

SUPPLEMENT TO “A UNIQUE COSTLY  
CONTEMPLATION REPRESENTATION”  
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This online supplement provides additional material to accompany the printed paper, henceforth denoted ES. In Section S.1, we review some mathematical background on Fenchel duality used in the proof of the representation and uniqueness results in ES. In Section S.2, we establish the necessity of the L-continuity axiom for the signed RFCC representation defined in Appendix C. Section S.3 includes the details of the proof of Proposition 1, which constructs the function  $V$  in a signed RFCC representation. Finally, in Section S.4, we give a proof of Equation (24), which is used in the proof of Theorem 6, characterizing constant cost functions in a signed RFCC representation.

S.1. MATHEMATICAL BACKGROUND: FENCHEL DUALITY

IN THIS SECTION we present some general mathematical results that are used to prove the representation and uniqueness theorems in ES. The results will center around a classic duality relationship from convex analysis. Throughout this section, let  $X$  be a real Banach space and let  $X^*$  denote the space of all continuous linear functionals on  $X$ .

We now introduce the standard definition of the subdifferential of a function.

DEFINITION S.1: Suppose  $C \subset X$  and  $f : C \rightarrow \mathbb{R}$ . For  $x \in C$ , the *subdifferential* of  $f$  at  $x$  is defined to be

$$\partial f(x) = \{x^* \in X^* : \langle y - x, x^* \rangle \leq f(y) - f(x) \text{ for all } y \in C\}.$$

The subdifferential is useful for the approximation of convex functions by affine functions. It is straightforward to show that  $x^* \in \partial f(x)$  if and only if the affine function  $h : X \rightarrow \mathbb{R}$  defined by  $h(y) = f(x) + \langle y - x, x^* \rangle$  satisfies  $h \leq f$  and  $h(x) = f(x)$ . It should also be noted that when  $X$  is infinite dimensional, it is possible to have  $\partial f(x) = \emptyset$  for some  $x \in C$ , even if  $f$  is convex. However, it is shown in Ergin and Sarver (2010) that if  $C \subset X$  is convex and  $f : C \rightarrow \mathbb{R}$  is Lipschitz continuous and convex, then  $\partial f(x) \neq \emptyset$  for all  $x \in C$ . The formal definition of Lipschitz continuity follows.

DEFINITION S.2: Suppose  $C \subset X$ . A function  $f : C \rightarrow \mathbb{R}$  is said to be *Lipschitz continuous* if there is some real number  $K$  such that for every  $x, y \in C$ ,  $|f(x) - f(y)| \leq K\|x - y\|$ . The number  $K$  is called a *Lipschitz constant* of  $f$ .

We now introduce the definition of the conjugate of a function.

DEFINITION S.3: Suppose  $C \subset X$  and  $f: C \rightarrow \mathbb{R}$ . The *conjugate* (or *Fenchel conjugate*) of  $f$  is the function  $f^*: X^* \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$f^*(x^*) = \sup_{x \in C} [\langle x, x^* \rangle - f(x)].$$

There is an important duality between  $f$  and  $f^*$ . Lemma S.1 summarizes certain properties of  $f^*$  that are useful in establishing this duality.<sup>47</sup>

LEMMA S.1: *Suppose  $C \subset X$  and  $f: C \rightarrow \mathbb{R}$ . Then the following statements hold:*

- (i)  $f^*$  is lower semicontinuous in the weak\* topology.
- (ii)  $f(x) \geq \langle x, x^* \rangle - f^*(x^*)$  for all  $x \in C$  and  $x^* \in X^*$ .
- (iii)  $f(x) = \langle x, x^* \rangle - f^*(x^*)$  if and only if  $x^* \in \partial f(x)$ .

PROOF: (i) For any  $x \in C$ , the mapping  $x^* \mapsto \langle x, x^* \rangle - f(x)$  is continuous in the weak\* topology. Therefore, for all  $\alpha \in \mathbb{R}$ ,  $\{x^* \in X^* : \langle x, x^* \rangle - f(x) \leq \alpha\}$  is weak\* closed. Hence,

$$\{x^* \in X^* : f^*(x^*) \leq \alpha\} = \bigcap_{x \in C} \{x^* \in X^* : \langle x, x^* \rangle - f(x) \leq \alpha\}$$

is closed for all  $\alpha \in \mathbb{R}$ . Thus  $f^*$  is lower semicontinuous.

(ii) For any  $x \in C$  and  $x^* \in X^*$ , we have

$$f^*(x^*) = \sup_{x' \in C} [\langle x', x^* \rangle - f(x')] \geq \langle x, x^* \rangle - f(x)$$

and, therefore,  $f(x) \geq \langle x, x^* \rangle - f^*(x^*)$ .

