is the standard deviation of $t\tilde{x}$

Notice that the mean has a first-order effect on the certainty equivalent, but the standard deviation only has a second-order effect.

Second-order risk aversion: (1) implies that if $\mu > 0$, some positive level $t > 0$ of investment will be optimal, no matter how large the variance σ of $\tilde{x}$ is and no matter now large the coefficient of risk aversion $A(w)$ is.

Outline

Rabin (2000) Paradox

TABLE I

IF AVERSE TO 50-50 LOSE \$100 / GAIN g BETS FOR ALL WEALTH LEVELS, WILL TURN DOWN 50-50 LOSE L / GAIN G BETS; G'S ENTERED IN TABLE.

		g		
L	\$101	\$105	\$110	\$125
\$400	400	420	550	1,250
\$600	600	730	990	∞
\$800	800	1,050	2,090	∞
\$1,000	1,010	1,570	∞	∞
\$2,000	2,320	∞	∞	∞
\$4,000	5,750	∞	∞	∞
\$6,000	11,810	∞	∞	∞
\$8,000	34,940	∞	∞	∞
\$10,000	∞	∞	∞	∞
\$20,000	∞	∞	∞	∞

- People tend to reject the small gambles (overwhelmingly so at the levels of $g = 101$ and 105)
- Rejecting the large-scale gambles is completely unrealistic.

Key to the Result: Reject at All Wealth Levels

- Suppose we know a risk averse individual would reject -\$100/+\$110 gamble at a $w = \$290,000$ (and we are told nothing else).
- It could be that that her utility function is very concave around this current wealth level (perhaps even kinked), but approximately linear elsewhere.
  - Reject any gamble proportional to -\$100/+\$110
  - Might accept gamble with proportionally larger gain, e.g., -\$4,000/+\$5,000
- The key to Rabin's conclusion is the stronger assumption that the individual would reject the -\$100/+\$110 bet at *every wealth level*.
- This implies that not only is her marginal utility dropping rapidly between \$289,900 and \$290,110, but also between $w - \$100$ and $w + \$110$ for any w. Cumulative effect is very large!

Theorem (A Rabin(-ish) Calibration Result)

Suppose an individual has expected-utility preferences with a Bernoulli utility function u that is concave. Suppose the individual would reject a 50-50 gamble with loss l and gain g at every wealth level. Then:

1. *If $(g/l)^m < 2$, then she will reject any 50-50 gamble with loss $L = mg$ and gain $G = ng$ provided*
$$n < f(l, g, m) \equiv \frac{\log\left(2 - \left(\frac{g}{l}\right)^m\right)}{\log\left(\frac{l}{g}\right)}.$$
2. *If instead $(g/l)^m > 2$, then the individual will reject the gamble with loss $L = mg$ regardless of the possible gain.*

Example ($l = 100$ and $g = 105$)

1. $f(100, 105, 9) = 16.43$, so will reject gamble with loss $L = (9)(105) = 945$ and gain $G = (16)(105) = 1,680$.
2. $(105/100)^{15} = 2.08 > 2$, so will reject any gamble with loss $L > (15)(105) = 1,575$.

Proof i

First, note that rejection of the small gambles implies

$$u(w) \geq \frac{1}{2}u(w - l) + \frac{1}{2}u(w + g)$$
$$\iff u(w) - u(w - l) \geq u(w + g) - u(w)$$

Second, note that by concavity,

$$u(w) - u(w - g) \geq \left(\frac{g}{l}\right)[u(w) - u(w - l)]$$

Putting these together we get our key equation for calibrating the rate of decline of marginal utility:

$$u(w) - u(w - g) \geq \left(\frac{g}{l}\right)[u(w + g) - u(w)] \qquad (2)$$

Proof ii

Since (2) is true at every wealth level,

$$u(w) - u(w - g) \geq \left(\frac{g}{l}\right)[u(w + g) - u(w)]$$
$$u(w - g) - u(w - 2g) \geq \left(\frac{g}{l}\right)^2[u(w + g) - u(w)]$$
$$\vdots \qquad \vdots$$
$$u(w - (m-1)g) - u(w - mg) \geq \left(\frac{g}{l}\right)^m[u(w + g) - u(w)]$$

