ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 76, No. 2 (March, 2008), 263–305

ANTICIPATING REGRET: WHY FEWER OPTIONS MAY BE BETTER

TODD SARVER
Northwestern University, Evanston, IL 60208, U.S.A.

Econometrica, Vol. 76, No. 2 (March, 2008), 263–305

ANTICIPATING REGRET: WHY FEWER OPTIONS MAY BE BETTER

BY TODD SARVER[1]

We study preferences over menus which can be represented as if the agent selects an alternative from a menu and experiences regret if her choice is ex post inferior. Since regret arises from comparisons between the alternative selected and the other available alternatives, our axioms reflect the agent's desire to limit her options. We prove that our representation is essentially unique. We also introduce two measures of comparative regret attitudes and relate them to our representation. Finally, we explore the formal connection between the present work and the literature on temptation.

KEYWORDS: Regret, preference for commitment, subjective state space.

> I see it all perfectly; there are two possible situations—one can either do this or that. My honest opinion and my friendly advice is this: do it or do not do it—you will regret both.
>
> Soren Kierkegaard

1. INTRODUCTION

1.1. *Brief Overview*

PEOPLE OFTEN FACE DECISIONS in which they are not certain of the better course of action. These decisions could be of great consequence, such as whether to marry, take a new job, or move to a new city, and they could also be as simple as what to order for dinner at a restaurant. Even if an agent makes the best decision given the information available at the time, she may still feel a sense of loss or regret if she comes to find that another alternative would have been better. Such an agent may prefer to have fewer options so as to reduce the chance that her choice will be "wrong" ex post.

Following Dekel, Lipman, and Rustichini (2001), we investigate preferences over menus of lotteries. We consider an agent who chooses a menu in period 0 and subsequently selects an alternative (lottery) from that menu in period 1. Our interpretation is that the agent makes both of these decisions prior to the resolution of some subjective uncertainty and then experiences regret if her chosen alternative is ex post inferior to another alternative on the menu. We show that by simply observing the agent's preference over menus, we can determine whether her choices can be modeled *as if* she anticipates regret. Our main result is a representation theorem for what we refer to as *regret preferences*.

[1]I wish to thank Haluk Ergin, Peter Klibanoff, Marco LiCalzi, Massimo Marinacci, Jawwad Noor, Marciano Siniscalchi, numerous seminar participants, a co-editor, and three anonymous referees for helpful comments. I am particularly grateful to Bart Lipman for detailed comments and for his encouragement when this project was in its formative stages.

Since regret arises in our model from ex post comparisons between the alternative selected and the other available alternatives, a regret preference will reflect the agent's desire to limit her options. Our main axiom identifies precisely when restricting options is beneficial to the agent. Formally, suppose there are two lotteries, p and q, such that the agent prefers the menu containing only p to the menu containing only q. That is, $\{p\} \succsim \{q\}$. We refer to the agent's preference over singleton menus as her *commitment preference* since $\{p\} \succsim \{q\}$ implies the agent would choose p over q if she could commit in period 0. The commitment preference reflects the agent's ex ante expectation of the value of each alternative. We assume that in period 1 the agent will continue to rank alternatives according to her commitment preference and hence will choose p over q when choosing from any menu containing both lotteries.[2] Therefore, if $\{p\} \succsim \{q\}$, then p "dominates" q in the sense that q cannot add value to any menu that already contains p. This property leads to our main axiom, which we refer to as *dominance*: If $\{p\} \succsim \{q\}$ and $p \in A$, then $A \succsim A \cup \{q\}$.

One can further understand the dominance axiom by contrasting our regret model with the standard model. In the standard model, the agent does not experience regret and therefore values a menu based on its best element. For such an agent, if $\{p\} \succsim \{q\}$ for some $p \in A$, then q is no better than the best element in A and, therefore, $A \sim A \cup \{q\}$. The dominance axiom relaxes this condition to allow for $A \succ A \cup \{q\}$. Intuitively, even if there is a $p \in A$ that is expected to be better than q ex ante (that is, $\{p\} \succsim \{q\}$), there may be some state in which q gives a higher ex post utility than p (or any other element of A). Therefore, for an agent who experiences regret, adding q to the menu A will strictly increase her regret in this state. Thus an agent who anticipates regret may strictly prefer A to $A \cup \{q\}$.

We identify a regret preference with the dominance axiom and three additional axioms. These axioms are variations of the standard expected-utility axioms: weak order (the preference is complete and transitive), continuity, and independence.

1.2. *Preview of Results*

We now describe the functional form identified by our representation theorem. The agent faces some subjective uncertainty that affects her future tastes. We model this uncertainty using a probability measure μ over a set of possible ex post utility functions $\mathcal{U}$. We refer to the utility functions $u \in \mathcal{U}$ as states and we impose the restriction that each $u \in \mathcal{U}$ be a von Neumann–Morgenstern expected-utility function. Note that both $\mathcal{U}$ and μ arise as part

[2] This assumption is important for isolating regret from other factors that may lead the agent to want to limit her alternatives, such as temptation. Intuitively, temptation may cause the agent to *knowingly* pick a suboptimal alternative from a menu. To distinguish from temptation, we focus on regret where the agent's choices from menus in period 1 are precisely the alternatives that she would want to commit to in period 0, that is, those that she expects will be optimal ex ante.

of the representation—they are not directly observable, but instead must be elicited from the agent's preferences. For any realized state $u \in \mathcal{U}$ and any lottery p, the agent's ex post utility is denoted by $u(p)$.

In our representation, ex post regret for a realization of the agent's tastes u is proportional to the difference between the maximum ex post utility attainable from the menu A under u and the actual utility attained from the agent's choice. That is, if the agent chooses p from the menu A, then her ex post regret in state u is

$$R(p, A, u) = K\Big[\max_{q\in A} u(q) - u(p)\Big],$$

where $K \geq 0$. The constant K can be thought of as representing the strength of regret. The agent's preference over menus is represented as if she chooses p from the menu A to maximize the ex ante expectation of utility minus regret[3]:

$$V(A) = \max_{p\in A} \int_{\mathcal{U}} [u(p) - R(p, A, u)]\mu(du).$$

Note that the agent experiences no regret when A is a singleton menu. That is, $R(p, A, u) = 0$ if $A = \{p\}$. This property of the representation may give the impression that the agent does not regret her choice of menu in our model. However, as we discuss in detail in Section 4.3, our model can be thought of as describing the *additional* regret (on top of the regret associated with menu choice) that the agent experiences because of her choice from a menu and the effect it has on her preference over menus.

We now give a numerical example to illustrate the role of regret in this representation.

EXAMPLE 1: Suppose an agent is going to make a reservation at one of several restaurants. Restaurant 1 serves a beef dish (b) and a chicken dish (c). Restaurant 2 serves only the beef dish, and restaurant 3 serves only the chicken dish. Therefore, the set of alternatives is $Z = \{b, c\}$, and the menus are $A_1 = \{b, c\}$, $A_2 = \{b\}$, and $A_3 = \{c\}$.

Suppose the agent has two possible ex post utility functions, so that $\mathcal{U} = \{u_1, u_2\}$. Also, suppose $\mu(\{u_1\}) = \mu(\{u_2\}) = 0.5$, $K \geq 0$, and the expected-utility functions u_1 and u_2 take values on Z given by

	u_1	u_2
b	4	2
c	1	4

[3]Since only the agent's preference over menus is observed, her choice from a menu is part of the interpretation of the model.

In words, the agent is uncertain whether she will like beef or chicken best, but she likes beef better on average. As illustrated above, the agent experiences no regret from a singleton menu. Therefore, $V(\{b\}) = 3$ and $V(\{c\}) = 2.5$. The ex post regret for the menu $A_1 = \{b, c\}$ and each selection from this menu is summarized by

	u_1	u_2
b	0	$2K$
c	$3K$	0

Choosing beef from the menu A_1 leads to an ex ante expectation of utility minus regret of $3 - K$, whereas choosing chicken yields $2.5 - 1.5K$. The agent will therefore choose beef from the menu A_1, and hence $V(\{b, c\}) = 3 - K$.[4]

If $\succsim$ is the preference induced by this representation, then we have $\{b\} \succsim \{c\}$ and, as required by the dominance axiom, $\{b\} \succsim \{b, c\}$. However, note that $V(\{b, c\})$ may be larger or smaller than $V(\{c\})$, depending on the value of K. Intuitively, adding beef to a menu of just chicken gives the agent a better alternative, but also introduces regret. Which of these effects is stronger depends on the parameters of the model. We return to this issue when we discuss comparative regret attitudes in Section 3.3.

The remainder of the paper is organized as follows. We discuss our model in detail in Section 2, presenting our representation in Section 2.1 and our axioms in Section 2.2. Our main results are contained in Section 3. We present our representation theorem and sketch its proof in Section 3.1, and we present our uniqueness results in Section 3.2. In Section 3.3, we introduce two measures of comparative regret attitudes and discuss implications for our representation. Section 4 contains a discussion of related models and extensions of our model. In Section 4.1, we relate our model to the existing literature on regret, including the "regret theory" introduced by Bell (1982) and Loomes and Sugden (1982). We explain how our model differs from this literature in terms of both primitives and the type of behavior used to identify regret. In Section 4.2, we give a detailed discussion of the relationship between regret and temptation. Since temptation is similar to anticipated regret in that both may cause an agent to benefit from commitment, we make the relationship between the two more precise by comparing a generalization of our representation with the temptation representations of Gul and Pesendorfer (2001) and Dekel, Lipman, and Rustichini (2007). Finally, in Section 4.3, we discuss an important extension our model in which the agent is allowed to regret both her choice of alternative and her choice of menu, and we find that our current model of regret is consistent with this more general model.

[4]Note that beef being the optimal choice from A_1 is consistent with our earlier claim that the agent will choose from a menu according to her commitment preference. In Section 2.1, we verify that this is a general property of our representation.

2. THE MODEL

Let Z be a finite set of prizes, and let $\Delta(Z)$ denote the set of all probability distributions on Z, endowed with the Euclidean metric d.[5] Let $\mathcal{A}$ denote the collection of all closed subsets of $\Delta(Z)$. We refer to the sets $A \in \mathcal{A}$ as menus and we endow $\mathcal{A}$ with the Hausdorff metric, defined by

$$d_h(A, B) = \max\Big\{\max_{p\in A}\min_{q\in B} d(p, q), \max_{p\in B}\min_{q\in A} d(p, q)\Big\}.$$

For any $A, B \in \mathcal{A}$ and $\alpha \in [0, 1]$, define the convex combination of these two menus by $\alpha A + (1 - \alpha)B = \{\alpha p + (1 - \alpha)q : p \in A \text{ and } q \in B\}$. The primitive of our model is a binary relation $\succsim$ on $\mathcal{A}$, representing the agent's preference over menus.

We have in mind an agent facing a two-period decision problem. The agent chooses a menu in period 0 and subsequently selects a lottery from that menu in period 1. However, we do not explicitly model the agent's period 1 choice, leaving it as part of the interpretation of the agent's period 0 preference.

2.1. *Representation*

We model the agent's uncertainty about her future tastes using a probability measure over a set of possible ex post utility functions, which we refer to as states. We impose the restriction that each ex post utility function in our representation be a von Neumann–Morgenstern expected-utility function. Since expected-utility functions on $\Delta(Z)$ are equivalent to vectors in $\mathbb{R}^Z$, we will use the notation $u(p)$ and $p \cdot u$ interchangeably. Moreover, since expected-utility functions are only unique up to an affine transformation, it is possible to impose a normalization on the set of ex post utility functions in our representation. Define the set of *normalized (nonconstant) expected-utility functions* on $\Delta(Z)$ to be

$$\mathcal{U} = \left\{ u \in \mathbb{R}^Z : \sum_{z\in Z} u_z = 0, \sum_{z\in Z} u_z^2 = 1 \right\}. \tag{1}$$

For any $\hat{u} \in \mathbb{R}^Z$ (i.e., any expected-utility function), there exist $\alpha \geq 0$, $\beta \in \mathbb{R}$, and $u \in \mathcal{U}$ such that $\hat{u} = \alpha u + \beta$. Therefore, modulo an affine transformation, $\mathcal{U}$ contains all possible ex post expected-utility functions.

Using this canonical state space, we define a regret representation as follows:

[5]Since Z is finite, the topology generated by d is equivalent to the topology of weak convergence on $\Delta(Z)$.

DEFINITION 1: A *regret representation* is a pair (μ, K) that consists of a finite (and countably additive) Borel probability measure μ on $\mathcal{U}$ and a constant $K \geq 0$ such that $\succsim$ is represented by the function $V : \mathcal{A} \to \mathbb{R}$ defined by

$$V(A) = \max_{p \in A} \int_{\mathcal{U}} [u(p) - R(p, A, u)] \mu(du), \tag{2}$$

where

$$R(p, A, u) = K\Big[\max_{q \in A} u(q) - u(p)\Big]. \tag{3}$$

The interpretation of this representation is just as in the Introduction. The agent has subjective uncertainty about her future tastes, and when evaluating a menu, the agent anticipates the following: She will select an item from the menu ex ante (before the subjective state is realized) and she will experience regret after the state is realized if her choice is ex post inferior. Her value for the menu is therefore based on her expectation of utility minus regret.

Note that the normalization placed on $\mathcal{U}$ in Equation (1) imposes no real restrictions on the representation. Any representation of the form given in Equations (2) and (3), where $\mathcal{U}$ is allowed to be a (nonnormalized) set of ex post expected-utility functions (as in Example 1), can, after appropriate rescaling of the measure, be written as a regret representation with $\mathcal{U}$ defined by Equation (1). Thus our representation theorem would continue to hold without the normalization of $\mathcal{U}$. We impose this normalization for the simple reason that it makes the statement of our uniqueness results more straightforward.

When we discussed our axioms in the Introduction, we made the assumption that the agent chooses out of a menu according to her commitment preference. For the (interpreted) choice of lottery in our representation to be consistent with this assumption, it must be that a lottery is a maximizer in Equation (2) if and only if it solves $\max_{p \in A} V(\{p\})$. It is easily seen that a regret representation indeed satisfies this property when Equations (2) and (3) are combined and rewritten as[6]

$$V(A) = \max_{p \in A}\Big[(1 + K) \int_{\mathcal{U}} u(p)\mu(du)\Big] - K \int_{\mathcal{U}} \max_{q \in A} u(q)\mu(du). \tag{4}$$

Intuitively, the agent in our model chooses a lottery to maximize the expectation of her utility minus regret. However, the lottery that maximizes her

[6]Although the agent's period 1 choice of lottery is part of the interpretation of the representation, it is a fairly straightforward exercise to add a choice correspondence to the primitives of the model, thus formally capturing the agent's choice from each menu. The choice from menus given by this correspondence will be consistent with our interpretation of the representation if and only if the alternatives chosen from each menu are those that maximize the commitment preference.

expected utility also minimizes the expectation of her regret. Therefore, although regret may cause the agent in our model to sometimes prefer committing to smaller menus, it does not "distort" the agent's choice from a menu. This observation touches upon an issue that we will discuss in more detail in Section 4.1: While existing models of regret (see Bell (1982), Loomes and Sugden (1982, 1987), Sugden (1993), Hayashi (2007)) only identify regret through its distorting effect on an agent's choice of alternative *from* a menu, by instead examining preferences over menus, we are able to identify regret even when it does not distort choice from menus.