(iii) By the definition of the subdifferential,  $x^* \in \partial f(x)$  if and only if

$$(S.1) \quad \langle y, x^* \rangle - f(y) \leq \langle x, x^* \rangle - f(x)$$

for all  $y \in C$ . By the definition of the conjugate, Equation (S.1) holds if and only if  $f^*(x^*) = \langle x, x^* \rangle - f(x)$ , which is equivalent to  $f(x) = \langle x, x^* \rangle - f^*(x^*)$ . *Q.E.D.*

Suppose that  $C \subset X$  is convex and  $f: C \rightarrow \mathbb{R}$  is Lipschitz continuous and convex. As noted above, this implies that  $\partial f(x) \neq \emptyset$  for all  $x \in C$ . Therefore, by parts (ii) and (iii) of Lemma S.1, we have

$$(S.2) \quad f(x) = \max_{x^* \in X^*} [\langle x, x^* \rangle - f^*(x^*)]$$

<sup>47</sup>For a complete discussion of the relationship between  $f$  and  $f^*$ , see Ekeland and Turnbull (1983) or Holmes (1975). A finite-dimensional treatment can be found in Rockafellar (1970).

for all  $x \in C$ .<sup>48</sup> To establish the existence of a minimal set of measures in Theorem 3 in ES, it is useful to establish that under certain assumptions, there is a minimal compact subset of  $X^*$  for which Equation (S.2) holds. Let  $C_f$  denote the set of all  $x \in C$  for which the subdifferential of  $f$  at  $x$  is a singleton:

$$(S.3) \quad C_f = \{x \in C : \partial f(x) \text{ is a singleton}\}.$$

Let  $\mathcal{N}_f$  denote the set of functionals contained in the subdifferential of  $f$  at some  $x \in C_f$ :

$$(S.4) \quad \mathcal{N}_f = \{x^* \in X^* : x^* \in \partial f(x) \text{ for some } x \in C_f\}.$$

Finally, let  $\mathcal{M}_f$  denote the closure of  $\mathcal{N}_f$  in the weak\* topology:

$$(S.5) \quad \mathcal{M}_f = \overline{\mathcal{N}_f}.$$

**THEOREM S.1—Ergin and Sarver (2010):** *Suppose (i)  $X$  is a separable Banach space, (ii)  $C$  is a closed and convex subset of  $X$  containing the origin such that  $\text{span}(C)$  is dense in  $X$ , and (iii)  $f : C \rightarrow \mathbb{R}$  is Lipschitz continuous and convex. Then  $\mathcal{M}_f$  is weak\* compact and for any weak\* compact  $\mathcal{M} \subset X^*$ ,*

$$\mathcal{M}_f \subset \mathcal{M} \iff f(x) = \max_{x^* \in \mathcal{M}} [\langle x, x^* \rangle - f^*(x^*)] \quad \forall x \in C.$$

The intuition for Theorem S.1 is fairly simple. We already know from Lemma S.1 that for any  $x \in C_f$ ,  $f(x) = \max_{x^* \in \mathcal{N}_f} [\langle x, x^* \rangle - f^*(x^*)]$ . Ergin and Sarver (2010) showed that under the assumptions of Theorem S.1,  $C_f$  is dense in  $C$ . Therefore, it can be shown that for any  $x \in C$ ,

$$f(x) = \sup_{x^* \in \mathcal{N}_f} [\langle x, x^* \rangle - f^*(x^*)] = \max_{x^* \in \mathcal{M}_f} [\langle x, x^* \rangle - f^*(x^*)].$$

In addition, if  $\mathcal{M}$  is a weak\* compact subset of  $X^*$  and  $\mathcal{M}_f$  is not a subset of  $\mathcal{M}$ , then there exists  $x^* \in \mathcal{N}_f$  such that  $x^* \notin \mathcal{M}$ . That is, there exists  $x \in C_f$  such that  $\partial f(x) = \{x^*\}$  and  $x^* \notin \mathcal{M}$ . Therefore, Lemma S.1 implies  $f(x) > \max_{x^* \in \mathcal{M}} [\langle x, x^* \rangle - f^*(x^*)]$ .

In the proof of Theorem 3 in ES, we construct an RFCC representation in which  $\mathcal{M}_f$ , for a certain function  $f$ , is the set of measures. The  $\Rightarrow$  direction in Theorem S.1 is used to show that this set is sufficient for representing the preference. However, the function  $f$  is only pinned down up to a positive affine transformation by the preference. Therefore, to prove that  $\mathcal{M}_f$  is minimal in

<sup>48</sup>This is a slight variation of the classic Fenchel–Moreau theorem. The standard version of this theorem states that if  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous and convex, then  $f(x) = f^{**}(x) \equiv \sup_{x^* \in X^*} [\langle x, x^* \rangle - f^*(x^*)]$ . See, for example, Proposition 1 in Ekeland and Turnbull (1983, p. 97).

the sense of Definition 2, we need a slightly stronger version of the  $\Leftarrow$  direction in Theorem S.1. The following proposition imposes an additional “consistency” assumption on the functionals in  $\mathcal{M}_f$  and  $\mathcal{M}$  that allows us to obtain the necessary result.

**PROPOSITION S.1:** *Suppose  $X$  is a Banach space,  $C$  is a convex subset of  $X$ , and  $f: C \rightarrow \mathbb{R}$ . Let  $\mathcal{M} \subset X^*$  be weak\* compact. Assume that there exists  $\bar{x} \in X$  such that  $\langle \bar{x}, x^* \rangle = \langle \bar{x}, y^* \rangle \neq 0$  for any  $x^*, y^* \in \mathcal{M}_f \cup \mathcal{M}$ . Define  $g: C \rightarrow \mathbb{R}$  by*

$$g(x) = \max_{x^* \in \mathcal{M}} [\langle x, x^* \rangle - f^*(x^*)].$$

*Then if there exists  $\alpha > 0$  and  $\beta \in \mathbb{R}$  such that  $g = \alpha f + \beta$ , it must be that  $\mathcal{M}_f \subset \mathcal{M}$ .*

