

Correlated Equilibrium in Evolutionary Models with Subpopulations*

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Abstract

We study a version of the multipopulation replicator dynamics, where each population is comprised of multiple subpopulations. We establish that correlated equilibrium is a natural solution concept in this setting. Specifically, we show that every correlated equilibrium is equivalent to a stationary state in the replicator dynamics of some subpopulation model. We also show that every interior stationary state, Lyapunov stable state, or limit of an interior solution is equivalent to a correlated equilibrium. We provide an example with a Lyapunov stable limit state whose equivalent correlated equilibrium lies outside the convex hull of the set of Nash equilibria. Finally, we prove that if the matching distribution is a product measure, a state satisfying any of the three conditions listed above is equivalent to a Nash equilibrium.

Keywords: correlated equilibrium, evolutionary game theory, replicator dynamics, subpopulations.

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1 Introduction

The concept of correlated equilibrium has received much attention in the literature since its introduction by Aumann in 1974. An important focus of the discussion has involved the relationship and applicability of the correlated equilibrium solution concept to the rationalistic and evolutionary frameworks.¹ The correlated equilibrium is an extension of the Nash equilibrium solution concept, allowing for correlation between the actions of different players. A common interpretation for the correlating device is an exogenous instrument that sends to players potentially correlated signals, the distribution of which is common knowledge. Each player chooses an optimal strategy given the strategies of her opponents and her knowledge of the signal distribution.

A key feature of evolutionary and learning frameworks is that they permit us to relax the assumptions about knowledge and rationality of players. In fact, a useful characteristic of evolutionary models is that they provide a test of the sensitivity of game theoretic results to these typical assumptions. The appropriateness of the Nash equilibrium solution concept in models with evolutionary dynamics has been well established in the literature. However, how to interpret the correlated equilibrium solution concept in a this setting is not as well understood.

We develop a dynamic evolutionary model that gives rise to correlated equilibria. As noted above, a correlated equilibrium can be thought of as a Nash equilibrium in a game where rational players observe external signals for which they know the joint distribution. Such a game is mathematically equivalent to a game where players have multiple types, the distribution of types is known to each player, and players can condition their actions on their realized types. We assert that a game with types has a natural interpretation with large populations that are partitioned into subpopulations. Rather than matching particular types of each player role, the matching occurs over the subpopulations of each player population. Correlation in how the individuals are chosen from among these subpopulations generates replicator dynamics that produce correlated equilibria.

Specifically, we show that for every correlated equilibrium, there exists some subpopulation breakdown and matching distribution for which there is an equivalent stationary state under the replicator dynamics. We also show that for stationary states that are interior for all subpopulations, there exists an equivalent correlated equilibrium. Furthermore, we show that any Lyapunov stable state or any limit to an interior solution may be represented by a correlated equilibrium. We then discuss an additional restriction that generates Nash equilibria in this subpopulation model, and we discuss the relationship of

¹For a discussion of the epistemic foundations of correlated equilibrium, see Aumann (1987). For a survey of literature in this area, see Brandenburger (1992).

this model to the standard replicator dynamics.

We believe that the subpopulation interpretation fits naturally in many contexts. The replicator dynamics with subpopulations appears especially well suited to locational models. Players in one geographic region may be matched more frequently with opponents from nearby regions. Hence, if the populations are divided into discrete neighborhoods, the matching distribution can reflect such proximities.² As we will show, correlated equilibrium is an appropriate solution concept for such games, even if the agents are assumed to behave in a purely evolutionary manner.

This paper is certainly not the first to analyze correlated equilibria in the context of evolutionary-like behavior. Hart and Mas-Colell (2000), for example, apply correlated equilibria to adaptive models. Their work is similar to ours in that it emphasizes the dynamic mechanisms for which the play converges to a correlated equilibrium. However, agents in Hart and Mas-Colell's model exhibit learning and regret in that they weigh their past actions against alternatives. Rather than assuming specific behavior on the part of agents, we show that correlated equilibria can arise simply from the way in which individuals are matched for play. Even with the most basic evolutionary dynamics, the replicator dynamics, we obtain a representation of correlated equilibrium. The replicator dynamics have the advantage of both simplicity and widespread applicability. A model of replicator dynamics can apply to purely biological contexts, where one imagines agents as pre-programmed to play a pure strategy and hence unable to exhibit learning or regret. The replicator dynamics can also be interpreted as an aggregate representation of certain types of boundedly rational social learning.³ By deriving a representation of correlated equilibrium directly in terms of the replicator dynamics, we ensure that our results can be useful in a wide variety of contexts.