2.2. *Axioms*

We impose four axioms on preferences. The first three are standard axioms in the setting of preferences over menus:

AXIOM 1—Weak Order: $\succsim$ *is complete and transitive*.

AXIOM 2—Strong Continuity:
1. von Neumann–Morgenstern (vNM) Continuity: *If* $A \succ B \succ C$, *then there exist* $\alpha, \bar{\alpha} \in (0, 1)$ *such that*

$$\alpha A + (1-\alpha)C \succ B \succ \bar{\alpha} A + (1-\bar{\alpha})C.$$

2. Lipschitz (L) Continuity: *There exist menus* $A^*, A_* \in \mathcal{A}$ *and* $M > 0$ *such that for every* $A, B \in \mathcal{A}$ *and* $\alpha \in (0, 1)$ *with* $d_h(A, B) \leq \alpha/M$,

$$(1-\alpha)A + \alpha A^* \succsim (1-\alpha)B + \alpha A_*.$$

AXIOM 3—Independence: *If* $A \succ B$, *then for all* C *and all* $\alpha \in (0, 1]$,

$$\alpha A + (1-\alpha)C \succ \alpha B + (1-\alpha)C.$$

We refer the reader to Dekel, Lipman, and Rustichini (2001) and Dekel, Lipman, Rustichini, and Sarver (2007, henceforth DLRS) for a discussion of these axioms. The independence axiom was also discussed by Gul and Pesendorfer (2001) in their model of temptation.

The following axiom allows for the possibility of regret:

AXIOM 4—Dominance: *If* $\{p\} \succsim \{q\}$ *and* $p \in A$, *then* $A \succsim A \cup \{q\}$.

As we discussed in the Introduction, in a standard model, the agent values a menu based on its best element. For such an agent, if $\{p\} \succsim \{q\}$ for some $p \in A$, then $A \sim A \cup \{q\}$. The dominance axiom relaxes this condition to allow for $A \succ A \cup \{q\}$. Intuitively, adding an alternative to a menu that is not ultimately chosen in period 1 cannot increase the utility of the agent. Moreover, since we assume that the agent's period 1 choice is made according to her commitment preference, $\{p\} \succsim \{q\}$ implies the agent does at least as well by choosing p from the menu $A \cup \{q\}$ as she does by choosing q. Therefore, the addition of q to the menu A cannot benefit the agent. However, it can hurt the agent if there is a state in which q is better than every element of A, as the agent would experience increased regret in this state.

3. MAIN RESULTS

3.1. *Representation Theorem*

The following is our main representation theorem:

THEOREM 1: *A preference $\succsim$ has a regret representation if and only if it satisfies weak order, strong continuity, independence, and dominance.*

Given this result, we refer to a preference that satisfies weak order, strong continuity, independence, and dominance as a *regret preference*. In the remainder of this section, we prove the necessity of the axioms in Theorem 1 and sketch the intuition behind the sufficiency of the axioms. The formal proof of sufficiency is contained in Appendix C.1.

The first step in establishing Theorem 1 is to note that the regret representation is a special case of what Dekel, Lipman, and Rustichini (2001) refer to as an additive expected-utility (EU) representation. Taking $\mathcal{U}$ as defined in Equation (1), we define the following (normalized) version of their representation:

DEFINITION 2: An *additive EU representation* is a finite (and countably additive) signed Borel measure μ on $\mathcal{U}$ such that $\succsim$ is represented by the function $V : \mathcal{A} \to \mathbb{R}$ defined by

$$V(A) = \int_{\mathcal{U}} \max_{p \in A} u(p)\, \mu(du). \tag{5}$$

The following is the Dekel, Lipman, and Rustichini (2001) representation theorem as presented in the Supplemental material to DLRS (2007)[7]:

THEOREM 2: *A preference $\succsim$ has an additive EU representation if and only if it satisfies weak order, strong continuity, and independence.*

[7]Although Definition 2 differs slightly from the original definition of the additive EU representation given in Dekel, Lipman, and Rustichini (2001), it is easily verified that the two formulations are equivalent. Moreover, the proof of Theorem 2 contained in the Supplemental material to DLRS (2007) uses precisely the normalized form of the representation defined above.

In light of Theorem 2, the necessity of weak order, strong continuity, and independence in Theorem 1 is established by showing that the regret representation is a special case of the additive EU representation:

LEMMA 1: *Any regret representation can be written as an additive EU representation.*

PROOF: For ease of manipulation, we will work with the formulation of the regret representation given in Equation (4). Note that $(1+K)\int_{\mathcal{U}} u(\cdot)\,\mu(du)$ is itself an expected-utility function and therefore must equal $\alpha\bar{u}+\beta$ for some $\bar{u}\in\mathcal{U}$, $\alpha\geq 0$, and $\beta\in\mathbb{R}$. By the definition of $\mathcal{U}$ it follows that $\beta=0$, and hence Equation (4) can be written as

$$V(A)=\max_{p\in A}\alpha\bar{u}(p)-K\int_{\mathcal{U}}\max_{q\in A}u(q)\,\mu(du).$$

Therefore, define a new signed measure ν, for any Borel set $E\subset\mathcal{U}$, by

$$\nu(E)=\begin{cases}-K\mu(E), & \text{if } \bar{u}\notin E,\\ \alpha-K\mu(E), & \text{if } \bar{u}\in E.\end{cases}$$

Then, the above expression simplifies to

$$V(A)=\int_{\mathcal{U}}\max_{p\in A}u(p)\nu(du),$$

which completes the proof. *Q.E.D.*

The necessity of the dominance axiom is established by the following lemma:

LEMMA 2: *A preference $\succsim$ with a regret representation must satisfy dominance.*

PROOF: Suppose $\succsim$ has a regret representation, formulated as in Equation (4), and suppose $A\in\mathcal{A}$ and $q\in\Delta(Z)$ are such that there exists $p\in A$ with $\{p\}\succsim\{q\}$. Notice that $\{p\}\succsim\{q\}$ if and only if $V(\{p\})\geq V(\{q\})$ if and only if

$$\int_{\mathcal{U}}u(p)\mu(du)\geq\int_{\mathcal{U}}u(q)\mu(du).$$

Therefore, the addition of $\{q\}$ to the menu A leaves the first term of Equation (4) unchanged:

$$\max_{\hat{p}\in A}\left[(1+K)\int_{\mathcal{U}}u(\hat{p})\mu(du)\right]=\max_{\hat{p}\in A\cup\{q\}}\left[(1+K)\int_{\mathcal{U}}u(\hat{p})\mu(du)\right].$$

Clearly, we also have $\max_{\hat{p}\in A} u(\hat{p}) \le \max_{\hat{p}\in A\cup\{q\}} u(\hat{p})$ for all $u \in \mathcal{U}$. Thus the second term of Equation (4) becomes weakly smaller (i.e., more negative) whenever more items are added to a menu, which implies $V(A) \ge V(A\cup\{q\})$ or, equivalently, $A \succsim A\cup\{q\}$. *Q.E.D.*

We now give the intuition behind the sufficiency of the axioms in Theorem 1. From Theorem 2, we know that if $\succsim$ satisfies weak order, strong continuity, and independence, then it has an additive EU representation. We want to show that if $\succsim$ also satisfies dominance, then this representation can be written as a regret representation. For simplicity, we will assume that the support of the measure μ in the additive EU representation is finite, but the proof in the Appendix deals with the more general case. If the support of μ is finite, then Equation (5) can be written as

$$V(A) = \sum_{u\in \mathrm{supp}(\mu)} \mu(u) \max_{p\in A} u(p).$$

Note that μ is a signed measure, so some states may be given negative weight. Thus $\mathrm{supp}(\mu)$ can be indexed by a pair of finite sets I^+ and I^-, where $\mu(u_i) > 0$ for $i \in I^+$ and $\mu(u_i) < 0$ for $i \in I^-$. Letting $\alpha_i = |\mu(u_i)| > 0$, Equation (5) becomes

$$V(A) = \sum_{i\in I^+} \alpha_i \max_{p\in A} u_i(p) - \sum_{i\in I^-} \alpha_i \max_{p\in A} u_i(p).$$

Therefore, the sign of μ is very important, as it determines whether adding options to a menu has a positive or negative effect on utility in a given state. We refer to the states indexed by I^+ as positive states and to the states indexed by I^- as negative states. Negative states will turn out to be key to capturing regret.

Define $v: \Delta(Z) \to \mathbb{R}$ by $v(p) = V(\{p\})$ for $p \in \Delta(Z)$. Thus v represents the agent's preference over singletons (i.e., her commitment preference).[8] We will show that dominance implies that any positive state in this additive EU representation can be written as an affine transformation of v. Then some simple algebraic manipulations will yield the finite-state version of Equation (4).

Our first step is to show that any positive state must have the same level curves as v and be increasing in the same direction as v. Since v and each of the u_i are expected-utility functions, their level curves are linear. Now consider any state i such that u_i and v do not have the same level curves. We can always choose a menu $A \in \mathcal{A}$ and a lottery $q \in \Delta(Z)$ such that adding q to A increases the maximum value of u_i, but does not increase the maximum value of v or

[8]Note that our notation differs from that of Gul and Pesendorfer (2001), who used v to indicate a temptation ranking.

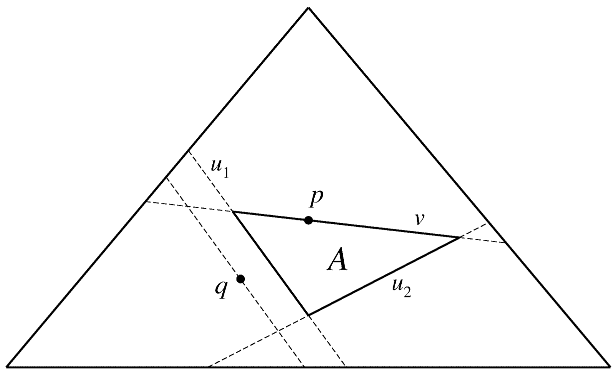

FIGURE 1.—Dominance.

u_j for $j \neq i$. This is illustrated for a two-state additive EU representation in Figure 1. The line labeled u_1 indicates the level curve for u_1 that is tangent to the set A—this is the highest level curve for u_1 attainable under the menu A. Notice that q gives a higher value for u_1 than is possible under A, but adding q to A does not increase the maximum value of v or u_2.

Take any $p \in A$ that maximizes v over A. Since adding q to A does not increase the maximum value of v, we have that $v(p) \geq v(q)$. This is also illustrated in Figure 1. Then the definition of v implies that $\{p\} \succsim \{q\}$ and, therefore, dominance implies that $V(A) \geq V(A \cup \{q\})$. Since the only state that attains a different maximum expected utility under A and $A \cup \{q\}$ is i, it must be that $i \in I^-$. That is, i must be a negative state. Thus, as claimed, any positive state must have the same level curves as v and be increasing in the same direction. Note that by the definition of $\mathcal{U}$ this implies there can be at most one positive state. If we denote the single positive state by 0, then we have shown that V is of the form

$$V(A) = \alpha_0 \max_{p \in A} u_0(p) - \sum_{i \in I^-} \alpha_i \max_{p \in A} u_i(p).$$

Moreover, since v and $\alpha_0 u_0$ are affine, share the same indifference curves, and are increasing in the same direction, it is a standard result that $\alpha_0 u_0 = \alpha v + \beta$ for some $\alpha \geq 0$ and $\beta \in \mathbb{R}$ (see Lemma 14). For expositional purposes, we make the simplifying assumption that $\alpha > 1$ and $\beta = 0$. Therefore, we have

$$V(A) = \max_{p \in A}[\alpha v(p)] - \sum_{i \in I^-} \alpha_i \max_{p \in A} u_i(p).$$

Taking any $p \in \Delta(Z)$, this equation implies $v(p) = \alpha v(p) - \sum_{i \in I^-} \alpha_i u_i(p)$, and hence $(\alpha - 1)v = \sum_{i \in I^-} \alpha_i u_i$. Let $\hat{V} = (\alpha - 1)V$ and let $K = \alpha - 1$. Then we have

$$\hat{V}(A) = \max_{p \in A}[\alpha(\alpha - 1)v(p)] - (\alpha - 1)\sum_{i \in I^-} \alpha_i \max_{p \in A} u_i(p)$$
$$= \max_{p \in A}\left[(1 + K)\sum_{i \in I^-} \alpha_i u_i(p)\right] - K\sum_{i \in I^-} \alpha_i \max_{p \in A} u_i(p).$$

After normalizing the α_i's to be probabilities by dividing by $\sum_{i \in I^-} \alpha_i$, we obtain a finite-state version of Equation (4).

3.2. *Uniqueness of the Representation*

As in the previous section, let v denote the utility function for singleton menus induced by a regret representation. That is, for a regret representation (μ, K), define the function $v : \Delta(Z) \to \mathbb{R}$, for $p \in \Delta(Z)$, by

$$v(p) = V(\{p\}) = \int_{\mathcal{U}} u(p)\mu(du). \tag{6}$$

Also define $r : \mathcal{A} \to \mathbb{R}$, for $A \in \mathcal{A}$, by

$$\begin{aligned} r(A) &= \min_{p \in A} \int_{\mathcal{U}} R(p, A, u)\mu(du) \\ &= \min_{p \in A} \int_{\mathcal{U}} K\Big[\max_{q \in A} u(q) - u(p)\Big]\mu(du). \end{aligned} \tag{7}$$

The function $r(A)$ represents the minimal expected regret that the agent can experience when faced with the menu A. As we discussed in Section 2.1, in a regret representation the lottery that maximizes the expectation of utility also minimizes expected regret. Thus for any menu $A \in \mathcal{A}$, the agent will choose $p \in A$ to maximize $v(p)$, and

$$V(A) = \max_{p \in A} v(p) - r(A).$$

The functions v and r will be useful for understanding the uniqueness properties of our representation (and for analyzing comparative regret attitudes in Section 3.3). In this section, we first consider the extent to which the functions v and r are identified. Then we turn to the uniqueness of the underlying parameters that determine v and r.

Our first uniqueness result is that for any regret preference, the v and r that arise from a regret representation are unique up to a common scalar multiple.