Thus, for $L = mg$,

$$u(w) - u(w - L) = \sum_{i=1}^{m}[u(w - (i-1)g) - u(w - ig)]$$
$$\geq \sum_{i=1}^{m}\left(\frac{g}{l}\right)^i[u(w + g) - u(w)]$$

Proof iii

Using (2) and working in the other direction,

$$
\begin{aligned}
u(w+2g) - u(w+g) &\leq \left(\frac{l}{g}\right)[u(w+g) - u(w)] \\
u(w+3g) - u(w+2g) &\leq \left(\frac{l}{g}\right)^2[u(w+g) - u(w)] \\
&\vdots \qquad\qquad \vdots \\
u(w+ng) - u(w+(n-1)g) &\leq \left(\frac{l}{g}\right)^{n-1}[u(w+g) - u(w)]
\end{aligned}
$$

Thus, for $G = ng$,

$$
\begin{aligned}
u(w+G) - u(w) &= \sum_{i=1}^{n} [u(w+ig) - u(w+(i-1)g)] \\
&\leq \sum_{i=1}^{n} \left(\frac{l}{g}\right)^{i-1}[u(w+g) - u(w)]
\end{aligned}
$$

Proof iv

$$
\begin{aligned}
u(w) - u(w-L) &\geq \sum_{i=1}^{m} \left(\frac{g}{l}\right)^{i}[u(w+g) - u(w)] \\
u(w+G) - u(w) &\leq \sum_{i=1}^{n} \left(\frac{l}{g}\right)^{i-1}[u(w+g) - u(w)]
\end{aligned}
$$

Therefore, reject 50-50 lose $L = mg$ or gain $G = ng$ gamble whenever

$$
\begin{aligned}
&\sum_{i=1}^{n} \left(\frac{l}{g}\right)^{i-1} < \sum_{i=1}^{m} \left(\frac{g}{l}\right)^{i} \\
\iff &\frac{1 - \left(\frac{l}{g}\right)^n}{1 - \left(\frac{l}{g}\right)} < \frac{\left(\frac{g}{l}\right)^{m+1} - \left(\frac{g}{l}\right)}{\left(\frac{g}{l}\right) - 1} = \frac{\left(\frac{g}{l}\right)^{m} - 1}{1 - \left(\frac{l}{g}\right)} \\
\iff &\left(\frac{l}{g}\right)^n > 2 - \left(\frac{g}{l}\right)^m
\end{aligned}
$$

Taking logs and rearranging gives the formula in the theorem:

$$
n < \frac{\log\left(2 - \left(\frac{g}{l}\right)^m\right)}{\log\left(\frac{l}{g}\right)}.
$$

Rabin Paradox: Accounting for DARA

TABLE II

TABLE I REPLICATED, FOR INITIAL WEALTH LEVEL \$290,000, WHEN l/g BEHAVIOR IS ONLY KNOWN TO HOLD FOR $w \leq$ \$300,000.

L	g: \$101	\$105	\$110	\$125
\$400	400	420	550	1,250
\$600	600	730	990	36,000,000,000
\$800	800	1,050	2,090	90,000,000,000
\$1,000	1,010	1,570	718,190	160,000,000,000
\$2,000	2,320	69,930	12,210,880	850,000,000,000
\$4,000	5,750	635,670	60,528,930	9,400,000,000,000
\$6,000	11,510	1,557,360	180,000,000	89,000,000,000,000
\$8,000	19,290	3,058,540	510,000,000	830,000,000,000,000
\$10,000	27,780	5,503,790	1,300,000,000	7,700,000,000,000,000
\$20,000	85,750	71,799,110	160,000,000,000	540,000,000,000,000,000,000

- The small bets may become more attractive at extremely high wealth levels if decreasing absolute risk aversion. (Recall: CRRA$\Longrightarrow$DARA)
- The paradox is robust to DARA, as the table above shows.