**PROOF:** Note that  $f^*$  is weak\* lower semicontinuous by part (i) of Lemma S.1. Therefore, by part (ii) of Theorem S.3 (below) applied to  $g$ , we have that  $\mathcal{M}_g \subset \mathcal{M}$ . It is easy to show that  $g = \alpha f + \beta$  implies  $\mathcal{M}_g = \alpha \mathcal{M}_f$ . Thus, if we take any  $x^* \in \mathcal{M}_f$ , then  $\alpha x^* \in \alpha \mathcal{M}_f = \mathcal{M}_g \subset \mathcal{M}$ . We therefore have  $\langle \bar{x}, x^* \rangle = \langle \bar{x}, \alpha x^* \rangle \neq 0$ , which can be possible only if  $\alpha = 1$ . This implies that  $\mathcal{M}_f = \mathcal{M}_g \subset \mathcal{M}$ . *Q.E.D.*

For the next result, assume that  $X$  is a Banach lattice.<sup>49</sup> Let  $X_+ = \{x \in X : x \geq 0\}$  denote the *positive cone* of  $X$ . A function  $f: C \rightarrow \mathbb{R}$  on a subset  $C$  of  $X$  is *monotone* if  $f(x) \geq f(y)$  whenever  $x, y \in C$  are such that  $x \geq y$ . A continuous linear functional  $x^* \in X^*$  is *positive* if  $\langle x, x^* \rangle \geq 0$  for all  $x \in X_+$ . The following proposition establishes that under suitable conditions, if  $f$  is monotone, then the functionals in  $\mathcal{M}_f$  are all positive. See Section S.1.1 for the proof.

**THEOREM S.2:** *Suppose  $C$  is a convex subset of a Banach lattice  $X$ , such that at least one of the following conditions holds:*

- (i)  $x \vee x' \in C$  for any  $x, x' \in C$ , or
- (ii)  $x \wedge x' \in C$  for any  $x, x' \in C$ .

*Let  $f: C \rightarrow \mathbb{R}$  be Lipschitz continuous, convex, and monotone. Then, the functionals in  $\mathcal{M}_f$  are positive.*

The following result will be used in the proof of Theorem 4 in ES to establish the uniqueness of the RFCC representation. See Section S.1.2 for the proof.

**THEOREM S.3:** *Suppose  $X$  is a Banach space and  $C$  is a convex subset of  $X$ . Let  $\mathcal{M}$  be a weak\* compact subset of  $X^*$  and let  $c: \mathcal{M} \rightarrow \mathbb{R}$  be weak\* lower semicontinuous. Define  $f: C \rightarrow \mathbb{R}$  by*

$$(S.6) \quad f(x) = \max_{x^* \in \mathcal{M}} [\langle x, x^* \rangle - c(x^*)].$$

<sup>49</sup>See Aliprantis and Border (1999, p. 302) for a definition of Banach lattices.

Then the following statements hold:

- (i) The function  $f$  is Lipschitz continuous and convex.
- (ii) For all  $x \in C$ , there exists  $x^* \in \partial f(x)$  such that  $x^* \in \mathcal{M}$  and  $f^*(x^*) = c(x^*)$ . In particular, this implies  $\mathcal{N}_f \subset \mathcal{M}$ ,  $\mathcal{M}_f \subset \mathcal{M}$ , and  $f^*(x^*) = c(x^*)$  for all  $x^* \in \mathcal{N}_f$ .
- (iii) If  $C$  is also compact (in the norm topology), then  $f^*(x^*) = c(x^*)$  for all  $x^* \in \mathcal{M}_f$ .

### S.1.1. Proof of Theorem S.2

We will show that under the conditions stated in the theorem, for every  $x \in C$ , there exists a positive  $x^* \in \partial f(x)$ . This will imply in particular that for  $x \in C_f$ , we have  $\partial f(x) = \{x^*\}$  for some positive  $x^* \in X^*$ . Therefore, the functions in  $\mathcal{N}_f$  are positive, and since the set of positive functionals is weak\* closed in  $X^*$ , every  $x^* \in \overline{\mathcal{M}_f} = \overline{\mathcal{N}_f}$  is also positive.

The remainder of the proof consists of showing that for every  $x \in C$ , there exists a positive  $x^* \in \partial f(x)$ . We begin by introducing the standard definition of the epigraph of a function  $f: C \rightarrow \mathbb{R}$ :

$$\text{epi}(f) = \{(x, t) \in C \times \mathbb{R} : t \geq f(x)\}.$$

Note that  $\text{epi}(f) \subset X \times \mathbb{R}$  is a convex set because  $f$  is convex with a convex domain  $C$ . Now, for any  $x \in C$ , define

$$H(x) = \{(y, t) \in X \times \mathbb{R} : t < f(x) - K\|y - x\|\}.$$

It is easily seen that  $H(x)$  is nonempty and convex. Also, since  $\|\cdot\|$  is necessarily continuous,  $H(x)$  is open (in the product topology). Note that  $\text{epi}(f) \cap H(x) = \emptyset$  for any  $x \in C$ . To see this, suppose  $(y, t) \in \text{epi}(f)$ , so that  $t \geq f(y)$ . By Lipschitz continuity, we have  $f(y) \geq f(x) - K\|y - x\|$ . Therefore,  $t \geq f(x) - K\|y - x\|$ , which implies  $(y, t) \notin H(x)$ .