THEOREM 3: *Two regret representations (μ, K) and (μ', K') represent the same preference $\succsim$ if and only if there exists $\alpha > 0$ such that $v' = \alpha v$ and $r' = \alpha r$.*

PROOF: *Only if*. We have already proved that any regret representation is an additive EU representation (see Lemma 1). The mixture space theorem can be applied to this setting to show that the function V given by an additive EU representation is unique up to a positive affine transformation.[9] Therefore, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $V' = \alpha V + \beta$, which implies $v' = \alpha v + \beta$. Treating v as a vector in $\mathbb{R}^Z$ and recalling the definition of $\mathcal{U}$, note that

$$(8) \qquad \sum_z v_z = \sum_z \left[\int_{\mathcal{U}} u_z \, \mu(du) \right] = \int_{\mathcal{U}} \left[\sum_z u_z \right] \mu(du) = 0.$$

Similarly, $\sum_z v'_z = 0$. It follows that $\beta = 0$.[10] Finally, for all $A \in \mathcal{A}$,

$$r'(A) = \max_{p \in A} v'(p) - V'(A) = \alpha \left[\max_{p \in A} v(p) - V(A) \right] = \alpha r(A).$$

If. If $v' = \alpha v$ and $r' = \alpha r$, then an argument similar to that given above shows that $V' = \alpha V$, and hence V' and V represent the same preference. *Q.E.D.*

To obtain a uniqueness result for the parameters (μ, K) in a regret representation, we now restrict attention to regret representations in which regret plays a nontrivial role, that is, there is some menu A for which $r(A) > 0$. Before presenting our main uniqueness result, we note an interesting relationship between this property of the representation and the following axiom:

AXIOM 5—Monotonicity: *If $A \subset B$, then $B \succsim A$.*

Kreps (1979) interpreted this axiom in terms of an agent who is uncertain of her tastes when she chooses a menu, but realizes her tastes before she chooses an alternative from a menu. Thus the agent has a preference for flexibility. However, in our representation the agent must choose from a menu before she realizes her tastes. Therefore, flexibility is not valuable in our model, and it can be harmful if additional options increase regret. We see that dominance and monotonicity have a trivial intersection in the sense that an agent with a regret preference that also satisfies monotonicity neither values flexibility nor experiences regret:

LEMMA 3: *Suppose $\succsim$ is a regret preference and hence has a regret representation. Then $r(A) = 0$ for all $A \in \mathcal{A}$ if and only if $\succsim$ satisfies monotonicity.*

[9] See the proof of Proposition 2 in Dekel, Lipman, and Rustichini (2001) for a detailed explanation.

[10] To see this, let $p^* = (1/|Z|, \dots, 1/|Z|)$. Then $\sum_z v_z = 0$ implies $v(p^*) = 0$, and $\sum_z v'_z = 0$ implies $v'(p^*) = 0$. Therefore, $0 = v'(p^*) = \alpha v(p^*) + \beta = \beta$.

PROOF: *Only if*. Suppose $r(A) = 0$ for all $A \in \mathcal{A}$. If $B \subset C$, then

$$V(B) = \max_{p \in B} v(p) \leq \max_{p \in C} v(p) = V(C),$$

and hence $C \succsim B$.

If. We prove by establishing the contrapositive. Suppose there exists $A \in \mathcal{A}$ such that $r(A) > 0$. We want to show that monotonicity is violated. Let $\bar{p} \in \arg\max_{p \in A} v(p)$. Then $\{\bar{p}\} \subset A$ and

$$V(\{\bar{p}\}) = v(\bar{p}) = \max_{p \in A} v(p) > \max_{p \in A} v(p) - r(A) = V(A),$$

violating monotonicity. *Q.E.D.*

Given this relationship between r and the monotonicity axiom, we say that a regret preference is *nontrivial* if it violates monotonicity. That is, $\succsim$ is a *nontrivial regret preference* if there exist $A, B \in \mathcal{A}$ such that $A \subset B$ and $A \succ B$.

Our uniqueness theorem for the parameters (μ, K) will have two parts, depending on whether or not the agent is indifferent between all singleton menus, that is, whether or not the following axiom is satisfied:

AXIOM 6—Singleton Nontriviality: *There exist* $p, q \in \Delta(Z)$ *with* $\{p\} \succ \{q\}$.

To state our uniqueness result for the case when $\succsim$ satisfies singleton nontriviality, we need to define the norm of an expected-utility function. Take any regret representation (μ, K) and define v by Equation (6). Define the norm of v, denoted $\|v\|$, to be the Euclidean norm of v when considered as a vector in $\mathbb{R}^Z$. The following lemma provides a technical result that is needed for the subsequent theorem.

LEMMA 4: *Suppose* (μ, K) *represents the preference* $\succsim$ *and* v *is defined by Equation* (6). *Then* $\|v\| \neq 0$ *if and only if* $\succsim$ *satisfies singleton nontriviality, and in this case* $\frac{v}{\|v\|} \in \mathcal{U}$.

See Appendix C.2 for the proof.

We can now state our main uniqueness result. Let $\delta_{v/\|v\|}$ denote the Dirac (probability) measure concentrated at $\frac{v}{\|v\|}$. Note that Lemma 4 is needed to ensure that this is a measure on $\mathcal{U}$.

THEOREM 4: *Suppose* (μ, K) *represents a nontrivial regret preference* $\succsim$.

1. *If* $\succsim$ *satisfies singleton nontriviality, then* (μ', K') *also represents* $\succsim$ *if and only if there exists* $0 < \alpha < \frac{1}{1-\|v\|}$ *such that*

$$\mu' = \alpha\mu + (1-\alpha)\delta_{v/\|v\|} \quad \text{and} \quad K' = \left[1 + \frac{1-\alpha}{\alpha\|v\|}\right]K. \tag{9}$$

2. *If* $\succsim$ *violates singleton nontriviality, then* (μ', K') *also represents* $\succsim$ *if and only if* $\mu' = \mu$ *and* $K' > 0$.

See Appendix C.3 for the proof.

To interpret part 1 of Theorem 4, suppose (μ, K) is a regret representation for a preference $\succsim$ that satisfies singleton nontriviality. Defining v as in Equation (6), consider the probability measure $\delta_{v/\|v\|}$. Note that if an agent has a regret representation with measure $\delta_{v/\|v\|}$, then this agent has no subjective uncertainty, as she knows with probability 1 that her ex post preference will be given by $\frac{v}{\|v\|}$. Therefore, regardless of the constant in the representation, this agent will experience no regret: Given any $A \in \mathcal{A}$, she will choose $p \in A$ to maximize $p \cdot \frac{v}{\|v\|}$, and this will be the ex post "wrong" choice with probability 0. Now, consider (μ', K') defined as in Equation (9) for some $\alpha \in (0, 1)$.[11] As α gets smaller, μ' puts greater probability mass on a single point, $\frac{v}{\|v\|}$. Therefore, the probability of ex post regret becomes smaller as α decreases, so K' must be increased to compensate. This intuition is confirmed by Equation (9), as it is readily seen that K' increases as α decreases.

Part 1 of Theorem 4 highlights the difficulty of identifying the strength of regret, K, in our model. The agent's preference over menus indicates the combined effect of her strength of regret and degree of uncertainty, but it is not possible to completely differentiate these two factors. As the previous paragraph illustrates, we are unable to distinguish between an agent who is fairly certain of her future tastes but has a strong sense of regret and an agent who is more uncertain of her tastes but has a weaker sense of regret. However, Theorem 4 implies that these two effects are jointly identified when $\succsim$ satisfies singleton nontriviality. In particular, if (μ, K) and (μ', K') are two regret representations for a preference satisfying singleton nontriviality, then $\mu = \mu'$ if and only if $K = K'$.

To interpret part 2 of Theorem 4, note that when an agent has a preference $\succsim$ that violates singleton nontriviality, the sole objective of this agent is to minimize expected regret. Because the agent has no utility function over singletons against which to measure this regret, it is impossible to pin down the constant K in the representation. However, in this case the measure in the representation is uniquely identified.

[11]Note that Theorem 4 actually allows for $\alpha > 1$. In other words, it is possible to shift probability mass away from $\frac{v}{\|v\|}$. However, the restriction that $\alpha < \frac{1}{1-\|v\|}$ is needed to ensure that $K' > 0$. In addition, since the measure in a regret representation is required to be a probability measure, μ' must be nonnegative and, hence, there is also an implicit restriction that $\alpha \leq \frac{1}{1-\mu(\{v/\|v\|\})}$.

3.3. *Comparing Regret Attitudes*

In this section, we present two measures of regret attitudes of a preference. The first is a comparative measure of the incidence of regret, and the second is a comparative measure of the strength of regret.

For a regret preference $\succsim$, we say that A *dominates* B with respect to $\succsim$ if for all $q \in B$ there exists a $p \in A$ such that $\{p\} \succsim \{q\}$. A relatively straightforward consequence of our axioms is that whenever A dominates B with respect to a regret preference $\succsim$, we have $A \succsim A \cup B$ (see Lemma 11 in Appendix C.1). Thus adding a dominated set of alternatives to a menu can never be an improvement. When the addition of a dominated set leads to an increase in regret, the original menu must be strictly preferred to the addition of this set to the menu. Therefore, by observing how often adding dominated sets of alternatives to a menu leads to a strict decrease in utility, we can determine how regret prone a preference is:

DEFINITION 3: Suppose $\succsim_1$ and $\succsim_2$ are two regret preferences. We say that $\succsim_1$ *is more regret prone than* $\succsim_2$ if for all $A, B \in \mathcal{A}$ such that A dominates B with respect to $\succsim_1$ and $\succsim_2$,

$$A \succ_2 A \cup B \quad \Longrightarrow \quad A \succ_1 A \cup B.$$

The following theorem characterizes this measure of regret attitudes in terms of our representation. As in the previous section, $\succsim$ is a nontrivial regret preference if there exist menus $A, B \in \mathcal{A}$ such that $A \subset B$ and $A \succ B$. Singleton nontriviality and the function v are also defined as in the previous section.[12]

THEOREM 5: *Suppose* (μ_1, K_1) *and* (μ_2, K_2) *represent the nontrivial regret preferences* $\succsim_1$ *and* $\succsim_2$*, respectively, and suppose these preferences satisfy singleton nontriviality. Then* $\succsim_1$ *is more regret prone than* $\succsim_2$ *if and only if*

$$\operatorname{supp}(\mu_2) \subset \operatorname{supp}(\mu_1) \cup \left\{\frac{v_1}{\|v_1\|}\right\} \cup \left\{\frac{v_2}{\|v_2\|}\right\}. \tag{10}$$

See Appendix C.4 for the proof.

Suppose (μ_1, K_1) represents a nontrivial regret preference $\succsim_1$. Adding a set of alternatives B to a menu A that does not increase the maximum value of v_1 cannot increase the utility of the menu. In addition, if the alternatives in B increase the maximum value of u for some $u \in \operatorname{supp}(\mu_1)$, then with positive probability the agent will regret not choosing one of these new alternatives.

[12]Singleton nontriviality can be dropped from Theorem 5 if Equation (10) is replaced with the following more general condition: If $u \in \operatorname{supp}(\mu_2)$ is not a positive affine transformation of v_1 or v_2, then $u \in \operatorname{supp}(\mu_1)$.

Therefore, a measure with a larger support leads to an increased incidence of regret.

We now introduce a comparative measure of the strength of regret of a preference[13]:

DEFINITION 4: Suppose $\succsim_1$ and $\succsim_2$ are two regret preferences. We say that $\succsim_1$ *is more regret averse than* $\succsim_2$ if for all $A \in \mathcal{A}$ and $p \in \Delta(Z)$,

$$\{p\} \succ_2 A \quad \Longrightarrow \quad \{p\} \succ_1 A.$$

Using singleton menus as benchmarks for comparison, this measure considers the trade-off between adding potentially better alternatives to a menu and the regret that could arise from having more alternatives from which to choose. To illustrate more concretely, consider again Example 1. Recall that $Z = \{b, c\}$ and the regret representation satisfied $V(\{b\}) = 3$, $V(\{c\}) = 2.5$, and $V(\{b, c\}) = 3 - K$. Suppose $\succsim$ is the preference induced by this representation. Then $\{b\} \succsim \{c\}$ and, as required by the dominance axiom, $\{b\} \succsim \{b, c\}$. However, dominance is silent about which of $\{c\}$ or $\{c, s\}$ will be preferred. Clearly, this will depend on the value of K (i.e., the strength of regret) in the representation: $\{b, c\} \succsim \{c\}$ for small K and $\{c\} \succ \{b, c\}$ for large K. Hence, adding the "better" alternative b to the menu $\{c\}$ is beneficial precisely when regret is weaker, and this intuition is generalized in Definition 4.

Our notion of comparative regret aversion is similar in spirit to the literature on comparative ambiguity aversion. Both Epstein (1999) and Ghirardato and Marinacci (2002) defined comparative ambiguity aversion by comparing arbitrary acts to unambiguous acts in the same manner that we compare arbitrary menus to singleton menus (although each uses a different definition of what constitutes an unambiguous act). Ahn (2007) considered ambiguity aversion in the current preferences-over-menus framework and proposed a measure of comparative ambiguity aversion that is almost identical to our Definition 4.

The following theorem examines the relationship between regret aversion and the regret representation. Note that v and r are defined as in Equations (6) and (7), respectively.[14]

[13]Note that if $\succsim_1$ is more regret averse than $\succsim_2$, then both preferences have the same commitment preference: For any $p, q \in \Delta(Z)$, taking $A = \{q\}$ in Definition 4 gives the condition $\{p\} \succ_2 \{q\} \Longrightarrow \{p\} \succ_1 \{q\}$, which when combined with our other axioms implies $\{p\} \succ_2 \{q\} \Longleftrightarrow \{p\} \succ_1 \{q\}$.

[14]Singleton nontriviality is needed in Theorem 6 to ensure that comparative regret aversion is an informative measure. For example, suppose $\succsim_1$ and $\succsim_2$ are two regret preferences. Also, for $i = 1, 2$, suppose that $\{p\} \sim_i \{q\}$ for all $p, q \in \Delta(Z)$ and $A \prec_i \{p\}$ for any nonsingleton $A \in \mathcal{A}$. If $|Z| \geq 3$, then there are many different regret preferences that satisfy these conditions. However, since $A \prec_i \{p\}$ for all $p \in \Delta(Z)$ and for all nonsingleton $A \in \mathcal{A}$, Definition 4 does not get any traction. That is, we find both that $\succsim_1$ is more regret averse than $\succsim_2$ and that $\succsim_2$ is more regret averse than $\succsim_1$, even though we have hardly specified any properties of these regret preferences.

THEOREM 6: *Suppose* $\succsim_1$ *and* $\succsim_2$ *are two regret preferences that satisfy singleton nontriviality*, *and suppose* (μ_1, K_1) *and* (μ_2, K_2) *are regret representations for* $\succsim_1$ *and* $\succsim_2$, *respectively*. *The following statements are equivalent*:

1. $\succsim_1$ *is more regret averse than* $\succsim_2$.
2. *There exists* $\alpha > 0$ *such that* $v_2 = \alpha v_1$ *and* $r_2 \leq \alpha r_1$.

See Appendix C.5 for the proof.

The interpretation of this result is straightforward. If $\succsim_1$ is more regret averse than $\succsim_2$, then modulo transformation by a scalar multiple, r_1 is larger than r_2. Thus an agent with preference $\succsim_1$ expects to experience more regret from any menu than an agent with preference $\succsim_2$.[15]

4. DISCUSSION AND EXTENSIONS

We have presented a model of regret in which preferences over menus are represented by an easily interpreted functional form: It is as if the agent chooses a single alternative from a menu prior to the resolution of her subjective uncertainty and experiences regret if her choice is "wrong" ex post. We proved that our representation is essentially unique, and we introduced two measures of comparative regret attitudes.

We conclude by discussing two related areas of research and presenting an important extension of our model. In Section 4.1, we discuss the related literature on regret. In Section 4.2, we relate our model of regret to the literature on temptation. Finally, in Section 4.3, we present an extension of our model that allows the agent to regret her choice of menu.