The model of Cripps (1991) is closer in spirit to the present work. Cripps shows that for a class of evolutionary models, strict correlated equilibria can be interpreted in a framework of evolutionary stability. In particular, the set of evolutionary stable strategies in simple contests is identical to the set of strict correlated equilibria. Cripps's use of player roles is similar to our use of subpopulations. We choose to explore these issues in a dynamic setting, rather than with a static solution concept such as evolutionary stability. We also offer a more detailed comparison to the standard evolutionary model, highlighting what attributes give rise correlated equilibria outside of the set of Nash equilibria. Based on the relationship between evolutionary stability and stability in replicator dynamics, we are confident that our model of subpopulation

²Which matching distributions are plausible in geographic contexts is an interesting question for further research. In the present work, however, we impose no such restrictions, and our results apply to arbitrary matching distributions.

³For example, see Schlag (1998) or Björnerstedt and Weibull (1996).

matching in a context of evolutionary stability would generate results that are compatible with those of Cripps.

Mailath, Samuelson, and Shaked (1997) present an approach that leads to a characterization similar to ours. They generalize the concept of Nash equilibrium to an equilibrium with local interactions and demonstrate equivalencies between this equilibrium concept and correlated equilibrium. Their equilibrium with local interactions is effectively a pure strategy Nash equilibrium of a transformed version of the original game, in which they create a finite population of individuals for each player role and match individuals for play according to an exogenously given probability distribution. Correlation in how individuals are matched allows for correlation in strategies played at the aggregate level, and hence Nash equilibria of the transformed game would be correlated equilibria of the original game. Our approach is the natural analog of that of Mailath *et al.* (1997) in a dynamic evolutionary framework. Intuitively, given their main proposition, along with appropriate results for standard replicator dynamics, one would expect equivalencies similar to those that we demonstrate.⁴ Instead, we prefer to establish our results directly from the mechanics of our subpopulation replicator dynamics. In addition to finding the direct approach more accessible, we find that it helps us to better demonstrate the necessity of certain assumptions to our results.

The remainder of the paper is structured as follows. In Section 2, we define a model of subpopulation replicator dynamics and develop notation that we will use throughout the paper. In Section 3, we demonstrate the connection between stationarity in the subpopulation replicator dynamics and correlated equilibrium. In Section 4, we discuss how Lyapunov stable states and the limits of interior solutions are related to correlated equilibria. We describe in Section 5 an additional restriction that generates Nash equilibria.

2 Model and Notation

Let $\Gamma = (I, (S_i)_{i \in I}, (\pi_i)_{i \in I})$ be a finite, n -player normal-form game. The set of players, or in the evolutionary context, the set of n player populations is $I = \{1, \dots, n\}$. Let the finite set of pure strategies for each player population be denoted S_i and let $S \equiv \mathbf{X}_{i \in I} S_i$ be the set of possible pure-strategy profiles. Each player population has a payoff function $\pi_i: S \rightarrow \mathbb{R}$.

As is standard, we denote a profile of strategies excluding that of player i as $s_{-i} \in S_{-i} \equiv \mathbf{X}_{j \neq i} S_j$. Furthermore, (k, s_{-i}) is a strategy profile in which player

⁴To explicitly make this connection, one would first have to allow individuals in Mailath *et al.*'s model to play mixed strategies and extend their equivalency result to mixed-strategy Nash equilibria of the transformed game.

i plays strategy $k \in S_i$ and her opponents play $s_{-i} \in S_{-i}$. We will carry this notation over to the various functions and subpopulations defined below.

2.1 Subpopulations and Replicator Dynamics

Each player population i can be partitioned into a finite set of subpopulations indexed by a set M_i .⁵ For every subpopulation $h \in M_i$, let the *subpopulation state* $x_{i,h}$ be a point in the mixed-strategy simplex ΔS_i so that $x_{i,h}^k$ is the share of the subpopulation h that plays pure strategy $k \in S_i$. The subpopulation state gives the distribution of play in subpopulation h of population i .

One subpopulation from each population is selected for play. Let $M \equiv \prod_{i \in I} M_i$ be the set of possible subpopulation matches and let $m \in M$ be a vector of subpopulations that characterizes a match. Define $x_m = (x_{i,m_i})_{i \in I}$, so that x_m describes the distribution of play in the active subpopulation of each population under the match m . Then, let the *population state* be $x = (x_m)_{m \in M}$ and let X be the set of all population states. The population state fully describes the distribution of play under all subpopulation matches. Furthermore, define $x_{m_{-i}} = (x_{j,m_j})_{j \in I: j \neq i}$.

Let η be an exogenous probability distribution over M . For every subpopulation match $m \in M$, $\eta(m)$ gives the probability of that match. For any game Γ , we will call the pair (M, η) a *subpopulation setting* of Γ . Note that the probability that subpopulation $m_i \in M_i$ plays is $\eta_i(m_i) = \sum_{m_{-i} \in M_{-i}} \eta(m_i, m_{-i})$. One can interpret $\eta_i(m_i)$ as the share of population i in subpopulation m_i . In standard models of replicator dynamics, the probability distribution of which individual in each of the populations will play is uniform, so we would have the probability that the individuals come from subpopulations m_1, m_2, \dots, m_n equal to $\eta_1(m_1)\eta_2(m_2) \cdots \eta_n(m_n)$. However, in our model there is the potential for correlation between which subpopulation of population i plays and which subpopulations of the other populations play: that is, η might not be a product measure.