Define  $I(x) = H(x) + X_+ \times \{0\}$ . Then  $I(x) \subset X \times \mathbb{R}$  is convex as the sum of two convex sets and it has a nonempty interior since it contains the nonempty open set  $H(x)$ . We will show that if either of the conditions stated in the theorem is satisfied, then  $\text{epi}(f) \cap I(x) = \emptyset$  for any  $x \in C$ .

- *Case 1— $x \vee x' \in C$  for any  $x' \in C$ :* Suppose for a contradiction that  $(x', t) \in \text{epi}(f) \cap I(x)$ . Then  $x' \in C$ , and there exist  $y \in X$  and  $z \in X_+$  such that  $x' = y + z$  and  $t < f(x) - K\|y - x\|$ . Let  $\bar{x} = x \vee x' \in C$ . Note that  $|\bar{x} - x'| = \bar{x} - x' = (x - x')^+$  and  $x - y = x - x' + z \geq x - x'$ . Hence,

$$|x - y| \geq (x - y)^+ \geq (x - x')^+ = |\bar{x} - x'|.$$

Since  $X$  is a Banach lattice, the above inequality implies that  $\|x - y\| \geq \|\bar{x} - x'\|$ . Monotonicity of  $f$  implies that  $f(\bar{x}) \geq f(x)$ . Thus,  $t < f(x) - K\|y - x\| \leq f(\bar{x}) - K\|x' - \bar{x}\|$ , which implies that  $(x', t) \in H(\bar{x})$ , a contradiction to our previous observation that  $\text{epi}(f) \cap H(\bar{x}) = \emptyset$ . Thus,  $\text{epi}(f) \cap I(x) = \emptyset$ .

• *Case 2*— $x \wedge x' \in C$  for any  $x' \in C$ : Suppose again for a contradiction that  $(x', t) \in \text{epi}(f) \cap I(x)$ . Then  $x' \in C$ , and there exist  $y \in X$  and  $z \in X_+$  such that  $x' = y + z$  and  $t < f(x) - K\|y - x\|$ . Let  $\bar{x} = x \wedge x' \in C$ . Note that  $|x - \bar{x}| = x - \bar{x} = (x - x')^+$  and  $x - y = x - x' + z \geq x - x'$ . Therefore,

$$|x - y| \geq (x - y)^+ \geq (x - x')^+ = |x - \bar{x}|,$$

which implies  $\|x - y\| \geq \|x - \bar{x}\|$ . Thus,  $t < f(x) - K\|y - x\| \leq f(x) - K\|\bar{x} - x\|$ , which implies  $(\bar{x}, t) \in H(x)$ . Monotonicity of  $f$  implies that  $f(\bar{x}) \leq f(x') \leq t$ , which implies  $(\bar{x}, t) \in \text{epi}(f)$ , a contradiction to  $\text{epi}(f) \cap H(x) = \emptyset$ . Thus,  $\text{epi}(f) \cap I(x) = \emptyset$ .

Fix any  $x \in C$ . We have shown that under either condition (i) or (ii),  $I(x)$  and  $\text{epi}(f)$  are disjoint sets. Since both sets are convex and  $I(x)$  has nonempty interior, a version of the separating hyperplane theorem (see Theorem 5.50 in Aliprantis and Border (1999)) implies there exists a nonzero continuous linear functional  $(x^*, \lambda) \in X^* \times \mathbb{R}$  that separates  $I(x)$  and  $\text{epi}(f)$ . That is, there exists a scalar  $\delta$  such that

$$(S.7) \quad \langle y, x^* \rangle + \lambda t \leq \delta, \quad \text{if } (y, t) \in \text{epi}(f),$$

$$(S.8) \quad \langle y, x^* \rangle + \lambda t \geq \delta, \quad \text{if } (y, t) \in I(x).$$

Clearly, we cannot have  $\lambda > 0$ . Also, if  $\lambda = 0$ , then Equation (S.8) implies  $x^* = 0$ . This would contradict  $(x^*, \lambda)$  being a nonzero functional. Therefore,  $\lambda < 0$ . Without loss of generality, we can take  $\lambda = -1$ , for otherwise we could renormalize  $(x^*, \lambda)$  and  $\delta$  by dividing by  $|\lambda|$ .

Since  $(x, f(x)) \in \text{epi}(f)$ , we have  $\langle x, x^* \rangle - f(x) \leq \delta$ . For all  $\varepsilon > 0$ , we have  $(x, f(x) - \varepsilon) \in H(x) \subset I(x)$ , which implies  $\langle x, x^* \rangle - f(x) + \varepsilon \geq \delta$ . Therefore,  $\langle x, x^* \rangle - f(x) = \delta$  and thus for all  $y \in C$ ,  $\langle y, x^* \rangle - f(y) \leq \delta = \langle x, x^* \rangle - f(x)$ . Equivalently, we can write  $\langle y - x, x^* \rangle \leq f(y) - f(x)$ . Thus,  $x^* \in \partial f(x)$ .

It only remains to show that  $x^*$  is positive. Let  $y \in X_+$ . Then, for any  $\varepsilon > 0$ ,  $(x + y, f(x) - \varepsilon) \in I(x)$ . By Equation (S.8),

$$\langle x + y, x^* \rangle - f(x) + \varepsilon \geq \delta = \langle x, x^* \rangle - f(x)$$

and hence  $\langle y, x^* \rangle \geq -\varepsilon$ . Since the latter holds for all  $\varepsilon > 0$  and  $y \in X_+$ , we have that  $\langle y, x^* \rangle \geq 0$  for all  $y \in X_+$ . Therefore,  $x^*$  is positive.