4.1. *Related Models of Regret*

The present work is similar in spirit to so-called regret theory (see Bell (1982), Loomes and Sugden (1982, 1987), Sugden (1993)). However, our emphasis is quite different. Like our model, classic regret theory posits that the agent will experience regret if, after the resolution of uncertainty, her choice is inferior to another alternative. It is then assumed that the anticipation of this regret will affect the agent's decisions, an assumption we also make. However, despite these intuitive similarities, our approach has a very different foundation. Sugden (1993) gave an axiomatic treatment of regret theory by considering preferences over acts that are conditional on the feasible set of alternatives. Hence, the emphasis of his regret theory is on the effect of anticipated regret on the agent's choice *from* a menu. In contrast, the present work is set in a

[15]This does not necessarily imply that the first agent has a larger strength of regret, K, than the second. Recall from Section 3.2 that an agent's expected regret depends on the combination of her strength of regret and degree of uncertainty, and these two factors cannot be separately identified.

preferences-over-menus framework, and thus our focus is on the effect of regret on the agent's choice *of* menu.

Recall from Section 2.1 that the agent in our model is assumed to choose from a menu according to her commitment preference. Hence, the agent's (interpreted) choice from menus in our model is the same as that of a standard expected-utility maximizer, and thus by only examining the agent's choice from menus, it is impossible to determine whether or not the agent anticipates regret. In contrast, in classic regret theory, regret takes a nonlinear form which allows it to be identified by observing choice from menus. Although models of nonlinear regret are both interesting and plausible, we find the preferences-over-menus framework appealing because it also allows for the identification of simple linear forms of regret, such as the current model.

Regret was also studied by Hayashi (2007), who obtained a representation for a choice correspondence from menus of acts in which the agent minimizes her maximum regret.[16] His representation is a generalization of the minimax regret theory of Savage (1954). Aside from the different primitives, there are two main distinctions between these models and the present model: First, in models of minimax regret, the agent's only decision criterion is the minimization of maximum regret. In contrast, in our representation, the agent compromises between two objectives, minimizing regret and maximizing ex post expected utility. Second, the agent in our model is Bayesian, whereas the pessimism of the agent in models of minimax regret can be interpreted in terms of ambiguity. Specifically, the representation of Hayashi (2007) involves multiple priors and, in the case of a single prior, reduces to a standard model of utility maximization.

4.2. *Temptation or State-Dependent Regret?*

In this section, we show how the temptation representations of Gul and Pesendorfer (2001) and Dekel, Lipman, and Rustichini (2007) are related to what we refer to as a state-dependent regret representation. Each of the temptation representations we discuss is a special case of the additive EU representation.[17] In addition, each representation has a finite state space, so for ease of comparison, we restrict attention to finite-state-space representations throughout this section.

[16]Related representations in that framework were also considered by Stoye (2007).

[17]Technically, the expected-utility functions in these temptation representations are not required to be in the set $\mathcal{U}$. Nonetheless, it is straightforward to verify that these representations could be normalized and written in the form of Definition 2. We should also note that Gul and Pesendorfer (2001) allowed for a more general domain by permitting the set of prizes, Z, to be any compact metric space.

One of the representations considered by Gul and Pesendorfer (2001) is the *self-control representation*, which they defined by

$$V(A) = \max_{p \in A}[v(p) + t(p)] - \max_{q \in A} t(q),$$

where $v : \Delta(Z) \to \mathbb{R}$ and $t : \Delta(Z) \to \mathbb{R}$ are expected-utility functions. As above, v represents the commitment preference since $V(\{p\}) = v(p)$ for any $p \in \Delta(Z)$. The function t represents the temptation ranking.[18] This representation can easily be understood in terms of self-control costs. If we let $c(p, A) = \max_{q \in A} t(q) - t(p)$, then the self-control representation can be written as

$$V(A) = \max_{p \in A}[v(p) - c(p, A)].$$

Thus the temptation ranking determines the cost of self-control. Under the interpretation that the agent chooses an item from the menu A, the presence of self-control costs can alter this decision. In contrast, if an agent has a regret representation as in Section 2.1, then the presence of regret may cause the agent to select a different menu, but it will not alter her choice from a given menu.

Gul and Pesendorfer (2001) proved that a preference $\succsim$ has a self-control representation if and only if it satisfies weak order, continuity (see Axiom 10 in Appendix C.1), independence, and set betweenness, which they defined as follows[19]:

AXIOM 7—Set Betweenness: *If* $A \succsim B$, *then* $A \succsim A \cup B \succsim B$.

Dekel, Lipman, and Rustichini (2007) proposed several generalizations of the self-control representation. One of the representations they considered is the *no-uncertainty representation*, which they defined by

$$V(A) = \max_{p \in A}\left[v(p) + \sum_{i \in I} t_i(p)\right] - \sum_{i \in I} \max_{q \in A} t_i(q),$$

where I is a finite index set, and where $v : \Delta(Z) \to \mathbb{R}$ and $t_i : \Delta(Z) \to \mathbb{R}$, $i \in I$, are expected-utility functions. This representation is so named because, in contrast to some of the representations they considered, an agent with this representation has no uncertainty about what her future temptations will be. This

[18] As mentioned earlier, we use different notation than Gul and Pesendorfer (2001). They used u in the place of our v and used v in the place of our t.

[19] The astute reader will notice that strong continuity is not included in this list of axioms, even though it is still a necessary condition. As explained in Dekel, Lipman, and Rustichini (2007), the combination of weak order, continuity, independence, and any axiom that guarantees a finite state space (such as set betweenness) implies strong continuity.

representation can also be understood in terms of self-control costs since we can write

$$V(A) = \max_{p \in A} [v(p) - c(p, A)],$$

where $c(p, A)$ is now defined by

$$c(p, A) = \sum_{i \in I} \Big[\max_{q \in A} t_i(q) - t_i(p) \Big].$$

This representation generalizes the self-control representation by allowing multiple temptations to occur simultaneously.

Dekel, Lipman, and Rustichini (2007) proved that a preference $\succsim$ has a no-uncertainty representation if and only if it satisfies weak order, continuity, independence, a finiteness axiom (to guarantee that the state space is finite), and positive set betweenness, which they defined as follows:

AXIOM 8—Positive Set Betweenness: *If* $A \succsim B$, *then* $A \succsim A \cup B$.

Clearly, set betweenness implies positive set betweenness, which is necessary since the no-uncertainty representation is a generalization of the self-control representation. We now consider an alternative interpretation for preferences that satisfy positive set betweenness, and hence this alternative interpretation also applies to preferences that satisfy set betweenness. Recall that the interpretation of our original regret representation is that the agent is not certain about her future tastes and experiences regret when she makes an ex post inferior choice. If the agent can be uncertain about her tastes, could she not also be uncertain about the strength of her regret? Thus it seems reasonable to incorporate the constant K into her subjective uncertainty by allowing it to depend on the state.

DEFINITION 5: A *state-dependent regret representation* is a function $V : \mathcal{A} \to \mathbb{R}$ that represents $\succsim$ such that

$$V(A) = \max_{p \in A} \sum_{i \in I} \alpha_i \Big[u_i(p) - K_i \Big[\max_{q \in A} u_i(q) - u_i(p) \Big] \Big], \tag{11}$$

where I is a finite index set, $u_i \in \mathcal{U}$, $K_i \geq 0$, $\alpha_i \geq 0$, and $\sum_{i \in I} \alpha_i = 1$.

Notice that Equation (11) is simply the finite-state version of the regret representation defined in Section 2.1, except that the constant K is now replaced by a state-dependent constant K_i. It is also worth noting that, in contrast to the state-independent regret discussed in previous sections, the presence of state-dependent regret may alter the agent's choice from a menu. For example, if an agent has a disproportionately strong feeling of regret in the state where the lottery $q \in A$ is optimal, then she may select that lottery even if another lottery $p \in A$ is better in terms of her commitment preference.

The following theorem shows that preferences that satisfy positive set betweenness can be interpreted in terms of regret.

THEOREM 7: *A preference $\succsim$ has a no-uncertainty representation if and only if it has a state-dependent regret representation.*

PROOF: *If*. Take $v = \sum_{i\in I} \alpha_i u_i$ and $t_i = \alpha_i K_i u_i$.

Only if. Suppose $\succsim$ has a no-uncertainty representation with index set I. Let $\hat{I} = I \cup \{0\}$, where, without loss of generality, we suppose $0 \notin I$. By the definition of $\mathcal{U}$, for each $i \in I$, there exist $u_i \in \mathcal{U}$, $\alpha_i \geq 0$, and $\beta_i \in \mathbb{R}$ such that $t_i = \alpha_i u_i + \beta_i$. Also, there exist $u_0 \in \mathcal{U}$, $\alpha_0 \geq 0$, and $\beta_0 \in \mathbb{R}$ such that $v - \sum_{i\in I} t_i = \alpha_0 u_0 + \beta_0$. If we let $K_i = 1$ for $i \in I$ and let $K_0 = 0$, then

$$\begin{aligned}
V(A) &= \max_{p\in A}\left[v(p) + \sum_{i\in I} t_i(p)\right] - \sum_{i\in I} \max_{q\in A} t_i(q) \\
&= \max_{p\in A}\left[\sum_{i\in\hat{I}}(\alpha_i u_i(p) + \beta_i) + \sum_{i\in\hat{I}} K_i(\alpha_i u_i(p) + \beta_i)\right] \\
&\quad - \sum_{i\in\hat{I}} \max_{q\in A} K_i(\alpha_i u_i(q) + \beta_i) \\
&= \max_{p\in A} \sum_{i\in\hat{I}} \alpha_i\left[u_i(p) - K_i\left[\max_{q\in A} u_i(q) - u_i(p)\right]\right] + \sum_{i\in\hat{I}} \beta_i.
\end{aligned}$$

Dropping the constant $\sum_{i\in\hat{I}} \beta_i$ and normalizing the α_i's to be probabilities yields a state-dependent regret representation. *Q.E.D.*

This result is not a statement that regret is a better interpretation than temptation for preferences that satisfy positive set betweenness, and it is perfectly plausible that both play a role in such preferences. Rather, it simply illustrates that we are unable to perfectly disentangle these two factors in the absence of additional information: Under the interpretation of temptation, adding more items to a menu may be harmful to the agent if they make self-control more costly, while under the interpretation of regret, these additional items are harmful if they cause the agent ex post regret. Note that incorporating period 1 choice into the model will also not serve to differentiate between these two effects since the agent's (interpreted) period 1 choice from menus is the same under both a no-uncertainty representation and its equivalent state-dependent regret representation.

Given these observations, it is clear that distinguishing between regret and temptation is a subtle issue. Moreover, the boundary between the two effects seems highly sensitive to the precise definitions of regret and temptation that one considers. For example, an agent whose preference satisfies positive set betweenness (or dominance) may rank a menu as worse than every alternative in that menu. That is, there may be a menu A such that $\{p\} \succ A$ for all $p \in A$. While such a ranking seems perfectly reasonable for an agent who anticipates regret, whether this property is plausible for an agent suffering from temptation depends on how temptation is defined. If temptation is restricted to be self-control costs that occur at the time of a decision, then an agent suffering from temptation would arguably never exhibit such a ranking: If an agent suffering from temptation decides to "give in" to temptation, then she would not need to exert self-control, so her utility from the menu would be that of whatever alternative she obtains. However, if one considers a richer definition of temptation, for instance, by allowing guilt to be a part of temptation, then the ranking above would be possible.[20]

We close this section with an interesting observation. Ignoring the issue of finite versus infinite state spaces, the regret representation is a special case of the state-dependent regret representation. Therefore, positive set betweenness should be implied by the axioms that characterize the regret representation. We see that this is indeed the case.[21]

LEMMA 5: *If $\succsim$ is a regret preference, then it satisfies positive set betweenness.*

See Appendix C.6 for the proof.

4.3. *Regretting the Choice of Menu*

An important property of a regret representation is that there is no regret associated with a singleton menu. That is, $R(p, A, s) = 0$ if $A = \{p\}$. This property may give the impression that the agent in our model cannot regret her choice of menu, which is somewhat troubling. To illustrate, recall that in Example 1 the agent expects to prefer beef (b) to chicken (c) and hence she will select b when choosing from $\{b, c\}$. However, chicken is better than beef in state u_2, so in that state she will regret choosing beef from this menu. Therefore, she prefers the menu $\{b\}$ to the menu $\{b, c\}$, as there is no regret associated with the menu $\{b\}$. It is natural to ask why choosing the restaurant that

[20]Guilt is qualitatively similar to regret in that it is also a negative feeling associated with not choosing the "optimal" alternative. The difference is that ex post regret arises in our model after making a "mistake," whereas guilt arises after succumbing to temptation.

[21]This result is not immediately implied by the relationship between the representations because of the finiteness issue. The regret representation is actually not a special case of the no-uncertainty representation since it allows for an infinite state space.

only serves beef prevents the agent from experiencing regret ex post. For instance, if she chooses the menu $\{b\}$ and then finds that c is better ex post, would she not regret choosing the menu $\{b\}$ instead of either $\{b, c\}$ or $\{c\}$?

Although it seems perfectly reasonable for an agent to regret her choice of menu as well as her choice from a menu, the fundamentals of our model only allow for the identification of regret associated with the latter choice. As we argue below, to identify the regret associated with the choice of menu, we would need a richer primitive. Since regret from the choice of menu is unidentified in our framework, our regret representation can be thought of as simply making the normalization that it is zero. In this section, we illustrate these ideas by discussing a simple extension of our model.

In our model, we use preferences over menus to identify the regret associated with a choice from a menu. To identify regret associated with the choice of menu, we can look at an agent's preference over menus of menus of lotteries. We now introduce one possible extension of our representation to this framework. As above, let $\mathcal{A}$ denote the set of all closed subsets of $\Delta(Z)$. Let $\mathbf{A}$ denote a generic closed subset of $\mathcal{A}$. Thus we have $A \in \mathbf{A} \subset \mathcal{A}$. Suppose the agent has a preference over closed subsets of $\mathcal{A}$ that is represented by[22]

$$V_0(\mathbf{A}) = \max_{A\in\mathbf{A}} \max_{p\in A} \int_{\mathcal{U}} [u(p) - R_0(p, A, \mathbf{A}, u)]\mu(du),$$

where

$$\begin{aligned} &R_0(p, A, \mathbf{A}, u) \\ &\quad = K_0\Big[\max_{B\in\mathbf{A}} \max_{q\in B} u(q) - u(p)\Big] + K_1\Big[\max_{q\in A} u(q) - u(p)\Big]. \end{aligned}$$

The first term of R_0 represents the agent's ex post regret for her choice of menu, while the second term represents the agent's regret for her choice from that menu. Notice that if $K_0 = 0$, then this representation simplifies to $V_0(\mathbf{A}) = \max_{A\in\mathbf{A}} V(A)$, where V is defined as in the definition of the regret representation (see Equation (2)) with parameters (μ, K_1).