We restrict attention to matching distributions for which $\eta_i(m_i) > 0$ for all $i \in I$ and $m_i \in M_i$. This assumption is without loss of generality because any subpopulation that never plays can simply be eliminated from the model. However, the *joint* distribution η need not give positive probability to every subpopulation match. For instance, one can imagine a subpopulation setting with two opposing subpopulations that never meet. In our examples, we often assign some matches probability zero for simplicity; however, the propositions hold for general matching distributions.

Note that for a given subpopulation match m , the conditional probability that a certain pure-strategy profile s is played is $\prod_{i \in I} x_{i,m_i}^{s_i}$. For a given subpopulation match m , the expected utility function for a member of subpopulation

⁵In many situations, it is convenient to think of a set of subpopulations as a subset of the natural numbers: $M_i = \{1, \dots, |M_i|\}$.

m_i of population i is:

$$u_i(x_m) = \sum_{s \in S} \left[\prod_{j \in I} x_{j, m_j}^{s_j} \right] \pi_i(s) \quad (1)$$

Let e_i^k be the point in population i 's mixed-strategy simplex that puts probability one on pure strategy k . When one fixes a pure strategy k for a subpopulation m_i , the utility function is written as $u_i(e_i^k, x_{m_{-i}})$. Note that:

$$u_i(e_i^k, x_{m_{-i}}) = \sum_{s_{-i} \in S_{-i}} \left[\prod_{j \neq i} x_{j, m_j}^{s_j} \right] \pi_i(k, s_{-i}) \quad (2)$$

Let \dot{x}_{i, m_i}^k denote the derivative of the share of subpopulation m_i playing strategy k with respect to time. The η -replicator dynamics for any $i \in I$, $m_i \in M_i$, and $k \in S_i$ are defined:⁶

$$\dot{x}_{i, m_i}^k = \sum_{m_{-i} \in M_{-i}} \eta(m_{-i} | m_i) \left[u_i(e_i^k, x_{m_{-i}}) - u_i(x_m) \right] x_{i, m_i}^k \quad (3)$$

A population state is *stationary* in the η -replicator dynamics if $\dot{x}_{i, m_i}^k = 0$ for all $i \in I$, $m_i \in M_i$, and $k \in S_i$. A population state is *interior* if $x_{i, m_i}^k \gg 0$ for all $i \in I$, $m_i \in M_i$. One should note that this definition of interiority requires that the average play in all subpopulations is a completely mixed strategy.

2.2 Correlated Equilibrium

We will use the following definition of a correlated equilibrium:

Definition 1 A correlated equilibrium in Γ is a probability distribution ψ on S such that for all $i \in I$ and for any $f_i: S_i \rightarrow S_i$,

$$\sum_{s \in S} \psi(s) \left[\pi_i(f_i(s_i), s_{-i}) - \pi_i(s) \right] \leq 0. \quad (4)$$

One can conceptualize a correlated equilibrium as a random signal that recommends that the players play the strategy profile s with probability $\psi(s)$. Each player i receives the recommendation for her own strategy; however, she is unaware of the recommendations given to the other players. In this sense, the recommendation given to a player gives her information about the state of the world, but does not necessarily fully reveal the state of the world. Each player i can condition her action on this information by choosing a mapping

⁶All the results presented in this paper are robust to scalings of the subpopulation replicator dynamics by positive-valued functions of m_i . In particular, one could scale the replicator dynamics in Equation (3) by $\eta_i(m_i)$ so that the conditional distribution $\eta(m_{-i} | m_i)$ would be replaced with the joint matching distribution $\eta(m_i, m_{-i})$.

$f_i: S_i \rightarrow S_i$. The distribution ψ constitutes a correlated equilibrium if following the recommendation s_i is weakly preferred by all players to any other strategy given by $f_i(s_i)$. Let $\psi_i(s_i)$ be the marginal probability that strategy s_i is recommended to player i , so that $\psi_i(s_i) = \sum_{s_{-i} \in S_{-i}} \psi(s_i, s_{-i})$. Note that ψ need not be a product measure: it is not necessarily true that $\psi(s) = \prod_{i \in I} \psi_i(s_i)$. At any population state, the average play over all subpopulation matches is a distribution on S of particular interest:

Definition 2 *Given a game Γ and a subpopulation setting (M, η) , a probability distribution ψ on S , not necessarily a correlated equilibrium, is equivalent to a population state x if and only if for all $s \in S$:*

$$\psi(s) = \sum_{m \in M} \eta(m) \left[\prod_{i \in I} x_{i, m_i}^{s_i} \right] \quad (5)$$

3 Stationarity

An important result of standard replicator dynamics is the relationship between Nash equilibria and interior stationary states. It has been shown that every Nash equilibrium is stationary in the replicator dynamics and that any interior stationary state is a Nash equilibrium.⁷ We now show that there exists a similar relationship between correlated equilibria and stationarity in our subpopulation model.