### S.1.2. Proof of Theorem S.3

(i) First, note that a solution to Equation (S.6) must exist for every  $x \in C$  since  $\mathcal{M}$  is weak\* compact,  $\langle x, \cdot \rangle$  is weak\* continuous, and  $c$  is weak\* lower semicontinuous. As the maximum of a collection of affine functions,  $f$  is obviously convex. To see that  $f$  is Lipschitz continuous, note that by the weak\* compactness of  $\mathcal{M}$  and Alaoglu's theorem (see Theorem 6.25 in Aliprantis and Border (1999)), there exists  $K \geq 0$  such that  $\|x^*\| \leq K$  for all  $x^* \in \mathcal{M}$ .

Fix any  $x, y \in C$  and let  $x^* \in \mathcal{M}$  be a solution to Equation (S.6) at  $x$ . Thus,  $f(x) = \langle x, x^* \rangle - c(x^*)$ . Since  $f(y) \geq \langle y, x^* \rangle - c(x^*)$ , we have

$$f(x) - f(y) \leq \langle x - y, x^* \rangle \leq \|x - y\| \|x^*\| \leq \|x - y\| K.$$

A similar argument shows  $f(y) - f(x) \leq \|x - y\| K$  and hence  $|f(x) - f(y)| \leq \|x - y\| K$ . Thus,  $f$  is Lipschitz continuous.

(ii) Fix any  $x \in C$ . Let  $x^* \in \mathcal{M}$  be a solution to Equation (S.6), so  $f(x) = \langle x, x^* \rangle - c(x^*)$ . For any  $y \in C$ , we have  $f(y) \geq \langle y, x^* \rangle - c(x^*)$  and hence  $f(y) - f(x) \geq \langle y - x, x^* \rangle$ . Therefore,  $x^* \in \partial f(x)$ . By (iii) in Lemma S.1, this implies that

$$\langle x, x^* \rangle - f^*(x^*) = f(x) = \langle x, x^* \rangle - c(x^*),$$

so  $f(x^*) = c(x^*)$ . To see the other claims, take any  $x^* \in \mathcal{N}_f$ . Then there exists  $x \in C_f$  such that  $\partial f(x) = \{x^*\}$ . By the preceding arguments, this implies  $x^* \in \mathcal{M}$  and  $f^*(x^*) = c(x^*)$ . Since this is true for any  $x^* \in \mathcal{N}_f$ , we have  $\mathcal{N}_f \subset \mathcal{M}$ . Since  $\mathcal{M}$  is weak\* closed,  $\mathcal{M}_f = \overline{\mathcal{N}_f} \subset \mathcal{M}$ .

(iii) We first show that  $c(x^*) \geq f^*(x^*)$  for any  $x^* \in \mathcal{M}$ . Fix any  $x^* \in \mathcal{M}$ . Then, by the definition of  $f$ ,

$$\begin{aligned} f(x) &\geq \langle x, x^* \rangle - c(x^*) \quad \forall x \in C \\ \implies c(x^*) &\geq \langle x, x^* \rangle - f(x) \quad \forall x \in C \\ \implies c(x^*) &\geq \sup_{x \in C} [\langle x, x^* \rangle - f(x)] = f^*(x^*). \end{aligned}$$

Since  $\mathcal{M}_f \subset \mathcal{M}$  by (ii), this implies  $c(x^*) \geq f^*(x^*)$  for all  $x^* \in \mathcal{M}_f$ . Therefore, it remains only to show that  $c(x^*) \leq f^*(x^*)$  for all  $x^* \in \mathcal{M}_f$ . Fix any  $x^* \in \mathcal{M}_f$ . Since  $\mathcal{M}_f = \overline{\mathcal{N}_f}$ , there exists a net  $\{x_d^*\}_{d \in D}$  in  $\mathcal{N}_f$  that converges to  $x^*$  in the weak\* topology. Recall from (ii) that  $f^*(x^*) = c(x^*)$  for all  $x^* \in \mathcal{N}_f$ . Therefore,

$$c(x^*) \leq \liminf_d c(x_d^*) = \liminf_d f^*(x_d^*),$$

where the inequality follows from lower semicontinuity of  $c$  (see Theorem 2.39 in Aliprantis and Border (1999)). The proof is completed by showing that  $\liminf_d f^*(x_d^*) \leq f^*(x^*)$ . To see this, first note that by the definition of the limit inferior, there exists a subnet of  $\{f^*(x_d^*)\}_{d \in D}$  that converges to  $\liminf_d f^*(x_d^*)$ . Without loss of generality, assume that the net itself converges to  $\liminf_d f^*(x_d^*)$ , so  $\lim_d f^*(x_d^*) = \liminf_d f^*(x_d^*)$ . Since  $C$  is compact and  $f$  is continuous (by (1)), for each  $d \in D$  there exists  $x_d \in C$  such that

$$f^*(x_d^*) = \sup_{x \in C} [\langle x, x_d^* \rangle - f(x)] = \langle x_d, x_d^* \rangle - f(x_d).$$

Since  $C$  is compact, the net  $\{x_d\}_{d \in D}$  must have a subnet that converges to some limit  $x \in C$ . Again, without loss of generality, assume that the net itself converges, so  $\lim_d x_d = x$ .