Suppose $\mathcal{C}_0$ is a choice correspondence that describes the agent's period 0 choice of menu, and suppose $\mathcal{C}_0$ is consistent with the period 0 choice suggested by the above representation. Thus $\mathcal{C}_0$ is defined by

$$\mathcal{C}_0(\mathbf{A}) = \arg\max_{A\in\mathbf{A}} \Big\{ \max_{p\in A} \int_{\mathcal{U}} [u(p) - R_0(p, A, \mathbf{A}, u)]\mu(du) \Big\},$$

[22] We are introducing this extension of our representation for comparison purposes only and, therefore, will not present an axiomatic treatment.

which simplifies to

$$\mathcal{C}_0(\mathbf{A}) = \arg\max_{A\in\mathbf{A}}\left\{\max_{p\in A}\left[(1+K_0+K_1)\int_{\mathcal{U}} u(p)\mu(du)\right] - K_1\int_{\mathcal{U}}\max_{p\in A} u(p)\mu(du)\right\}.$$

Let $K = K_1/(1+K_0)$, which implies $1+K = (1+K_0+K_1)/(1+K_0)$. Since dividing the above expression for $\mathcal{C}_0$ by the constant $1+K_0$ does not affect the arg max, we have

$$\mathcal{C}_0(\mathbf{A}) = \arg\max_{A\in\mathbf{A}}\left\{\max_{p\in A}\left[(1+K)\int_{\mathcal{U}} u(p)\mu(du)\right] - K\int_{\mathcal{U}}\max_{p\in A} u(p)\mu(du)\right\}.$$

Notice that if V is defined as in Equation (4), then $\mathcal{C}_0(\mathbf{A}) = \arg\max_{A\in\mathbf{A}} V(A)$. Thus the preference over menus induced by $\mathcal{C}_0$ has a regret representation (μ, K), and hence our regret representation is consistent with a model in which the agent also regrets her choice of menu. The constant K in our representation can be thought of as the additional regret associated with the agent's choice from a menu. However, notice that $K \neq K_1$ unless $K_0 = 0$. Therefore, to be more precise, we should say that our model identifies the regret associated with choice from a menu relative to the regret associated with prior decisions.

When $K_0 > 0$, the intuition for why $K = K_1/(1+K_0) < K_1$ is fairly straightforward. When choosing between two menus, one of which has a "better" alternative and the other of which has "fewer" alternatives, the influences of regret from the choice of menu and regret from the choice of alternative are conflicting. The latter may be decreased by choosing the menu with fewer options, but choosing this menu can only increase the former. Thus the larger the constant K_0, the more the agent will prefer menus with better alternatives to menus with fewer alternatives, giving the impression (when only the preference over menus is observed) that her strength of regret for choice from menus is smaller.

It is interesting to note that a parallel issue arises in the Gul and Pesendorfer (2001) model of temptation. In their model, it is assumed that the agent is not affected by temptations that lie in the future, and hence she does not suffer from temptation in period 0. If the agent is, in fact, affected by future temptations in period 0, then by only observing the agent's preference over menus, the "strength" of her period 1 temptation will appear to be smaller than is actually the case. The intuition for this misidentification is similar to the intuition behind the misidentification in our regret model: The more the agent is affected by temptations that lie in the future, the more the agent's preference in period 0 already reflects the influence of temptation and, hence, the less she will value commitment in period 0.

In the temptation literature, these issues have been resolved by considering dynamic models. Gul and Pesendorfer (2004) developed a dynamic model of temptation in which it is axiomatized that the agent does not suffer from future temptations, and Noor (2007a, 2007b) axiomatized a dynamic model that allows for future temptations. Similar extensions of our regret model to a dynamic framework may allow for the strength of regret at each stage to be correctly identified. Aside from resolving the identification issue, this type of extension of our model could have interesting implications for scenarios in which the agent makes decisions in stages over time.

Dept. of Economics, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, U.S.A.; tsarver@northwestern.edu.

Manuscript received November, 2006; final revision received October, 2007.

APPENDIX A: SUPPORT FUNCTIONS AND CONVEX SETS

In this section, we establish some preliminary results that will be used in our subsequent proofs. First, we show that there is a natural relationship between the sets in $\mathcal{A}$ and a certain class of continuous functions know as the *support functions*. Dekel, Lipman, and Rustichini (2001) used this relationship in establishing their additive EU representation, and some of their intermediate results regarding support functions will be relevant for establishing our representation theorem. Therefore, we also discuss those results in this section.

Before proceeding, define $\mathcal{A}^c \subset \mathcal{A}$ to be the set of convex menus. Then, for all $A \in \mathcal{A}$, we have $\text{co}(A) \in \mathcal{A}^c$, where $\text{co}(A)$ denotes the convex hull of A. It will be useful to establish our representations first on this set of convex menus. The following axiom will then allow us to easily extend our representation from $\mathcal{A}^c$ to $\mathcal{A}$:

AXIOM 9—Indifference to Randomization (IR): *For every* $A \in \mathcal{A}$, $A \sim \text{co}(A)$.

It is proved in Lemma 6 of the Supplemental material to DLRS (2007) that any preference that satisfies weak order and independence also satisfies IR. Thus the preferences we consider all satisfy IR. Therefore, $A \sim \text{co}(A)$ and so the utility of A can be defined to be that of $\text{co}(A)$. Note that for any $u \in \mathbb{R}^Z$ (i.e., any expected-utility function) and any $A \in \mathcal{A}$, we have

$$\max_{p \in A} p \cdot u = \max_{p \in \text{co}(A)} p \cdot u.$$

Thus if we establish our representations on $\mathcal{A}^c$ and apply the same functional form to $\mathcal{A}$, then the resulting function represents $\succsim$ on $\mathcal{A}$.

We now associate each $A \in \mathcal{A}^c$ with a continuous function. As in Section 2.1, let $\mathcal{U} = \{u \in \mathbb{R}^Z : \sum_{z\in Z} u_z = 0 \text{ and } \sum_{z\in Z} u_z^2 = 1\}$ be the set of *normalized (nonconstant) expected-utility functions* on $\Delta(Z)$. Let $C(\mathcal{U})$ be the set of all continuous functions on $\mathcal{U}$. When endowed with the supremum norm $\|\cdot\|_\infty$, $C(\mathcal{U})$ is a Banach space. Define an order $\geq$ on $C(\mathcal{U})$ by $f \geq g$ if $f(u) \geq g(u)$ for all $u \in \mathcal{U}$. We can map $\mathcal{A}^c$ into $C(\mathcal{U})$ by $A \mapsto \sigma_A$, where σ_A is defined for $u \in \mathcal{U}$ by

$$\sigma_A(u) = \max_{p\in A} p \cdot u.$$

The function σ_A is called the *support function* of A. For a more complete introduction to support functions, see Rockafellar (1970) or Schneider (1993). The following lemma lists some of the important properties of the mapping $A \mapsto \sigma_A$. Note that $\|\cdot\|$ denotes the Euclidean norm on $\Delta(Z)$.

LEMMA 6: *For any $A, B \in \mathcal{A}^c$:*

1. $A \subset B \iff \sigma_A \leq \sigma_B$.
2. $\sigma_{\alpha A+(1-\alpha)B} = \alpha\sigma_A + (1-\alpha)\sigma_B$.
3. *If $p^* = (1/|Z|, \ldots, 1/|Z|)$ and $C = \{p \in \Delta(Z) : \|p - p^*\| \leq \varepsilon\}$ for some $\varepsilon \in [0, 1/|Z|]$, then $\sigma_C = \varepsilon$. In particular, $\sigma_{\{p^*\}} = 0$.*

PROOF: *Parts 1 and 2*. These are standard results that can be found in Rockafellar (1970) or Schneider (1993).[23] For instance, in Schneider (1993), part 1 can be found on page 37 and part 2 follows from Theorem 1.7.5.

Part 3. Note that $p^* \cdot u = 0$ for any $u \in \mathcal{U}$, which follows from the requirement that $\sum_z u_z = 0$. Therefore, for any $u \in \mathcal{U}$ and $p \in C$, using the Cauchy–Schwarz inequality and the fact that $\|u\| = 1$, we have

$$p \cdot u = (p - p^*) \cdot u \leq \|p - p^*\| \cdot \|u\| = \|p - p^*\| \leq \varepsilon.$$

Now consider any $u \in \mathcal{U}$ and define $p = p^* + \varepsilon u$. For any $z \in Z$, we have $|p_z - p_z^*| \leq \|p - p^*\| \leq 1/|Z|$, which implies $p \in \Delta(Z)$. Clearly, we also have $\|p - p^*\| = \varepsilon$ and, hence, $p \in C$. Since $\|u\| = 1$, we have

$$p \cdot u = p^* \cdot u + \varepsilon\|u\| = \varepsilon.$$

Therefore, $\sigma_C(u) = \max_{p\in C}(p \cdot u) = \varepsilon$ for all $u \in \mathcal{U}$. *Q.E.D.*

Let $\Sigma = \{\sigma_A \in C(\mathcal{U}) : A \in \mathcal{A}^c\}$ be the set of support functions. The following representation result was proved in Dekel, Lipman, and Rustichini (2001) and the Supplemental material to DLRS (2007)[24]:

[23] The standard setting for support functions is the set of nonempty closed and convex subsets of $\mathbb{R}^n$. However, by imposing our normalizations on the domain of the support functions $\mathcal{U}$, the standard results are easily adapted to our setting of nonempty closed and convex subsets of $\Delta(Z)$.

[24] $W : \Sigma \to \mathbb{R}$ is Lipschitz continuous if there exists a constant κ such that $W(\sigma) - W(\sigma') \leq \kappa\|\sigma - \sigma'\|_\infty$ for all $\sigma, \sigma' \in \Sigma$, where $\|\cdot\|_\infty$ denotes the sup norm. Dekel, Lipman, and Rustichini

LEMMA 7: *If the preference $\succsim$ satisfies weak order, strong continuity, and independence, then there exists a Lipschitz continuous (with respect to the sup norm) linear functional $W : \Sigma \to \mathbb{R}$ such that for all $A, B \in \mathcal{A}^c$,*

$$W(\sigma_A) \geq W(\sigma_B) \iff A \succsim B.$$

Hence we have a representation for $\succsim$ on $\mathcal{A}^c$. This will serve as the starting point of the proof of our representation theorem in Appendix C.1. Dekel, Lipman, and Rustichini (2001) proceeded by showing that the linear functional W can be replaced with an integral over $\mathcal{U}$, yielding an additive EU representation as in Equation (5). However, before making this transformation, we will want to show that the dominance axiom implies that the functional W satisfies some additional properties, which will allow us to obtain our more specific representation.

We now discuss how the linear functional W will be transformed into an integral. The Riesz representation theorem states that a continuous linear functional on $C(\mathcal{U})$ can be written as an integral with respect to some measure. However, we are not yet able to apply the Riesz representation theorem since W is only defined on $\Sigma \subset C(\mathcal{U})$; we must first extend W to a continuous linear functional on $C(\mathcal{U})$. The following lemma shows that W has a unique continuous linear extension to $C(\mathcal{U})$.

LEMMA 8: *Any Lipschitz continuous linear functional $W : \Sigma \to \mathbb{R}$ has a unique continuous linear extension to $C(\mathcal{U})$.*

See Lemma 11 in the Supplemental material to DLRS (2007) for the proof.

We now give an outline of the construction used in DLRS (2007) because it will be important in subsequent proofs. When we translate the dominance axiom on $\succsim$ into properties of the functional W, we will need to show that these properties hold for the extension of W to $C(\mathcal{U})$.

First, define

$$H = \bigcup_{s \geq 0} s\Sigma = \{s\sigma \in C(\mathcal{U}) : s \geq 0 \text{ and } \sigma \in \Sigma\},$$

$$H^* = H - H = \{f \in C(\mathcal{U}) : f = f_1 - f_2 \text{ for some } f_1, f_2 \in H\}.$$

The following important properties of H^* are established in Lemma 10 of the Supplemental material to DLRS (2007): (i) For any $f \in H^*$, there exist $\sigma^1, \sigma^2 \in \Sigma$ and $s > 0$ such that $f = s(\sigma^1 - \sigma^2)$, and (ii) H^* is a dense linear subspace of $C(\mathcal{U})$. Using the first property, extend W to H^* by linearity. That is, if $f \in H^*$,

(2001) assumed the standard continuity axiom instead of strong continuity and obtained a continuous functional W. DLRS (2007) showed that if the preference satisfies the strong continuity axiom, then this functional is Lipschitz continuous.

then there exist $\sigma^1, \sigma^2 \in \Sigma$ and $s > 0$ such that $f = s(\sigma^1 - \sigma^2)$, so define the extension $\hat{W}: H^* \to \mathbb{R}$ by $\hat{W}(f) = s[W(\sigma^1) - W(\sigma^2)]$. DLRS (2007) proved that this extension is well defined and that it is the unique linear extension of W to H^*. In addition, they proved that the Lipschitz continuity of W on Σ implies that $\hat{W}$ is bounded. Therefore, apply the Hahn–Banach theorem (see Theorem 4, p. 223, of Royden (1988)) to conclude that $\hat{W}$ can be extended to a continuous linear functional $\bar{W}: C(\mathcal{U}) \to \mathbb{R}$. Since H^* is dense in $C(\mathcal{U})$, it follows that $\bar{W}$ is the unique continuous linear extension of $\hat{W}$ to $C(\mathcal{U})$.

APPENDIX B: DECOMPOSING BOUNDED LINEAR FUNCTIONALS

Our representation centers around how the measure from the additive EU representation is divided into a positive and negative measure. Equivalently, we are interested in how the continuous linear functional described in Appendix A decomposes into the difference of two positive linear functionals. The following lemma offers a construction of these two functionals.

LEMMA 9: *Suppose W is a bounded linear functional on $C(\mathcal{U})$. For each $f \in C(\mathcal{U})$ such that $f \geq 0$, define*

$$W^+(f) = \sup_{0 \leq g \leq f} W(g).$$

For arbitrary $f \in C(\mathcal{U})$, since there exists $M \geq 0$ such that $f + M \geq 0$, define

$$W^+(f) = W^+(f + M) - W^+(M).$$

Then W^+ is a well-defined (i.e., the choice of M is unimportant), positive linear functional. If we define $W^- = W^+ - W$, then W^- is also a positive linear functional.

See Section 13.5 of Royden (1988) for the proof.

APPENDIX C: PROOFS

C.1. *Proof of Theorem 1*

The necessity of the axioms was established in Lemmas 1 and 2. We now prove that if $\succsim$ satisfies weak order, strong continuity, independence, and dominance, then it has a regret representation. We begin by making a few manipulations of our axioms. First, we note the relationship between strong continuity and the following continuity axiom:

AXIOM 10—Continuity: *For all $A \in \mathcal{A}$, the sets $\{B \in \mathcal{A} : B \succsim A\}$ and $\{B \in \mathcal{A} : B \precsim A\}$ are closed (in the Hausdorff metric topology).*

It is easily verified that continuity implies vNM continuity. Continuity and L continuity are not directly comparable, but when combined with our other axioms, strong continuity is stronger than continuity:

LEMMA 10: *If* $\succsim$ *satisfies weak order, strong continuity, and independence, then it satisfies continuity.*

PROOF: This result follows from Theorem 2 and the continuity of the additive EU representation, which is proved in DLRS (2007).[25] *Q.E.D.*

The following stronger version of the dominance axiom will be important:

AXIOM 11—Strong Dominance: *If for every* $q \in B$ *there exists a* $p \in A$ *such that* $\{p\} \succsim \{q\}$, *then* $A \succsim A \cup B$.

Given Lemma 10, the following lemma establishes that in the presence of our other axioms, dominance and strong dominance are equivalent:

LEMMA 11: *Suppose* $\succsim$ *satisfies weak order and continuity. Then* $\succsim$ *satisfies dominance if and only if it satisfies strong dominance.*

PROOF: Obviously, strong dominance implies dominance. We want to show that dominance implies strong dominance. Suppose that $\succsim$ satisfies dominance, and that A and B are such that for all $q \in B$, the exists $p \in A$ such that $\{p\} \succsim \{q\}$.