Outline

Expected Utility and Asset Returns

Consider the following consumption-portfolio problem ($1 \neq \alpha > 0$):

$$
\max_{\substack{c_t,\, c_{t+1} \\ \xi^b,\, \xi^s}} \left(\frac{c_t^{1-\alpha}}{1-\alpha} + \beta \mathbb{E}_t\left[\frac{c_{t+1}^{1-\alpha}}{1-\alpha}\right] \right)
$$

subject to

$$
\begin{aligned}
c_t &= e_t - \xi^b - \xi^s \\
c_{t+1} &= e_{t+1} + \xi^b(1 + r_{t+1}^b) + \xi^s(1 + r_{t+1}^s)
\end{aligned}
$$

First-order conditions for interior solution:

$$
\begin{aligned}
1 &= \beta \mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}(1 + r_{t+1}^b)\right] \\
1 &= \beta \mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}(1 + r_{t+1}^s)\right]
\end{aligned}
$$

Puzzles (Mehra and Prescott (1985), Weil (1989), Kocherlakota (1996))

$$
\beta \mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}(1 + r_{t+1}^b)\right] = 1 \quad \text{and} \quad \mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}(r_{t+1}^s - r_{t+1}^b)\right] = 0
$$

- Equity Premium Puzzle: Any parameter values that satisfy $0 < \alpha \leq 10$, $0 < \beta < 1$, and $\mathbb{E}_t[r_{t+1}^b] \leq 4\%$ imply an equity premium of $\mathbb{E}_t[r_{t+1}^s - r_{t+1}^b] \leq 0.35\%$.
- What if larger values of α are allowed? For example, coefficient of relative risk aversion of $\alpha \approx 18$ gives an equity premium of 6%.
- Risk Free Rate Puzzle: Increasing α decreases the elasticity of intertemporal substitution, making r_{t+1}^b too large: To get a risk free rate of 1% given consumption growth of 1.8% requires $\beta > 1$.

Separating Risk Aversion and Intertemporal Substitution

Epstein and Zin (1989) and Weil (1989) used Kreps and Porteus (1978) preferences to separate intertemporal substitution from risk aversion (2-period version):

$$\frac{c_t^{1-\rho}}{1-\rho} + \beta \frac{\mathbb{E}_t\left[c_{t+1}^{1-\alpha}\right]^{\frac{1-\rho}{1-\alpha}}}{1-\rho}, \quad 1 \neq \rho \geq 0,\ 1 \neq \alpha \geq 0$$

Coefficient of relative risk aversion is α and elasticity of intertemporal substitution is $\frac{1}{\rho}$.

- Separation of these two parameters allows α to be large enough to explain the equity premium while keeping the risk free rate low.
- But still requires large coefficient of relative risk aversion ($\alpha \approx 18$). This is inconsistent with other evidence regarding willingness to large-scale idiosyncratic risks (think of occupational earnings risk, ownership of property or business).
- The equity premium remains a puzzle.

Outline

Risk Aversion in Different Domains

The Equity Premium Puzzle and Rabin's Paradox both have to do with relating risk aversion across different domains. Can summarize the issue using CRRA expected utility: $\mathbb{E}_t\left[\frac{c_{t+1}^{1-\alpha}}{1-\alpha}\right]$

Coefficients of Relate Risk Aversion
Needed in Various Domains

Domain	Scale of Risk	Implied CRRA
First-Order Risk Aversion	Infinitesimal	Infinite ($\alpha = \infty$)
Small Rabin Gambles	Small	Huge ($\alpha \geq 45$)
Stock Market Risk	Moderate	Large ($\alpha \approx 18$)
Large Rabin Gambles, Occupational Earnings Risk, Entrepreneurial Risk	Large	Small ($\alpha \leq 10$)

(CRRA utility is used for illustration, but the puzzles arise for any expected-utility preference.)

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