Proposition 1 *Consider a game Γ .*

- i. For any correlated equilibrium ψ , there exists a subpopulation setting (M, η) with a stationary state that is equivalent to ψ .*
- ii. For any subpopulation setting (M, η) , if x is an interior stationary state in the η -replicator dynamics, then the equivalent probability distribution ψ is a correlated equilibrium.*

The following lemma will be useful in the proof of this and subsequent propositions.

Lemma 1 *Let $x \in X$ and suppose that for all $i \in I$, $m_i \in M_i$, and $k \in S_i$:*

$$\sum_{m_{-i} \in M_{-i}} \eta(m_{-i} | m_i) \left[u_i(e_i^k, x_{m_{-i}}) - u_i(x_m) \right] \leq 0 \quad (6)$$

Then the probability distribution ψ that is equivalent to x is a correlated equilibrium.

⁷For more on the relationship between Nash equilibria and the standard replicator dynamics, see Weibull (1995).

Proof of Lemma 1: The equivalent distribution ψ is given by $\psi(s) = \sum_{m \in M} \eta(m) \prod_{i \in I} x_{i,m_i}^{s_i}$ for all $s \in S$. Consider any $i \in I$ and $m_i \in M_i$, and any mapping $f_i: S_i \rightarrow S_i$. For any s_i , $f_i(s_i)$ is fixed over s_{-i} and m_{-i} , so Equation (6) holds with $f_i(s_i)$ in place of k . We therefore have:

$$\sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \left[u_i(e_i^{f_i(s_i)}, x_{m_{-i}}) - u_i(x_m) \right] \leq 0 \quad (7)$$

Note that:

$$\begin{aligned} & \sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) u_i(e_i^{f_i(s_i)}, x_{m_{-i}}) \\ &= \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{s_{-i} \in S_{-i}} \left[\prod_{j \neq i} x_{j,m_j}^{s_j} \right] \pi_i(f_i(s_i), s_{-i}) \\ &= \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \sum_{s \in S} \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] \pi_i(f_i(s_i), s_{-i}) \end{aligned} \quad (8)$$

Furthermore, since $\sum_{s_i \in S_i} x_{i,m_i}^{s_i} = 1$, we have:

$$\begin{aligned} \sum_{s_i \in S_i} x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) u_i(x_m) &= \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) u_i(x_m) \\ &= \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \sum_{s \in S} \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] \pi_i(s) \end{aligned} \quad (9)$$

Using the result of Equation (7) with Equations (8) and (9), then multiplying by $\eta_i(m_i)$ and summing over m_i , we have:

$$\begin{aligned} 0 &\geq \sum_{m_i \in M_i} \eta_i(m_i) \left[\sum_{s \in S} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] [\pi_i(f_i(s_i), s_{-i}) - \pi_i(s)] \right] \\ &= \sum_{s \in S} \left[\sum_{m \in M} \eta(m) \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] [\pi_i(f_i(s_i), s_{-i}) - \pi_i(s)] \right] \\ &= \sum_{s \in S} \psi(s) [\pi_i(f_i(s_i), s_{-i}) - \pi_i(s)] \end{aligned} \quad (10)$$

Thus ψ is a correlated equilibrium.

Q.E.D.

Proof of Proposition 1: (i) The first claim is almost trivial if we are careful in our construction of the subpopulation setting. Let $M = S$ and $\eta = \psi$. In this specification, each population is divided into subpopulations that correspond to the available pure strategies for that population. Hence, the possible

subpopulation matches are the possible pure strategy profiles. Furthermore, we set the probability of each subpopulation match to the probability that the distribution ψ assigns to the strategy profile corresponding to that match.

Consider a population state x such that $x_{i,m_i} = e_i^{m_i}$ for all $i \in I$, $m_i \in M_i$. Here, each subpopulation is programmed to play the pure strategy to which it corresponds. Thus $\prod_{i \in I} x_{i,m_i}^{s_i} = 1_{\{m\}}(s)$ for all $m \in M$ and for all $s \in S$, where $1_{\{m\}}$ is the indicator function. Since all subpopulations are playing pure strategies, the population state x is trivially stationary. It remains only to show that x is equivalent to ψ . For any $s \in S$,

$$\sum_{m \in M} \eta(m) \left[\prod_{i \in I} x_{i,m_i}^{s_i} \right] = \sum_{m \in M} \eta(m) 1_{\{m\}}(s) = \eta(s) = \psi(s) \quad (11)$$

and hence x and ψ are equivalent.⁸

(ii) If x is an interior stationary state in the η -replicator dynamics, then for all $i \in I$, $m_i \in M_i$, and $k \in S_i$, we have $x_{i,m_i}^k > 0$ and therefore:

$$\sum_{m_{-i} \in M_{-i}} \eta(m_{-i} | m_i) \left[u_i(e_i^k, x_{m_{-i}}) - u_i(x_m) \right] = 0 \quad (12)$$

By Lemma 1, we therefore have that the equivalent probability distribution ψ is a correlated equilibrium. Q.E.D.