Note that since $\Delta(Z)$ is separable, we can choose a countable dense subset of B, say $B^* = \{q_1, q_2, \dots\}$. Define the sequence $\{B_n\}$ by $B_n = \{q_1, \dots, q_n\}$, so that $B_n \subset B_{n+1}$ for all n and $B^* = \bigcup_{n=1}^{\infty} B_n$. Then it is a standard result that $B_n \to \overline{B^*} = B$ in the Hausdorff metric topology. Now $q_1 \in B$ and, hence, there exists $p \in A$ such that $\{p\} \succsim \{q_1\}$. Thus dominance implies $A \succsim A \cup \{p_1\} = A \cup B_1$. Similarly, there exists $p \in A$ with $\{p\} \succsim \{q_2\}$, so dominance implies $A \cup B_1 \succsim A \cup B_1 \cup \{q_2\} = A \cup B_2$. Therefore, by transitivity $A \succsim A \cup B_2$. A simple induction argument extends this line of reasoning to show that $A \succsim A \cup B_n$ for all $n \in \mathbb{N}$.

Therefore, for each $n \in \mathbb{N}$, $A \cup B_n \in \{C \in \mathcal{A} : C \precsim A\}$. Since the lower contour sets are closed by the continuity assumption and $A \cup B_n \to A \cup B$, we have $A \succsim A \cup B$, as desired. *Q.E.D.*

From Lemma 7, we know that there exists a Lipschitz continuous linear functional $W : \Sigma \to \mathbb{R}$ such that for all $A, B \in \mathcal{A}^c$, $W(\sigma_A) \geq W(\sigma_B)$ if and only if

[25]The necessity of strong continuity was also discussed by DLRS (2007), and they showed by counterexample that weak order, continuity, and independence alone are not sufficient to guarantee the existence of an additive EU representation.

$A \succsim B$. Define a function $v : \Delta(Z) \to \mathbb{R}$ by $v(p) = W(\sigma_{\{p\}})$. By Lemma 6 and the linearity of W,

$$\begin{aligned} v(\alpha p + (1-\alpha)q) &= W(\sigma_{\{\alpha p + (1-\alpha)q\}}) = W(\alpha\sigma_{\{p\}} + (1-\alpha)\sigma_{\{q\}}) \\ &= \alpha W(\sigma_{\{p\}}) + (1-\alpha)W(\sigma_{\{q\}}) \\ &= \alpha v(p) + (1-\alpha)v(q). \end{aligned}$$

Hence, v is affine. By the definition of $\mathcal{U}$ (see Equation (1)), it contains the normalization of any affine function on $\Delta(Z)$. Therefore, there exist $\bar{u} \in \mathcal{U}$, $\bar{\alpha} \geq 0$, and $\bar{\beta} \in \mathbb{R}$ such that $v = \bar{\alpha}\bar{u} + \bar{\beta}$. If we take $p^* = (1/|Z|, \dots, 1/|Z|)$, then $\sigma_{\{p^*\}} = 0$ by Lemma 6 and, hence, $v(p^*) = W(0) = 0$. We also have $p^* \cdot \bar{u} = 0$ and thus $\bar{\beta} = 0$. Therefore, $v = \bar{\alpha}\bar{u}$.

We now use the strong dominance to obtain a particular property of W. Suppose $\bar{\alpha}\sigma_A(\bar{u}) = \bar{\alpha}\sigma_B(\bar{u})$ and $\sigma_A \leq \sigma_B$. Choose any $p \in \arg\max_{q \in A} q \cdot \bar{u}$. Then, for any $q \in B$,

$$\bar{\alpha}(q \cdot \bar{u}) \leq \bar{\alpha}\sigma_B(\bar{u}) = \bar{\alpha}\sigma_A(\bar{u}) = \bar{\alpha}(p \cdot \bar{u}),$$

which implies $v(p) \geq v(q)$ or, equivalently, $\{p\} \succsim \{q\}$. Strong dominance therefore implies $A \succsim A \cup B$. By Lemma 6, $\sigma_A \leq \sigma_B$ implies $A \subset B$, so $A \cup B = B$. Hence, $A \succsim B$, so we have $W(\sigma_A) \geq W(\sigma_B)$. We now want to generalize this property to all functions in $C(\mathcal{U})$. We say that $W : C(\mathcal{U}) \to \mathbb{R}$ has the *domination property* if for all $f, g \in C(\mathcal{U})$,[26]

$$\bar{\alpha}f(\bar{u}) = \bar{\alpha}g(\bar{u}) \quad \text{and} \quad f \leq g \quad \Longrightarrow \quad W(f) \geq W(g).$$

We have just demonstrated that W has the domination property on Σ. We now show that the extension of W to $C(\mathcal{U})$ described in Lemma 8 also has the domination property.

LEMMA 12: *If W is a Lipschitz continuous linear functional on Σ that has the domination property, then the unique continuous linear extension of W to $C(\mathcal{U})$ also has the domination property.*

PROOF: We will abuse notation slightly and let W also denote the extension of W to $C(\mathcal{U})$. We first show that the domination property holds on the set H^* defined in Appendix A. As noted in that section, H^* is a dense linear subspace of $C(\mathcal{U})$. Consider any $f, g \in H^*$ such that $\bar{\alpha}f(\bar{u}) = \bar{\alpha}g(\bar{u})$ and $f \leq g$. Since H^* is a linear subspace, $h = g - f \in H^*$. Also, note that $\bar{\alpha}h(\bar{u}) = 0$ and $h \geq 0$. Proving the domination property holds on H^* thus only requires that we show for $h \in H^*$, that $\bar{\alpha}h(\bar{u}) = 0$ and $h \geq 0$ implies $W(h) \leq 0$. Consider such an h.

[26]Note that if $\bar{\alpha} > 0$, then we can drop it from these conditions. We include $\bar{\alpha}$ so as to consider the two cases, $\bar{\alpha} = 0$ and $\bar{\alpha} > 0$, simultaneously.

As discussed in Appendix A, there exist $\sigma^1, \sigma^2 \in \Sigma$ and $s > 0$ such that $h = s(\sigma^1 - \sigma^2)$. The assumed properties of h imply $\bar{\alpha}\sigma^1(\bar{u}) = \bar{\alpha}\sigma^2(\bar{u})$ and $\sigma^1 \geq \sigma^2$. Since the domination property holds on Σ, this implies $W(\sigma^1) \leq W(\sigma^2)$. The linearity of W therefore implies $W(h) \leq 0$.

We now prove that the domination property holds on $C(\mathcal{U})$. Since $C(\mathcal{U})$ is a linear space, it again suffices to show that for $h \in C(\mathcal{U})$, $\bar{\alpha}h(\bar{u}) = 0$ and $h \geq 0$ implies $W(h) \leq 0$. Consider such an h. Since H^* is dense in $C(\mathcal{U})$, there exists a sequence $\{h_n\} \subset H^*$ such that $h_n \to h$ in the sup-norm topology. Without loss of generality, we can assume that $\bar{\alpha}h_n(\bar{u}) = 0$ and $h_n \geq 0$ for all n. To see this, consider the sequence $\{h'_n\} \subset H^*$ defined by

$$h'_n = \begin{cases} h_n, & \text{if } \bar{\alpha} = 0, \\ h_n - h_n(\bar{u}), & \text{if } \bar{\alpha} > 0. \end{cases}$$

We claim that $\bar{\alpha}h'_n(\bar{u}) = 0$ and $h'_n \to h$. If $\bar{\alpha} = 0$, then this is obvious. If $\bar{\alpha} > 0$, then $\bar{\alpha}h'_n(\bar{u}) = \bar{\alpha}[h_n(\bar{u}) - h_n(\bar{u})] = 0$ and

$$\|h'_n - h\|_\infty = \|(h_n - h_n(\bar{u})) - h\|_\infty \leq \|h_n - h\|_\infty + \|h_n(\bar{u})\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the sup norm on $C(\mathcal{U})$. Since $\bar{\alpha} > 0$, $h(\bar{u}) = 0$, so $h_n \to h$ implies $\|h_n(\bar{u})\|_\infty \to 0$. Clearly, $\|h_n - h\|_\infty \to 0$ because $h_n \to h$, and, therefore, $h'_n \to h$. Now consider the sequence $\{h''_n\} \subset H^*$ defined by $h''_n(u) = \max\{h'_n(u), 0\}$.[27] Since $h \geq 0$, it must be that $\|h''_n - h\|_\infty \leq \|h'_n - h\|_\infty$ and, therefore, $h''_n \to h$. Now, for all $n \in \mathbb{N}$, we have $h''_n \in H^*$, $\bar{\alpha}h''_n(\bar{u}) = 0$, and $h''_n \geq 0$. Hence, the domination property on H^* implies that $W(h''_n) \leq W(0) = 0$. By the continuity of W on $C(\mathcal{U})$, this implies $W(h) \leq 0$, which completes the proof. *Q.E.D.*

So far, we have shown that there exists a continuous linear functional W on $C(\mathcal{U})$ that has the domination property, and such that $W(\sigma_A) \geq W(\sigma_B)$ if and only if $A \succsim B$. Now, apply the technique described in Appendix B to write W as the difference of two positive linear functionals, $W = W^+ - W^-$. The domination property has an important implication for W^+.

LEMMA 13: *Suppose W has the domination property on $C(\mathcal{U})$. Then, for any $\sigma^1, \sigma^2 \in \Sigma$, $\bar{\alpha}\sigma^1(\bar{u}) \geq \bar{\alpha}\sigma^2(\bar{u}) \Longrightarrow W^+(\sigma^1) \geq W^+(\sigma^2)$.*

PROOF: First, consider any $f_1, f_2 \in C(\mathcal{U})$ such that $f_1, f_2 \geq 0$ and $\bar{\alpha}f_1(\bar{u}) \geq \bar{\alpha}f_2(\bar{u})$. Lemma 9 states that

$$W^+(f_2) = \sup_{0 \leq g \leq f_2} W(g).$$

[27]The functions in this sequence are elements of H^* because $0 \in H^*$ and H^* is a vector lattice, which implies it contains the pointwise maximum of any two of its elements. A proof that H^* is a vector lattice can be found in Lemma 11 of Dekel, Lipman, and Rustichini (2001).

Choose an arbitrary $g \in C(\mathcal{U})$ such that $0 \leq g \leq f_2$. Then define h by $h(u) = \min\{g(u), f_1(u)\}$. Clearly, $h \in C(\mathcal{U})$ and $0 \leq h \leq f_1$. Now $\bar{\alpha}g(\bar{u}) \leq \bar{\alpha}f_2(\bar{u}) \leq \bar{\alpha}f_1(\bar{u})$, and, hence, $\bar{\alpha}h(\bar{u}) = \bar{\alpha}g(\bar{u})$. Since we also have $h \leq g$, the domination property implies $W(g) \leq W(h) \leq W^+(f_1)$. The choice of $0 \leq g \leq f_2$ was arbitrary, so therefore $W^+(f_1) \geq W^+(f_2)$.

Now, consider any $\sigma^1, \sigma^2 \in \Sigma$ such that $\bar{\alpha}\sigma^1(\bar{u}) \geq \bar{\alpha}\sigma^2(\bar{u})$. Since each of these functions is bounded, there exists an $M \geq 0$ such that $\sigma^1 + M, \sigma^2 + M \geq 0$. Since $\bar{\alpha}(\sigma^1 + M)(\bar{u}) \geq \bar{\alpha}(\sigma^2 + M)(\bar{u})$, the results of the previous paragraph imply $W^+(\sigma^1 + M) \geq W^+(\sigma^2 + M)$. Then, by the definition of W^+,

$$\begin{aligned} W^+(\sigma^1) &= W^+(\sigma^1 + M) - W^+(M) \geq W^+(\sigma^2 + M) - W^+(M) \\ &= W^+(\sigma^2), \end{aligned}$$

which completes the proof. *Q.E.D.*

LEMMA 14: *Let X be a nonempty convex subset of a vector space, and let u_1 and u_2 be two affine functionals on X. There exist $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $u_2 = \alpha u_1 + \beta$ if and only if $u_1(x) \geq u_1(y) \Longrightarrow u_2(x) \geq u_2(y)$ for all $x, y \in X$.*

This is a standard result. See, for example, Corollary B.3 of Ghirardato, Maccheroni, and Marinacci (2004).

It is easily verified that the mapping $\sigma \mapsto \bar{\alpha}\sigma(\bar{u})$ defines a linear functional on Σ. Since W^+ is also linear, Lemmas 13 and 14 imply there exist $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $W^+(\sigma) = \alpha\bar{\alpha}\sigma(\bar{u}) + \beta$. Letting $\sigma = 0$, we see that $\beta = 0$. Thus for all $\sigma \in \Sigma$, we have

$$W(\sigma) = \alpha\bar{\alpha}\sigma(\bar{u}) - W^-(\sigma). \tag{12}$$

Although it may not be immediately obvious, we have almost obtained a regret representation. The remaining steps are completed in the following lemma.

LEMMA 15: *If W satisfies the condition given in Equation (12), then $\succsim$ has a regret representation.*

PROOF: First, note that we can write Equation (12) as

$$W(\sigma) = (2 + \alpha)\bar{\alpha}\sigma(\bar{u}) - [2\bar{\alpha}\sigma(\bar{u}) + W^-(\sigma)].$$

This step is only necessary to ensure that the coefficient of $\bar{\alpha}\sigma(\bar{u})$ is greater than unity. We will see shortly why this is important.