As in (i), note that by the compactness of  $\mathcal{M}$  and Alaoglu's theorem, there exists  $K \geq 0$  such that  $\|x_d^*\| \leq K$  for all  $d \in D$ . Since  $x_d \rightarrow x$ ,  $x_d^* \xrightarrow{w^*} x^*$  and  $\|x_d^*\| \leq K$  for all  $d \in D$ , we have

$$\begin{aligned} |\langle x_d, x_d^* \rangle - \langle x, x^* \rangle| &\leq |\langle x_d - x, x_d^* \rangle| + |\langle x, x_d^* - x^* \rangle| \\ &\leq \|x_d - x\| \|x_d^*\| + |\langle x, x_d^* - x^* \rangle| \\ &\leq \|x_d - x\| K + |\langle x, x_d^* - x^* \rangle| \rightarrow 0, \end{aligned}$$

so  $\langle x_d, x_d^* \rangle \rightarrow \langle x, x^* \rangle$ . Given this result and the continuity of  $f$ , we have

$$\begin{aligned} \liminf_d f^*(x_d^*) &= \lim_d f^*(x_d^*) = \lim_d [\langle x_d, x_d^* \rangle - f(x_d)] \\ &= \langle x, x^* \rangle - f(x) \leq f^*(x^*), \end{aligned}$$

which completes the proof.

## S.2. NECESSITY OF L-CONTINUITY

Throughout this section, we will continue to use the notation for support functions that was introduced in Appendix C.2.

**LEMMA S.2:** *Suppose  $(\mathcal{M}, c)$  is a signed RFCC representation, and define  $V : \mathcal{A} \rightarrow \mathbb{R}$  as in Equation (7) of ES. Then  $V$  is Lipschitz continuous and translation linear.*

**PROOF:** Define  $W : \Sigma \rightarrow \mathbb{R}$  by  $W(\sigma) = V(A_\sigma)$ . Then, for all  $\sigma \in \Sigma$ ,

$$W(\sigma) = \max_{\mu \in \mathcal{M}} [\langle \sigma, \mu \rangle - c(\mu)].$$

By part (i) of Theorem S.3,  $W$  is Lipschitz continuous. Therefore, the restriction of  $V$  to  $\mathcal{A}^c$  is Lipschitz continuous by parts (i) and (iii) of Lemma 5. Let  $K > 0$  be any Lipschitz constant of  $V|_{\mathcal{A}^c}$ . Take any  $A, B \in \mathcal{A}$ . It is easily verified that  $V(A) = V(\text{co}(A))$ ,  $V(B) = V(\text{co}(B))$  and  $d_h(\text{co}(A), \text{co}(B)) \leq d_h(A, B)$ . Hence,

$$\begin{aligned} |V(A) - V(B)| &= |V(\text{co}(A)) - V(\text{co}(B))| \\ &\leq K d_h(\text{co}(A), \text{co}(B)) \leq K d_h(A, B). \end{aligned}$$

Thus,  $V$  is Lipschitz continuous on all of  $\mathcal{A}$  with the same Lipschitz constant  $K$ .

To see that  $V$  is translation linear, note that by consistency, there exists  $v \in \mathbb{R}^Z$  such that for all  $\mu \in \mathcal{M}$  and for all  $p \in \Delta(Z)$ ,

$$\int_{\mathcal{U}} (u \cdot p) \mu(du) = v \cdot p.$$

Thus, for any  $A \in \mathcal{A}$  and  $\theta \in \Theta$  such that  $A + \theta \in \mathcal{A}$ ,

$$\begin{aligned} V(A + \theta) &= \max_{\mu \in \mathcal{M}} \left( \int_{\mathcal{U}} \max_{p \in A + \theta} (u \cdot p) \mu(du) - c(\mu) \right) \\ &= \max_{\mu \in \mathcal{M}} \left( \int_{\mathcal{U}} \max_{p \in A} (u \cdot p) \mu(du) + (v \cdot \theta) - c(\mu) \right) \\ &= V(A) + v \cdot \theta, \end{aligned}$$

so  $V$  is translation linear. *Q.E.D.*

Since the function  $V$  given by a signed RFCC representation must satisfy singleton nontriviality, the following lemma completes the proof of the necessity of L-continuity.

**LEMMA S.3:** *Suppose  $V: \mathcal{A} \rightarrow \mathbb{R}$  represents the preference  $\succsim$ . If  $V$  is Lipschitz continuous and translation linear, and there exists  $p^*, p_* \in \Delta(Z)$  such that  $V(\{p^*\}) > V(\{p_*\})$ , then  $\succsim$  satisfies L-continuity.*

**PROOF:** Translation linearity implies there exists  $v \in \mathbb{R}^Z$  such that for all  $A \in \mathcal{A}$  and  $\theta \in \Theta$  with  $A + \theta \in \mathcal{A}$ , we have  $V(A + \theta) = V(A) + v \cdot \theta$ . Let  $K > 0$  be any Lipschitz constant of  $V$ , let  $\theta^* \equiv p^* - p_*$ , and let  $M \equiv K/(v \cdot \theta^*) > 0$ . Then, for any  $A, B \in \mathcal{A}$ ,

$$V(B) - V(A) \leq K d_h(A, B) = M[v \cdot \theta^*] d_h(A, B).$$

Fix any  $A, B \in \mathcal{A}$  and  $\alpha \in (0, 1)$  with  $\alpha > M d_h(A, B)$ . Then, since  $d_h((1 - \alpha)B + \alpha\{p_*\}, (1 - \alpha)A + \alpha\{p_*\}) = (1 - \alpha)d_h(A, B)$ , and  $M d_h(A, B) < \alpha < 1$ , we have

$$\begin{aligned} &V((1 - \alpha)B + \alpha\{p_*\}) - V((1 - \alpha)A + \alpha\{p_*\}) \\ &\leq M[v \cdot \theta^*](1 - \alpha)d_h(A, B) \\ &< \alpha(1 - \alpha)[v \cdot \theta^*] < \alpha[v \cdot \theta^*]. \end{aligned}$$