One should note that our definition of interiority in a subpopulation model is more stringent than in a model without subpopulations. In particular, we require that *all* subpopulations play every strategy with strictly positive probability. However, this stronger definition is necessary to guarantee that the equivalent distribution is a correlated equilibrium.

For example, consider a game with the following normal-form representation:

	l	r
t	$1, 2$	$0, 0$
b	$0, 0$	$2, 1$

Suppose that each of the populations has two subpopulations: that is, let $M_1 = M_2 = \{1, 2\}$. Suppose that the subpopulation matches occur with the following probabilities:

	1	2
1	$1/2$	0
2	0	$1/2$

⁸One should note that for *any* distribution ψ over S , this same approach can be used to construct a population setting with an equivalent stationary state. However, we find this construction appealing because the stationary state corresponding to a correlated equilibrium has a nonpositive growth rate for every unplayed strategy. While in general this condition does not yield a characteristic stronger than stationarity, in the special case of a *strict* correlated equilibrium, one obtains an asymptotically stable state from this construction.

Consider the population state x with subpopulation states

$$\begin{aligned} x_{1,1} &= \left(\frac{1}{3}, \frac{2}{3}\right) & x_{1,2} &= (1, 0) \\ x_{2,1} &= \left(\frac{2}{3}, \frac{1}{3}\right) & x_{2,2} &= (0, 1) \end{aligned}$$

Notice that x is not an interior state by our definition even though the average play in each population is a completely mixed strategy. We will show that x is stationary, but does not correspond to a correlated equilibrium. As in the proof of Proposition 1, since the second subpopulation in each population is programmed to play a pure strategy, we have $\dot{x}_{1,2}^t = \dot{x}_{1,2}^b = 0$ and $\dot{x}_{2,2}^l = \dot{x}_{2,2}^r = 0$. For subpopulation 1 of population 1, we have:

$$\begin{aligned} \dot{x}_{1,1}^t &= \left[u_1(e_1^t, x_{2,1}) - u_1(x_{1,1}, x_{2,1}) \right] x_{1,1}^t \\ &= \left[\left(\frac{2}{3} \cdot \pi_1(t, l) \right) - \left(\frac{1}{3} \cdot \frac{2}{3} \cdot \pi_1(t, l) + \frac{2}{3} \cdot \frac{1}{3} \cdot \pi_1(b, r) \right) \right] \cdot \frac{1}{3} \\ &= \left[\frac{2}{3} - \left(\frac{2}{9} + \frac{4}{9} \right) \right] \cdot \frac{1}{3} \\ &= 0 \end{aligned}$$

and analogously for $\dot{x}_{1,1}^b$, $\dot{x}_{2,1}^l$, and $\dot{x}_{2,1}^r$. Therefore, x is a stationary state.

This population state is equivalent to the following distribution over strategy profiles:

	l	r
t	1/9	5/9
b	2/9	1/9

To see that this distribution is not a correlated equilibrium, consider a rational agent as the row player. If she is given the recommendation to play t , then she knows that column player was recommended to play l with probability $1/6$ and r with probability $5/6$. Her payoff from following the recommendation would be $1/6$, whereas if she deviated and played b , her payoff would be $5/3$. Hence, this distribution is not a correlated equilibrium.⁹

We should also discuss another important difference between our subpopulation model and the model without subpopulations. Proposition 1(ii) states that for any interior stationary state in the η -replicator dynamics, the equivalent distribution is a correlated equilibrium. Again, in a model without subpopulations, any interior stationary state in the replicator dynamics is a Nash equilibrium. In such a model, we also have that any state that is not stationary is not a Nash equilibrium. In our model Proposition 1(i) provides a partial converse to (ii), but the complete converse is not true. It is possible to have

⁹The fact that this distribution is not a correlated equilibrium requires that the condition of Lemma 1 be violated. One can show that indeed Equation (6) is violated in this example for the second subpopulation of each population.

a nonstationary state that is equivalent to a correlated equilibrium. The next example invalidates both the converses to Lemma 1 and Proposition 1(ii). Consider the previous game, again where each population has subpopulations $\{1, 2\}$ and suppose population matches occur with the following probabilities:

	1	2
1	1/3	0
2	0	2/3

Consider the population state x with subpopulation states

$$\begin{aligned} x_{1,1} &= (1, 0) & x_{1,2} &= (0, 1) \\ x_{2,1} &= \left(\frac{2}{3}, \frac{1}{3}\right) & x_{2,2} &= \left(\frac{2}{3}, \frac{1}{3}\right) \end{aligned}$$

It is straightforward to show that $\dot{x}_{2,1}^l = 4/9$ and $\dot{x}_{2,2}^r = 2/9$, so x is not stationary. However, x is equivalent to the correlated equilibrium with the following distribution:

	l	r
t	2/9	1/9
b	4/9	2/9

Notice that this distribution is a product measure. Any correlated equilibrium that is a product measure has randomization that is independent across players and is hence a Nash equilibrium. Thus x is actually equivalent to a Nash equilibrium.