It is straightforward to show that the mapping $f \mapsto f(\bar{u})$ is a positive linear functional on $C(\mathcal{U})$. Therefore, if we define $T: C(\mathcal{U}) \to \mathbb{R}$ by $T(f) = 2\bar{\alpha}f(\bar{u}) + W^-(f)$ for $f \in C(\mathcal{U})$, then T is also a positive linear functional on $C(\mathcal{U})$. Therefore, the Riesz–Markov theorem (see Section 13.4 of Royden

(1988)) implies there is a finite positive Borel measure μ on $\mathcal{U}$ such that for every $f \in C(\mathcal{U})$,

$$T(f) = \int_{\mathcal{U}} f(u)\mu(du).$$

Since $\sigma_A(u) = \max_{p\in A} p \cdot u = \max_{p\in A} u(p)$, for $A \in \mathcal{A}^c$ we have

$$W(\sigma_A) = (2+\alpha)\bar{\alpha}\sigma_A(\bar{u}) - \int_{\mathcal{U}} \max_{p\in A} u(p)\mu(du). \tag{13}$$

Recall that $\bar{\alpha}$ and $\bar{u}$ are defined so that $W(\sigma_{\{p\}}) = v(p) = \bar{\alpha}(p \cdot \bar{u})$ for any $p \in \Delta(Z)$. We also have $\sigma_{\{p\}}(u) = p \cdot u$; hence Equation (13) implies $(1 + \alpha)\bar{\alpha}(p \cdot \bar{u}) = \int_{\mathcal{U}} u(p)\mu(du)$. If we let $K = 1 + \alpha$, then Equation (13) can be written as

$$\begin{aligned} W(\sigma_A) &= (1+K)\bar{\alpha} \max_{p\in A}(p \cdot \bar{u}) - \int_{\mathcal{U}} \max_{q\in A} u(q)\,\mu(du) \\ &= \max_{p\in A}\biggl[\frac{1}{K}\int_{\mathcal{U}} u(p)\mu(du) + \int_{\mathcal{U}} u(p)\mu(du) \\ &\qquad - \int_{\mathcal{U}} \max_{q\in A} u(q)\mu(du)\biggr] \\ &= \frac{1}{K}\max_{p\in A}\int_{\mathcal{U}}\Bigl[u(p) - K\Bigl[\max_{q\in A} u(q) - u(p)\Bigr]\Bigr]\mu(du). \end{aligned} \tag{14}$$

It is clear from Equation (14) that (μ, K) is a regret representation for $\succsim$ if μ is a probability measure. If μ is not a probability measure, then it can be normalized as follows: If $\mu(\mathcal{U}) > 0$, then let $\hat{\mu} = \mu/\mu(\mathcal{U})$, and $(\hat{\mu}, K)$ is a regret representation for $\succsim$. If $\mu(\mathcal{U}) = 0$, then this type of renormalization is not possible. However, in this case, it is obvious from Equation (14) that $W(\sigma_A) = 0$ for all $A \in \mathcal{A}^c$. Thus $\succsim$ must be trivial in that $A \sim B$ for all $A, B \in \mathcal{A}$. Therefore, if we fix any $u \in \mathcal{U}$ and take a probability measure $\hat{\mu}$ such that $\hat{\mu}(\{u\}) = \hat{\mu}(\{-u\}) = 1/2$, then $(\hat{\mu}, 0)$ is a regret representation for this preference. *Q.E.D.*

C.2. *Proof of Lemma* 4

Clearly, if $\succsim$ satisfies singleton nontriviality, then $\|v\| \neq 0$. To see the opposite implication, suppose $\|v\| \neq 0$. Then, treating v as a vector in $\mathbb{R}^Z$, there must be some $\bar{z} \in Z$ such that $v_{\bar{z}} \neq 0$. Without loss of generality, suppose $v_{\bar{z}} > 0$. By Equation (8), $v_{\bar{z}} > 0$ implies there exists some $\hat{z} \in Z$ such that $v_{\hat{z}} < 0$. Let $p = \delta_{\bar{z}}$ (i.e., p is the lottery that yields $\bar{z}$ with probability 1) and $q = \delta_{\hat{z}}$. Then

$$v(p) = v_{\bar{z}} > v_{\hat{z}} = v(q),$$

so $\succsim$ satisfies singleton nontriviality.

It remains only to prove that $\|v\| \neq 0 \Longrightarrow \frac{v}{\|v\|} \in \mathcal{U}$. By Equation (8), $\sum_z \frac{v_z}{\|v\|} = 0$. Clearly, we also have $\|\frac{v}{\|v\|}\| = 1$ and, hence, $\frac{v}{\|v\|} \in \mathcal{U}$. *Q.E.D.*

C.3. *Proof of Theorem* 4

The following lemmas will be useful at several points in the proof of this theorem:

LEMMA 16: *If* (μ, K) *and* (μ', K') *represent the same preference and* $r' = \alpha r$, *then* $K'(1 - \|v'\|) = \alpha K(1 - \|v\|)$.

PROOF: Define $p^* = (1/|Z|, \dots, 1/|Z|)$ and let $C = \{p \in \Delta(Z) : \|p - p^*\| \leq \varepsilon\}$ for some $\varepsilon \in (0, 1/|Z|)$. We claim that $r(C) = K\varepsilon(1 - \|v\|)$ and $r'(C) = K'\varepsilon(1 - \|v'\|)$. By Lemma 6, we have $\max_{p\in C}(p \cdot u) = \sigma_C(u) = \varepsilon$ for all $u \in \mathcal{U}$. Therefore,

$$\begin{aligned} r(C) &= K\left[\int_{\mathcal{U}} \max_{p\in C}(p \cdot u)\mu(du) - \max_{p\in C} \int_{\mathcal{U}} (p \cdot u)\mu(du)\right] \\ &= K\left[\varepsilon - \max_{p\in C} v(p)\right]. \end{aligned}$$

If $\|v\| = 0$, then we have $r(C) = K\varepsilon = K\varepsilon(1 - \|v\|)$, as desired. If $\|v\| > 0$, then by Lemma 4, $\frac{v}{\|v\|} \in \mathcal{U}$. Therefore, Lemma 6 implies

$$\max_{p\in C} v(p) = \|v\| \max_{p\in C}\left(p \cdot \frac{v}{\|v\|}\right) = \|v\|\sigma_C\left(\frac{v}{\|v\|}\right) = \varepsilon\|v\|$$

and, hence, $r(C) = K\varepsilon(1 - \|v\|)$. A similar argument shows $r'(C) = K'\varepsilon(1 - \|v'\|)$. Since $r' = \alpha r$, we have $K'(1 - \|v'\|) = \alpha K(1 - \|v\|)$. *Q.E.D.*

LEMMA 17: *If* (μ, K) *represents a preference that has nontrivial regret*, *then* $\|v\| < 1$.

PROOF: If $\|v\| = 0$, then the assertion is obviously true, so suppose $\|v\| > 0$. We will first prove that $\|v\| \leq 1$ with equality if and only if $\mu(\{v/\|v\|\}) = 1$. To see this claim is true, note that for any $u \in \mathcal{U}$, the Cauchy–Schwarz inequality implies $v \cdot u \leq \|v\|\|u\| = \|v\|$ with equality if and only if $\frac{v}{\|v\|} = u$. Therefore,

$$\|v\| = \frac{v}{\|v\|} \cdot v = \int_{\mathcal{U}} \left(\frac{v}{\|v\|} \cdot u\right)\mu(du) \leq \int_{\mathcal{U}} 1\mu(du) = 1$$

with equality if and only if $\mu(\{v/\|v\|\}) = 1$. Note also that if $\mu(\{u\}) = 1$ for some $u \in \mathcal{U}$, then the preference induced by this representation must have trivial regret. Thus if the preference induced by (μ, K) has nontrivial regret, then $\|v\| < 1$. *Q.E.D.*

Theorem 4*, Part* 1*—if*. First, note that since $\succsim$ has nontrivial regret, Lemma 17 implies $1 - \|v\| > 0$ and hence $\frac{1}{1-\|v\|} > 0$. Suppose $\mu' = \alpha\mu + (1-\alpha)\delta_{v/\|v\|}$ and $K' = [1 + \frac{1-\alpha}{\alpha\|v\|}]K$ for some $0 < \alpha < \frac{1}{1-\|v\|}$. We will prove that (μ', K') and (μ, K) represent the same preference by appealing to Theorem 3. That is, we will find $\bar{\alpha} > 0$ such that $v' = \bar{\alpha}v$ and $r' = \bar{\alpha}r$. First, note that for any $p \in \Delta(Z)$,

$$\begin{aligned} v'(p) &= \int_{\mathcal{U}} (p \cdot u)\mu'(du) \\ &= \alpha \int_{\mathcal{U}} (p \cdot u)\mu(du) + (1-\alpha)\int_{\mathcal{U}} (p \cdot u)\delta_{v/\|v\|}(du) \\ &= \left[\alpha + \frac{1-\alpha}{\|v\|}\right] v(p). \end{aligned}$$

Thus, take $\bar{\alpha} = \alpha + \frac{1-\alpha}{\|v\|}$. Since $\alpha < \frac{1}{1-\|v\|}$, it follows that $\bar{\alpha} > 0$.

It remains only to prove that $r' = \bar{\alpha}r$. For any $A \in \mathcal{A}$,

$$\begin{aligned} r'(A) &= K'\left[\int_{\mathcal{U}} \max_{p\in A}(p \cdot u)\mu'(du) - \max_{p\in A}\int_{\mathcal{U}} (p \cdot u)\mu'(du)\right] \\ &= K'\left[\alpha \int_{\mathcal{U}} \max_{p\in A}(p \cdot u)\mu(du) + (1-\alpha)\int_{\mathcal{U}} \max_{p\in A}(p \cdot u)\delta_{v/\|v\|}(du) \right. \\ &\qquad \left. - \max_{p\in A} v'(p)\right] \\ &= K'\left[\alpha \int_{\mathcal{U}} \max_{p\in A}(p \cdot u)\mu(du) + \frac{1-\alpha}{\|v\|}\max_{p\in A} v(p) \right. \\ &\qquad \left. - \left[\alpha + \frac{1-\alpha}{\|v\|}\right]\max_{p\in A} v(p)\right] \\ &= K'\alpha\left[\int_{\mathcal{U}} \max_{p\in A}(p \cdot u)\mu(du) - \max_{p\in A} v(p)\right] = \alpha\frac{K'}{K} r(A). \end{aligned}$$

However, $K' = [1 + \frac{1-\alpha}{\alpha\|v\|}]K$ implies $\alpha\frac{K'}{K} = \alpha + \frac{1-\alpha}{\|v\|} = \bar{\alpha}$, as desired.

Theorem 4*, Part* 1*—only if*. We will first find $0 < \alpha < \frac{1}{1-\|v\|}$ such that $K' = [1 + \frac{1-\alpha}{\alpha\|v\|}]K$. By Theorem 3, there exist $\bar{\alpha} > 0$ such that $v' = \bar{\alpha}v$ and $r' = \bar{\alpha}r$. Since

$\|v\| \neq 0$ by singleton nontriviality, this implies $\bar{\alpha} = \frac{\|v'\|}{\|v\|}$. Thus $r' = \bar{\alpha} r = \frac{\|v'\|}{\|v\|} r$ and Lemma 16 implies

$$K'(1 - \|v'\|) = \frac{\|v'\|}{\|v\|} K(1 - \|v\|). \tag{15}$$

By assumption, the preference induced by (μ', K') and (μ, K) has nontrivial regret, so Lemma 17 implies $1 - \|v'\| > 0$ and $1 - \|v\| > 0$. Therefore, Equation (15) can be written as $K' = \frac{\|v'\|}{\|v\|}(\frac{1-\|v\|}{1-\|v'\|})K$. Now define $\alpha = \frac{1-\|v'\|}{1-\|v\|}$ and note that $0 < \alpha < \frac{1}{1-\|v\|}$. It is easily verified that $\frac{\|v'\|}{\|v\|} = \alpha + \frac{1-\alpha}{\|v\|}$ and, hence,

$$K' = \frac{\|v'\|}{\|v\|} \frac{1}{\alpha} K = \left(\alpha + \frac{1-\alpha}{\|v\|}\right) \frac{1}{\alpha} K = \left[1 + \frac{1-\alpha}{\alpha \|v\|}\right] K. \tag{16}$$

We now show that $\mu' = \alpha\mu + (1-\alpha)\delta_{v/\|v\|}$. Let $A \in \mathcal{A}^c$ be arbitrary. Since $r' = \frac{\|v'\|}{\|v\|} r$, we have

$$\begin{aligned} &K'\left[\int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu'(du) - \max_{p \in A} v'(p)\right] \\ &\quad = \frac{\|v'\|}{\|v\|} K\left[\int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu(du) - \max_{p \in A} v(p)\right]. \end{aligned}$$

By the first part of Equation (16), $\alpha = \frac{\|v'\|}{\|v\|} \frac{K}{K'}$ and, therefore, we have

$$\begin{aligned} &\int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu'(du) - \max_{p \in A} v'(p) \\ &\quad = \alpha \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu(du) - \alpha \max_{p \in A} v(p). \end{aligned}$$

Recall that $v' = \frac{\|v'\|}{\|v\|} v$. It is also easily verified that $\|v'\| - \alpha\|v\| = 1 - \alpha$. Therefore, the above equation implies

$$\begin{aligned} &\int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu'(du) \\ &\quad = \alpha \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu(du) + (\|v'\| - \alpha\|v\|) \max_{p \in A}\left(p \cdot \frac{v}{\|v\|}\right) \\ &\quad = \alpha \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu(du) + (1-\alpha) \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\delta_{v/\|v\|}(du) \\ &\quad = \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)[\alpha\mu + (1-\alpha)\delta_{v/\|v\|}](du). \end{aligned}$$

Using the definition of the support function from Appendix A, we have

$$\int_{\mathcal{U}} \sigma_A(u)\mu'(du) = \int_{\mathcal{U}} \sigma_A(u)[\alpha\mu + (1-\alpha)\delta_{v/\|v\|}](du).$$

Given this result, the following lemma proves that $\mu' = \alpha\mu + (1-\alpha)\delta_{v/\|v\|}$:

LEMMA 18: *If ν and ν' are two Borel probability measures on $\mathcal{U}$ and if $\int_{\mathcal{U}} \sigma_A(u)\nu(du) = \int_{\mathcal{U}} \sigma_A(u)\nu'(du)$ for all $A \in \mathcal{A}^c$, then $\nu = \nu'$.*

PROOF: As in Appendix A, let $\Sigma = \{\sigma_A \in C(\mathcal{U}) : A \in \mathcal{A}^c\}$. Define $W : \Sigma \to \mathbb{R}$ by

$$W(\sigma) = \int_{\mathcal{U}} \sigma(u)\nu'(du) - \int_{\mathcal{U}} \sigma(u)\nu(du)$$

for $\sigma \in \Sigma$. By assumption, we have $W(\sigma) = 0$ for all $\sigma \in \Sigma$. It is not difficult to show that W is Lipschitz continuous with Lipschitz constant $\nu'(\mathcal{U}) + \nu(\mathcal{U})$. Therefore, by Lemma 8, W has a unique continuous linear extension to $C(\mathcal{U})$. Obviously, $W_1 : C(\mathcal{U}) \to \mathbb{R}$, defined by $W_1(f) = 0$ for $f \in C(\mathcal{U})$, is a continuous linear extension of W. Another continuous linear extension of W is $W_2 : C(\mathcal{U}) \to \mathbb{R}$, defined for $f \in C(\mathcal{U})$ by

$$W_2(f) = \int_{\mathcal{U}} f(u)\nu'(du) - \int_{\mathcal{U}} f(u)\nu(du).$$

Since the extension of W must be unique, we know that $W_1 = W_2$ or, equivalently,

$$\int_{\mathcal{U}} f(u)\nu'(du) = \int_{\mathcal{U}} f(u)\nu(du) \tag{17}$$

for all $f \in C(\mathcal{U})$. It is a standard result that Equation (17) implies $\nu' = \nu$ (see Theorem 14.1 in Aliprantis and Border (1999)), which completes the proof. *Q.E.D.*

Theorem 4, Part 2—if. Let $\bar{\alpha} = \frac{K'}{K} > 0$. By Lemma 4, $\|v\| = 0$, and since $\mu' = \mu$, this implies $\|v'\| = 0$. Therefore, $v' = 0 = \bar{\alpha}v$. For any $A \in \mathcal{A}$,

$$r'(A) = K' \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu'(du) = \bar{\alpha}K \int_{\mathcal{U}} \max_{p \in A}(p \cdot u)\mu(du) = \bar{\alpha}r(A).$$

Applying Theorem 3, we conclude that (μ', K') also represents $\succsim$.