This implies

$$\begin{aligned} V((1 - \alpha)B + \alpha\{p_*\}) &< V((1 - \alpha)A + \alpha\{p_*\}) + \alpha[v \cdot \theta^*] \\ &= V((1 - \alpha)A + \alpha\{p^*\}) \end{aligned}$$

or, equivalently,  $(1 - \alpha)A + \alpha\{p^*\} \succ (1 - \alpha)B + \alpha\{p_*\}$ . *Q.E.D.*

## S.3. PROOF OF PROPOSITION 1

We start by establishing some preliminary lemmas that will be useful in proving Proposition 1. We first present one useful consequence of translation invariance (TI) defined in Section C.1.

LEMMA S.4: *Suppose  $\succsim$  satisfies weak order, continuity, and TI. If  $A \in \mathcal{A}$ ,  $\theta \in \Theta$ , and  $\alpha \in (0, 1)$  are such that  $A + \theta \in \mathcal{A}$ , then*

$$(S.9) \quad A \succsim A + \theta \iff A \succsim A + \alpha\theta \iff A + \alpha\theta \succsim A + \theta.$$

PROOF: We will make a simple induction argument. Suppose

$$A + \frac{m-1}{n}\theta \succsim A + \frac{m}{n}\theta$$

for some  $m, n \in \mathbb{N}$  with  $m < n$ . Then adding  $\frac{1}{n}\theta$  to each side of the above and applying TI yields

$$A + \frac{m}{n}\theta \succsim A + \frac{m+1}{n}\theta.$$

Now suppose that  $A \succsim A + \frac{1}{n}\theta$ . Then, using induction and the transitivity of  $\succsim$ , we obtain

$$(S.10) \quad A \succsim A + \frac{1}{n}\theta \succsim \dots \succsim A + \left(1 - \frac{1}{n}\right)\theta \succsim A + \theta.$$

A similar line of reasoning shows that if  $A \prec A + \frac{1}{n}\theta$ , then we obtain

$$(S.11) \quad A \prec A + \frac{1}{n}\theta \prec \dots \prec A + \left(1 - \frac{1}{n}\right)\theta \prec A + \theta.$$

In sum, Equations (S.10) and (S.11) imply that for any  $m, n \in \mathbb{N}$ ,  $1 \leq m < n$ , we have

$$\begin{aligned} A \succsim A + \frac{1}{n}\theta &\iff A \succsim A + \theta \iff A \succsim A + \frac{m}{n}\theta \\ &\iff A + \frac{m}{n}\theta \succsim A + \theta. \end{aligned}$$

This establishes Equation (S.9) for  $\alpha \in (0, 1) \cap \mathbb{Q}$ . The continuity of  $\succsim$  implies that Equation (S.9) holds for all  $\alpha \in (0, 1)$ . *Q.E.D.*

Although we do not assume that independence holds on  $\mathcal{A}$ , our other axioms imply that independence does hold for singleton menus.

AXIOM S.1—Singleton Independence: For all  $p, q, r \in \Delta(Z)$  and  $\lambda \in (0, 1)$ ,

$$\{p\} \succsim \{q\} \iff \lambda\{p\} + (1 - \lambda)\{r\} \succsim \lambda\{q\} + (1 - \lambda)\{r\}.$$

LEMMA S.5: If  $\succsim$  satisfies weak order, continuity, and TI, then it also satisfies singleton independence.

PROOF: Let  $\theta = q - p$  and  $\theta' = (1 - \lambda)(r - p)$ . Then

$$\begin{aligned} \{p\} \succsim \{q\} = \{p\} + \theta &\iff \{p\} \succsim \{p\} + \lambda\theta = (1 - \lambda)\{p\} + \lambda\{q\} \\ &\iff \lambda\{p\} + (1 - \lambda)\{r\} \succsim \lambda\{q\} + (1 - \lambda)\{r\}, \end{aligned}$$

where the first equivalence follows from Lemma S.4, and the second equivalence follows from TI,  $\{p\} + \theta' = \lambda\{p\} + (1 - \lambda)\{r\}$ , and  $(1 - \lambda)\{p\} + \lambda\{q\} + \theta' = \lambda\{q\} + (1 - \lambda)\{r\}$ . Therefore singleton independence is satisfied. *Q.E.D.*

In the remainder of this section, let  $\theta^*$  be as defined in Section C.1, that is,  $\theta^* \equiv p^* - p_*$ , where  $p^*$  and  $p_*$  come from the L-continuity axiom. We will utilize  $\theta^*$  a great deal in the construction of our representation, and the following lemma is an important property of  $\theta^*$ .