Proposition 1(i) states that for any correlated equilibrium there exists a subpopulation setting with a stationary state equivalent to this correlated equilibrium. The full converse, which we showed does not hold, would require that, for any population setting, any state equivalent to a correlated equilibrium must be stationary. There are two main reasons for the contrast between the subpopulation model and the model without subpopulations. The first is that since the η -replicator dynamics depend on the choice of subpopulations and their distribution, the same game can have many evolutionary representations, whereas models without subpopulations have unique replicator dynamics. The second is that in our subpopulation model, multiple states can be equivalent to the same correlated equilibrium. For instance, in the current example, the state x with subpopulation states $x_{1,1} = x_{1,2} = (\frac{1}{3}, \frac{2}{3})$ and $x_{2,1} = x_{2,2} = (\frac{2}{3}, \frac{1}{3})$ is stationary and equivalent to the same correlated equilibrium.

4 Stability and Limit States

Having discussed the relationship between correlated equilibria and rest points in the replicator dynamics of subpopulation models, we now turn to stable states in these replicator dynamics. As we shall see, the relationship between

correlated equilibria and stable states in the replicator dynamics of subpopulation models is similar to the that of Nash equilibria and stability under the standard replicator dynamics.¹⁰ Definition 3 is merely a reformulation of the standard dynamics solution concept for a model with subpopulations.

Definition 3 A solution through a point $x \in X$ to a system of η -replicator dynamics is a function $\xi(\cdot, x): \mathbb{R} \rightarrow X$ such that $\xi(0, x) = x$ and such that for all $t \in \mathbb{R}$, $i \in I$, $m_i \in M_i$, and $k \in S_i$:

$$\frac{d}{dt} \xi_{i,m_i}^k(t, x) = \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \left[u_i(e_i^k, \xi_{m_{-i}}(t, x)) - u_i(\xi_m(t, x)) \right] \xi_{i,m_i}^k(t, x) \quad (13)$$

One should note that the η -replicator dynamics in Equation (3) are defined by a polynomial in x . Since any polynomial is locally Lipschitz continuous, we have by the Picard-Lindelöf theorem that there exists a unique solution mapping $\xi: \mathbb{R} \times X \rightarrow X$. Given that the solution mapping is well-defined, we can utilize the concept of Lyapunov stability, which is defined as follows.

Definition 4 A state $x \in X$ is Lyapunov stable if every neighborhood U of x contains a neighborhood U^0 of x such that $\xi(t, y) \in U$ for all $y \in U^0 \cap X$ and $t \geq 0$.

The following result establishes the relationship between stability in the replicator dynamics and correlated equilibria.

Proposition 2 Consider a game Γ . Given a subpopulation setting (M, η) , if a population state $x \in X$ is Lyapunov stable in the η -replicator dynamics, then the equivalent distribution is a correlated equilibrium.

Proof: If x is Lyapunov stable, then it is stationary. We must show that any stationary state for which the equivalent distribution is not a correlated equilibrium is not Lyapunov stable.

Suppose that $x \in X$ is stationary and the equivalent distribution is not a correlated equilibrium. Then there exist some $i \in I$, $m_i \in M_i$, and $k \in S_i$ such that:

$$g_{i,m_i}^k(x) \equiv \sum_{m_{-i} \in M_{-i}} \eta(m_{-i}|m_i) \left[u_i(e_i^k, x_{m_{-i}}) - u_i(x_m) \right] > 0 \quad (14)$$

If this were not the case, then x would be equivalent to a correlated equilibrium by Lemma 1. Since x is stationary, this requires that $x_{i,m_i}^k = 0$.

Given this result, an argument analogous to that in the proof of Theorem 5.2(b) in Weibull (1995) shows that x is not Lyapunov stable. The intuition

¹⁰In the standard replicator dynamics, Lyapunov stable states and states that are limits to interior solutions are Nash equilibria. For a full discussion, see Weibull (1995).

here is that g_{i,m_i}^k is multilinear in x , and hence continuous, which implies that in a small enough neighborhood of x , g_{i,m_i}^k must be strictly positive. Thus, trajectories from interior states close to x will be pulled away from x . Q.E.D.

Another important relationship in this context is that limit states to interior solutions can also be represented by correlated equilibria. This result complements the previous proposition in that some limit states, such as saddle points, are not Lyapunov stable and some Lyapunov stable states are not limit states.

Proposition 3 *Consider a game Γ . Given a subpopulation setting (M, η) , if the population state x is the limit to some interior solution in the η -replicator dynamics, then the equivalent distribution is a correlated equilibrium.*

Proof: If $x^0 \in \text{int}(X)$ and $\xi(t, x^0) \xrightarrow{t \rightarrow \infty} x$, then it is straightforward to show that x is stationary. If x is not equivalent to a correlated equilibrium, then again by Lemma 1, there exists some $i \in I$, $m_i \in M_i$, and $k \in S_i$, such that $g_{i,m_i}^k(x) > 0$. Then by continuity, there exists $\delta > 0$ and some neighborhood V of x such that $g_{i,m_i}^k(y) \geq \delta$ for all $y \in V \cap X$.