Theorem 4, Part 2—only if. By Theorem 3, there exists $\bar{\alpha} > 0$ such that $v' = \bar{\alpha} v$ and $r' = \bar{\alpha} r$. Therefore, for any $A \in \mathcal{A}^c$,

$$K' \int_{\mathcal{U}} \max_{p\in A}(p\cdot u)\mu'(du) - K' \max_{p\in A} v'(p)$$
$$= \bar{\alpha} K \int_{\mathcal{U}} \max_{p\in A}(p\cdot u)\mu(du) - \bar{\alpha} K \max_{p\in A} v(p).$$

However, $\|v\| = \|v'\| = 0$ by Lemma 4, and hence $v = v' = 0$. By Lemma 16, $K' = \bar{\alpha} K > 0$. Therefore,

$$\int_{\mathcal{U}} \max_{p\in A}(p\cdot u)\mu'(du) = \int_{\mathcal{U}} \max_{p\in A}(p\cdot u)\mu(du).$$

In the language of support functions, we have

$$\int_{\mathcal{U}} \sigma_A(u)\mu'(du) = \int_{\mathcal{U}} \sigma_A(u)\mu(du)$$

for all $A \in \mathcal{A}^c$. Lemma 18 therefore implies $\mu' = \mu$. *Q.E.D.*

C.4. *Proof of Theorem 5*

If. Suppose Equation (10) holds. Let $A, B \in \mathcal{A}$ be any menus such that A dominates B with respect to $\succsim_1$ and $\succsim_2$, and such that $A \succ_2 A \cup B$.

Since $A \succ_2 A \cup B$, it must be that $\max_{p\in A} \bar{u}(p) < \max_{p\in A\cup B} \bar{u}(p)$ for some $\bar{u} \in \operatorname{supp}(\mu_2)$. It is easily verified that this implies there exists an open neighborhood $E \subset \mathcal{U}$ of $\bar{u}$ such that

$$\max_{p\in A} u(p) < \max_{p\in A\cup B} u(p) \quad \forall u \in E. \tag{18}$$

Since A dominates B with respect to $\succsim_1$ and $\succsim_2$, it must also be that $\max_{p\in A} v_1(p) = \max_{p\in A\cup B} v_1(p)$ and $\max_{p\in A} v_2(p) = \max_{p\in A\cup B} v_2(p)$. Therefore, $\bar{u}$ cannot equal $\frac{v_1}{\|v_1\|}$ or $\frac{v_2}{\|v_2\|}$, so by Equation (10), $\bar{u} \in \operatorname{supp}(\mu_1)$. By the definition of the support of a measure, $\mu_1(E) > 0$ and, hence, Equation (18) implies

$$\int_{\mathcal{U}} \max_{p\in A} u(p)\mu_1(du) < \int_{\mathcal{U}} \max_{p\in A\cup B} u(p)\mu_1(du).$$

Moreover, we have already shown that

$$\max_{p\in A} \int_{\mathcal{U}} u(p)\mu_1(du) = \max_{p\in A} v_1(p)$$
$$= \max_{p\in A\cup B} v_1(p) = \max_{p\in A\cup B} \int_{\mathcal{U}} u(p)\mu_1(du).$$

Finally, since $\succsim_1$ is a nontrivial regret preference, it must be that $K_1 > 0$. Therefore, using the formulation of the regret representation from Equation (4), it follows that $V_1(A) > V_1(A \cup B)$, that is, $A \succ_1 A \cup B$. Thus $\succsim_1$ is more regret prone than $\succsim_2$.

Only if. We prove by establishing the contrapositive. First, note that by Lemma 4, singleton nontriviality implies $\|v_1\| > 0$, $\|v_2\| > 0$, and $\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \in \mathcal{U}$. Now, suppose there exists $\bar{u} \in \text{supp}(\mu_2)$ such that $\bar{u} \notin \text{supp}(\mu_1) \cup \{\frac{v_1}{\|v_1\|}\} \cup \{\frac{v_2}{\|v_2\|}\}$. We will show that this implies that $\succsim_1$ is not more regret prone than $\succsim_2$.

Since the support of a measure is closed, there exists an open neighborhood $E \subset \mathcal{U}$ of $\bar{u}$ such that $E \cap [\text{supp}(\mu_1) \cup \{\frac{v_1}{\|v_1\|}\} \cup \{\frac{v_2}{\|v_2\|}\}] = \emptyset$. Therefore, $\mu_1(E) = 0$ and $\mu_2(E) > 0$. Fix any $\varepsilon > 0$. Since E is open, it is possible to choose a function $f \in C(\mathcal{U})$ such that $f(\bar{u}) > 2\varepsilon$ and $f(u) < -\varepsilon$ for all $u \notin E$. Define H^* as in Appendix A and recall that H^* is dense in $C(\mathcal{U})$ by Lemma 10 in the Supplemental material to DLRS (2007). Therefore, there exists $g \in H^*$ such that $\|f - g\|_\infty < \varepsilon$. It follows that

$$g(\bar{u}) \geq f(\bar{u}) - \|f - g\|_\infty > f(\bar{u}) - \varepsilon > \varepsilon$$

and

$$g(u) \leq f(u) + \|f - g\|_\infty < f(u) + \varepsilon < 0 \quad \forall u \notin E.$$

By Lemma 10 in the Supplemental material to DLRS (2007), there exist $A, B \in \mathcal{A}^c$ and $s > 0$ such that $g = s(\sigma_B - \sigma_A)$. If we let $\hat{g} = s(\sigma_{A \cup B} - \sigma_A)$, then it is easily verified that $\hat{g}(u) = \max\{g(u), 0\}$. Therefore, $\hat{g}(\bar{u}) = g(\bar{u}) > \varepsilon$ and $\hat{g}(u) = 0$ for $u \notin E$. Since $\hat{g}$ is continuous, $\mu_1(E) = 0$, and $\bar{u} \in \text{supp}(\mu_2)$, we have

$$\int_{\mathcal{U}} \hat{g}(u)\mu_1(du) = 0 \quad \text{and} \quad \int_{\mathcal{U}} \hat{g}(u)\mu_2(du) > 0.$$

This implies

$$\begin{aligned} \int_{\mathcal{U}} \max_{p \in A \cup B} u(p)\mu_1(du) &= \int_{\mathcal{U}} \sigma_{A \cup B}(u)\mu_1(du) = \int_{\mathcal{U}} \sigma_A(u)\mu_1(du) \\ &= \int_{\mathcal{U}} \max_{p \in A} u(p)\mu_1(du) \end{aligned}$$

and

$$\begin{aligned} \int_{\mathcal{U}} \max_{p \in A \cup B} u(p)\mu_2(du) &= \int_{\mathcal{U}} \sigma_{A \cup B}(u)\mu_2(du) > \int_{\mathcal{U}} \sigma_A(u)\mu_2(du) \\ &= \int_{\mathcal{U}} \max_{p \in A} u(p)\mu_2(du). \end{aligned}$$

Since $\hat{g}(u) = 0$ for $u \notin E$ and $\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|} \notin E$, we have $\sigma_{A\cup B}(\frac{v_1}{\|v_1\|}) = \sigma_A(\frac{v_1}{\|v_1\|})$ and $\sigma_{A\cup B}(\frac{v_2}{\|v_2\|}) = \sigma_A(\frac{v_2}{\|v_2\|})$. Therefore,

$$\max_{p\in A} \int_{\mathcal{U}} u(p)\mu_1(du) = \max_{p\in A} v_1(p) = \max_{p\in A\cup B} v_1(p) \\ = \max_{p\in A\cup B} \int_{\mathcal{U}} u(p)\mu_1(du)$$

and

$$\max_{p\in A} \int_{\mathcal{U}} u(p)\mu_2(du) = \max_{p\in A} v_2(p) = \max_{p\in A\cup B} v_2(p) \\ = \max_{p\in A\cup B} \int_{\mathcal{U}} u(p)\mu_2(du).$$

In addition, this implies that A dominates B with respect to $\succsim_1$ and $\succsim_2$. Since $\succsim_2$ is a nontrivial regret preference, it must be that $K_2 > 0$. Therefore, using the formulation of the regret representation from Equation (4), we have shown that $V_1(A) = V_1(A \cup B)$ and $V_2(A) > V_2(A \cup B)$. Thus A dominates B with respect to $\succsim_1$ and $\succsim_2$, $A \succ_2 A \cup B$, and $A \sim_1 A \cup B$. Therefore, $\succsim_1$ is not more regret prone than $\succsim_2$. *Q.E.D.*

C.5. *Proof of Theorem* 6

1 ⇒ 2. Since $\succsim_1$ is more regret averse than $\succsim_2$, for any $q, p \in \Delta(Z)$, we have

$$\{q\} \succsim_1 \{p\} \quad \Longrightarrow \quad \{q\} \succsim_2 \{p\}$$

or, equivalently,

$$v_1(q) \geq v_1(p) \quad \Longrightarrow \quad v_2(q) \geq v_2(p).$$

Since v_1 and v_2 are affine functions, Lemma 14 implies that there exist $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $v_2 = \alpha v_1 + \beta$. By the argument given in footnote 10, $\beta = 0$. Also, if $\alpha = 0$, then $v_2(p) = 0$ for all $p \in \Delta(Z)$, violating singleton nontriviality. Thus $\alpha > 0$.

We now show that $r_2 \leq \alpha r_1$. Let $A \in \mathcal{A}$ be arbitrary. We know by singleton triviality that there exist $p, q \in \Delta(Z)$ such that $\{p\} \succ_1 \{q\}$. Let $\hat{p} = \frac{1}{2}p + \frac{1}{2}q$. By independence, $\{p\} \succ_1 \{\hat{p}\} \succ_1 \{q\}$. By continuity, there exists $\lambda \in (0, 1)$ such that $\{p\} \succsim_1 \lambda A + (1-\lambda)\{\hat{p}\} \succsim_1 \{q\}$. Using continuity again, we know there exists $\bar{\lambda} \in [0, 1]$ such that

$$\lambda A + (1-\lambda)\{\hat{p}\} \sim_1 \bar{\lambda}\{p\} + (1-\bar{\lambda})\{q\}. \tag{19}$$

Let $\bar{p} = \bar{\lambda} p + (1 - \bar{\lambda})q$. Since $\succsim_1$ is more regret averse than $\succsim_2$, we have

$$\lambda A + (1-\lambda)\{\hat{p}\} \succsim_2 \{\bar{p}\}. \tag{20}$$

Equations (19) and (20) imply

$$\begin{aligned}
&\lambda[\alpha V_1(A)] + (1-\lambda)[\alpha v_1(\hat{p})] \\
&\quad = \alpha V_1[\lambda A + (1-\lambda)\{\hat{p}\}] = \alpha v_1(\bar{p}) = v_2(\bar{p}) \\
&\quad \leq V_2[\lambda A + (1-\lambda)\{\hat{p}\}] = \lambda V_2(A) + (1-\lambda)[\alpha v_1(\hat{p})],
\end{aligned}$$

which implies $V_2(A) \geq \alpha V_1(A)$. Therefore, for any $A \in \mathcal{A}$,

$$r_2(A) = \max_{p \in A} v_2(p) - V_2(A) \leq \max_{p \in A} [\alpha v_1(p)] - \alpha V_1(A) = \alpha r_1(A),$$

as desired.

2 ⇒ 1. Suppose $\{p\} \succ_2 A$ or, equivalently, $v_2(p) > V_2(A)$. Since $v_2 = \alpha v_1$ and $r_2 \leq \alpha r_1$, we have

$$\begin{aligned}
\alpha v_1(p) = v_2(p) > V_2(A) &= \max_{q \in A} v_2(q) - r_2(A) \\
&\geq \max_{q \in A} [\alpha v_1(q)] - \alpha r_1(A) = \alpha V_1(A).
\end{aligned}$$

Thus $\{p\} \succ_1 A$, so $\succsim_1$ is more regret averse than $\succsim_2$. *Q.E.D.*

C.6. *Proof of Lemma* 5

Defining the strong dominance axiom as in Appendix C.1, from Lemmas 10 and 11 we know that any regret preference satisfies strong dominance. Now, consider any $A, B \in \mathcal{A}$. We claim that either $A \succsim A \cup B$ or $B \succsim A \cup B$. For suppose $A \cup B \succ A$. Then, by strong dominance, there must be some $p \in B$ such that $\{p\} \succ \{q\}$ for all $q \in A$. However, applying strong dominance again, we have that $B \succsim A \cup B$. Thus either $A \succsim A \cup B$ or $B \succsim A \cup B$.

Suppose $A \succsim B$. To show that $\succsim$ satisfies positive set betweenness, we must prove that $A \succsim A \cup B$. From the previous paragraph, we have either $A \succsim A \cup B$ or $A \succsim B \succsim A \cup B$. Hence, $A \succsim A \cup B$. *Q.E.D.*

REFERENCES

AHN, D. S. (2007): "Ambiguity Without a State Space," *Review of Economic Studies*, forthcoming. [279]

ALIPRANTIS, C., AND K. BORDER (1999): *Infinite Dimensional Analysis*. Berlin: Springer-Verlag. [300]

BELL, D. E. (1982): "Regret in Decision Making Under Uncertainty," *Operations Research*, 30, 961–981. [266,269,280]

DEKEL, E., B. LIPMAN, AND A. RUSTICHINI (2001): "Representing Preferences with a Unique Subjective State Space," *Econometrica*, 69, 891–934. [263,269,270,275,288-290,294]
——— (2007): "Temptation-Driven Preferences," Mimeo, Boston University. [266,281-283]
DEKEL, E., B. LIPMAN, A. RUSTICHINI, AND T. SARVER (2007): "Representing Preferences with a Unique Subjective State Space: Corrigendum," *Econometrica*, 75, 591–600. [269,270,288-292, 302]
EPSTEIN, L. G. (1999): "A Definition of Uncertainty Aversion," *Review of Economic Studies*, 66, 579–608. [279]
GHIRARDATO, P., F. MACCHERONI, AND M. MARINACCI (2004): "Differentiating Ambiguity and Ambiguity Attitude," *Journal of Economic Theory*, 118, 133–173. [295]
GHIRARDATO, P., AND M. MARINACCI (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102, 251–289. [279]
GUL, F., AND W. PESENDORFER (2001): "Temptation and Self-Control," *Econometrica*, 69, 1403–1435. [266,269,272,281,282,287]
——— (2004): "Self-Control and the Theory of Consumption," *Econometrica*, 72, 119–158. [288]
HAYASHI, T. (2007): "Regret Aversion and Opportunity-Dependence," *Journal of Economic Theory*, forthcoming. [269,281]
KREPS, D. (1979): "A Preference for Flexibility," *Econometrica*, 47, 565–576. [275]
LOOMES, G., AND R. SUGDEN (1982): "Regret Theory: An Alternative Theory of Rational Choice Under Uncertainty," *Economic Journal*, 92, 805–824. [266,269,280]
——— (1987): "Some Implications of a More General Form of Regret Theory," *Journal of Economic Theory*, 41, 270–287. [269,280]
NOOR, J. (2007a): "Commitment and Self-Control," *Journal of Economic Theory*, 135, 1–34. [288]
——— (2007b): "Temptation, Welfare, and Revealed Preference," Mimeo, Boston University. [288]
ROCKAFELLAR, R. T. (1970): *Convex Analysis*. Princeton, NJ: Princeton University Press. [289]
ROYDEN, H. L. (1988): *Real Analysis*. Englewood Cliffs, NJ: Prentice-Hall. [291,295,296]
SAVAGE, L. (1954): *The Foundations of Statistics*. New York: Wiley. [281]
SCHNEIDER, R. (1993): *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge, U.K.: Cambridge University Press. [289]
STOYE, J. (2007): "Axioms for Minimax Regret Choice Correspondences," Mimeo, New York University. [281]
SUGDEN, R. (1993): "An Axiomatic Foundation for Regret Theory," *Journal of Economic Theory*, 60, 159–180. [269,280]