However, as in the proof of Theorem 5.2(c) in Weibull (1995), if $\xi(t, x^0)$ converges to x , then there exists some $T > 0$ such that $\xi(t, x^0) \in V \cap \text{int}(X)$ for all $t \geq T$. Since x is stationary and $g_{i,m_i}^k(x) \geq \delta$, it must be that $x_{i,m_i}^k = 0$. This implies $\frac{d}{dt} \xi_{i,m_i}^k(t, x^0) < 0$ for some $t \geq T$, which requires that $g_{i,m_i}^k(\xi(t, x^0)) < 0$. This contradicts $g_{i,m_i}^k(y) \geq \delta$ for all $y \in V \cap X$. Thus x is equivalent to a correlated equilibrium. Q.E.D.

We have so far shown several sufficient conditions for a state to be equivalent to a correlated equilibrium. One may ask whether these conditions are in fact sufficient for a state to be equivalent to a Nash equilibrium. We show here that this is not the case. The following is an example of a game with a Lyapunov stable state that is also the limit of an interior solution, but is not a Nash equilibrium.

Consider the following normal-form game, appearing in Aumann (1987):

	l	r
t	6, 6	2, 7
b	7, 2	0, 0

Suppose that each population has subpopulations $\{1, 2\}$ and that the population matches occur with the following probabilities:

	1	2
1	1/3	1/3
2	1/3	0

Consider the population state x with subpopulation states

$$\begin{array}{ll} x_{1,1} = (1, 0) & x_{1,2} = (0, 1) \\ x_{2,1} = (1, 0) & x_{2,2} = (0, 1) \end{array}$$

Notice that the unplayed action of each subpopulation of population 1 has a negative growth rate:

$$\begin{aligned} g_{1,1}^b(x) &= \frac{1}{2} [u_1(e_1^b, x_{2,1}) - u_1(x_{1,1}, x_{2,1})] + \frac{1}{2} [u_1(e_1^b, x_{2,2}) - u_1(x_{1,1}, x_{2,2})] \\ &= \frac{1}{2} [7 - 6] + \frac{1}{2} [0 - 2] = -\frac{1}{2} < 0 \end{aligned}$$

$$g_{1,2}^t(x) = u_1(e_1^t, x_{2,1}) - u_1(x_{1,2}, x_{2,1}) = -1 < 0$$

Similarly, $g_{2,1}^r(x) < 0$ and $g_{2,2}^l(x) < 0$. Since g is continuous, there exists some open ball V containing x such that these conditions also hold for any $y \in V \cap X$. This implies that x is Lyapunov stable, and in fact, x is the limit of the solution through any initial state $y \in V \cap \text{int}(X)$. Either of these conditions is sufficient for x to be equivalent to a correlated equilibrium. The equivalent correlated equilibrium has the distribution:

	l	r
t	1/3	1/3
b	1/3	0

However, this is clearly not a Nash equilibrium distribution. Furthermore, this distribution lies outside the convex hull of the Nash equilibria.

5 Nash Equilibrium

Given the previous example, one might ask what conditions are sufficient for a population state in this framework to be equivalent to a Nash equilibrium. One possibility is to consider a population setting (M, η) in which η is a product measure. In this case, dividing each population into subpopulations has no effect on the distribution of opponents that a member of a population will face: the conditional distribution $\eta(\cdot | m_i)$ is the same for all $m_i \in M_i$. Therefore, one would expect the same results as for standard replicator dynamics. Namely, the analogs of Proposition 1(ii), Proposition 2, and Proposition 3 should hold with Nash equilibrium substituted for correlated equilibrium. The following proposition shows that this is in fact the case.

Proposition 4 *Consider a game Γ . If (M, η) is a subpopulation setting such that η is a product measure, then for any population state x , the equivalent distribution ψ on S is also a product measure.*

As we discussed earlier, any correlated equilibrium that is product measure is also a Nash equilibrium. Thus Proposition 4, combined with the previous propositions, implies the desired results.

Proof: If η is a product measure, then for all $m \in M$, $\eta(m) = \prod_{i \in I} \eta_i(m_i)$. Therefore, for all $s \in S$,

$$\begin{aligned}
\psi(s) &= \sum_{m \in M} \eta(m) \left[\prod_{j \in I} x_{j,m_j}^{s_j} \right] = \sum_{m \in M} \left[\prod_{i \in I} \eta_i(m_i) \prod_{j \in I} x_{j,m_j}^{s_j} \right] \\
&= \sum_{m \in M} \left[\prod_{i \in I} \eta_i(m_i) x_{i,m_i}^{s_i} \right] = \prod_{i \in I} \left[\sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \right] \\
&= \prod_{i \in I} \psi_i(s_i) \tag{15}
\end{aligned}$$

The last equality follows from the fact that for any η , not necessarily a product measure, we have:

$$\begin{aligned}
\psi_i(s_i) &= \sum_{s_{-i} \in S_{-i}} \sum_{m \in M} \eta(m) x_{i,m_i}^{s_i} \left[\prod_{j \neq i} x_{j,m_j}^{s_j} \right] \\
&= \sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i} | m_i) \sum_{s_{-i} \in S_{-i}} \left[\prod_{j \neq i} x_{j,m_j}^{s_j} \right] \\
&= \sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \sum_{m_{-i} \in M_{-i}} \eta(m_{-i} | m_i) \\
&= \sum_{m_i \in M_i} \eta_i(m_i) x_{i,m_i}^{s_i} \tag{16}
\end{aligned}$$

Thus ψ is a product measure.

Q.E.D.

That we obtain the standard relationship between population states and Nash equilibria when η is a product measure is rather intuitive. In such a subpopulation setting, the subpopulation to which an individual is assigned has no effect on the distribution of his opponents. However, this does not imply that the division of a population into such subpopulations is inconsequential. If we consider the special case where a population is divided up into subpopulations such that the matching distribution is a product measure and such that within each population, each subpopulation is at the same subpopulation state, then our subpopulation replicator dynamics would be exactly the standard replicator dynamics. However, this result does not hold in general: if the matching distribution is a product measure, but the distribution of play differs among the subpopulations of a population, then our dynamics are not necessarily the same as when each population is comprised of only a single subpopulation.

The reason for this subtle difference is that our model treats the proportional division of a population into subpopulations as fixed. That is, even if one

subpopulation of a population is more successful than another, its relative size within the population remains constant. Therefore, if unsuccessful strategies are isolated within one subpopulation and successful strategies within another, then we would observe both of these strategies persisting in the population over time. However, this population state is not stable because a perturbation of these subpopulation states would cause the state of the first subpopulation to move toward the successful strategy. This example illustrates that, although the dynamics might differ from the standard replicator dynamics, a stable state must be a Nash equilibrium when η is a product measure.

6 Concluding Remarks

We have established in the preceding analysis the relationship between correlated equilibria and the replicator dynamics of evolutionary games with subpopulations. We showed that every correlated equilibrium is equivalent to a stationary state in the replicator dynamics of some subpopulation model. We also showed that every interior stationary state in a subpopulation model is equivalent to a correlated equilibrium. We found that any state that is Lyapunov stable or the limit of an interior solution is equivalent to a correlated equilibrium. We also provided an example with a Lyapunov stable limit state whose equivalent correlated equilibrium was outside the convex hull of the set of Nash equilibria. Finally, we demonstrated the connection to standard replicator dynamics and Nash equilibrium by considering matching distributions that are product measures.

There is, however, much more to investigate regarding this relationship. Our stability results indicate that some correlated equilibria are robust to exogenous shocks to the population state. However, one could also imagine having exogenous shocks to the subpopulation setting. For instance, in a locational context, some individuals could be exogenously moved from one subpopulation to another. How to model such perturbations and which correlated equilibria are robust to these perturbations is an interesting question for future research.

While exogenous changes in the subpopulation setting would likely be more manageable, endogenizing the subpopulation setting would be a more appropriate extension for many situations. One might expect in certain contexts that subpopulations whose distribution of play is concentrated around unsuccessful strategies would become proportionally smaller with respect to more successful subpopulations. In a biological setting, unsuccessful subpopulations might literally die out. In some economic contexts, we may want to allow for endogenous changes in the subpopulation setting in response to, for example, migration among subpopulations. We hope to extend our model to account for such endogeneity and investigate the effects on our main results.

Another interesting extension of our model would be to incorporate extensive-

form games into the subpopulation framework. The context of our model throughout has been the normal-form game, which is the common setting of evolutionary models. However, as argued in Cressman (2003), the extensive form matters to evolutionary games. Similar logic would suggest that before incorporating an extensive-form game into our subpopulation framework, we may need to modify our model to account for Cressman's results.¹¹

We believe that the main contribution of this paper is its interpretation of the correlated equilibrium, or more precisely the identification of a class of games for which the correlated equilibrium is a natural solution concept. We would like to emphasize the subpopulations framework and its intuitive relationship to the correlated equilibrium. While we formulate our results in terms of the replicator dynamics, we believe that similar results could be obtained if one were to employ learning and more sophisticated adaptation in this same framework.

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¹¹As we stated in the introduction, a correlated equilibrium can be thought of as a game in which players have multiple types and can condition their actions on their realized types. This in fact amounts to a very simple extensive-form game in which nature acts first to select types, after which players act simultaneously. However, since this extensive-form interpretation provides no additional strategic elements, nothing is gained from an explicit extensive-form treatment of normal-form games in our framework.

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