LEMMA S.6: Suppose  $\succsim$  satisfies weak order, strong continuity, and TI, and take  $\theta^* = p^* - p_*$ . Let  $A, B \in \mathcal{A}^\circ$  and  $\alpha, \beta \in \mathbb{R}$  be such that  $A \sim B$  and  $A + \alpha\theta^*, B + \beta\theta^* \in \mathcal{A}^c$ . Then

$$(S.12) \quad A + \alpha\theta^* \succsim B + \beta\theta^* \iff \alpha \geq \beta.$$

PROOF: We will first show that for any  $A \in \mathcal{A}^\circ$ , there exists  $\gamma > 0$  such that  $A + \gamma\theta^* \in \mathcal{A}^c$  and  $A + \gamma\theta^* \succ A$ . To see this, fix any  $A \in \mathcal{A}^\circ$ . It follows from part (i) of Lemma 4 that there exist  $A' \in \mathcal{A}^c$  and  $\gamma \in (0, 1)$  such that  $A = (1 - \gamma)A' + \gamma\{p_*\}$ .<sup>50</sup> By L-continuity we have

$$A + \gamma\theta^* = (1 - \gamma)A' + \gamma\{p^*\} \succ (1 - \gamma)A' + \gamma\{p_*\} = A.$$

Therefore, for any  $A \in \mathcal{A}^\circ$  and  $\alpha > 0$  such that  $A + \alpha\theta^* \in \mathcal{A}^c$ , take  $\gamma > 0$  such that  $A + \gamma\theta^* \in \mathcal{A}^c$  and  $A + \gamma\theta^* \succ A$ . Applying Lemma S.4 to  $A$  and  $\theta = \max\{\gamma, \alpha\}\theta^*$ , we have  $A + \alpha\theta^* \succ A$ . A similar argument shows that if  $A \in \mathcal{A}^\circ$  and  $\alpha < 0$  are such that  $A + \alpha\theta^* \in \mathcal{A}^c$ , then  $A \succ A + \alpha\theta^*$ .

Now, let  $A, B \in \mathcal{A}^\circ$  and  $\alpha, \beta \in \mathbb{R}$  be such that  $A \sim B$  and  $A + \alpha\theta^*, B + \beta\theta^* \in \mathcal{A}^c$ . We prove the equivalence from Equation (S.12) by considering three cases:

If  $\alpha = \beta$ , then  $\alpha\theta^* = \beta\theta^*$ . Hence by TI,  $A + \alpha\theta^* \sim B + \beta\theta^*$ .

<sup>50</sup>Take  $\varepsilon > 0$  as in Lemma 4, and let  $\gamma \equiv \varepsilon$  and  $A' \equiv \{q \in \mathbb{R}^Z : q = \frac{1}{1-\gamma}(p - \gamma p_*) \text{ for some } p \in A\}$ . It follows that  $A' \in \mathcal{A}^c$  and  $A = (1 - \gamma)A' + \gamma\{p_*\}$ .

If  $\alpha > \beta$ , there are three subcases to consider. First consider  $\alpha > \beta \geq 0$ , which implies  $0 < \alpha - \beta \leq \alpha$  and hence  $A + (\alpha - \beta)\theta^* \in \mathcal{A}^c$ . From the above arguments,  $A + (\alpha - \beta)\theta^* \succ A \sim B$ , so by TI,  $A + \alpha\theta^* = [A + (\alpha - \beta)\theta^*] + \beta\theta^* \succ B + \beta\theta^*$ . Similarly, if  $0 \geq \alpha > \beta$ , then  $\beta \leq \beta - \alpha < 0$  and hence  $B + (\beta - \alpha)\theta^* \in \mathcal{A}^c$ . From the above arguments,  $A \sim B \succ B + (\beta - \alpha)\theta^*$ , which implies by TI that  $A + \alpha\theta^* \succ [B + (\beta - \alpha)\theta^*] + \alpha\theta^* = B + \beta\theta^*$ . Finally,  $\alpha > 0 > \beta$  implies  $A + \alpha\theta^* \succ A \sim B \succ B + \beta\theta^*$ .

If  $\beta > \alpha$ , then by symmetric arguments  $B + \beta\theta^* \succ A + \alpha\theta^*$ . *Q.E.D.*

In the remainder of the section, let  $\mathcal{S}$ ,  $v$ ,  $\mathcal{A}^\circ$ , and the sequences  $\mathcal{A}_0, \mathcal{A}'_0, \mathcal{A}_1, \mathcal{A}'_1, \dots$  and  $V_0, V'_0, V_1, V'_1, \dots$  be as defined in Section C.1. We now present some important properties of  $\mathcal{A}_i$  and  $\mathcal{A}'_i$  that will be used to prove Lemmas S.8 and S.9.

LEMMA S.7: *For any  $i \geq 0$ , the following statements hold:*

(i) *If  $A \in \mathcal{A}_i$  and  $\theta \in \Theta$ , then there exists  $\bar{\alpha} > 0$  such that*

$$(S.13) \quad A + \alpha\theta \in \mathcal{A}_i \quad \forall \alpha \in [0, \bar{\alpha}].^{51}$$

(ii) *For all  $A, B \in \mathcal{A}'_i$  and  $C \in \mathcal{A}^\circ$ ,  $A \succsim C \succsim B$  implies  $C \in \mathcal{A}'_i$ .*

PROOF: (i) First, it follows immediately from part (i) of Lemma 4 that for any  $A \in \mathcal{A}^\circ$  and  $\theta \in \Theta$ , there exists  $\bar{\alpha} > 0$  such that

$$(S.14) \quad A + \alpha\theta \in \mathcal{A}^\circ \quad \forall \alpha \in [0, \bar{\alpha}].$$

We now prove by induction. To verify the property on  $\mathcal{A}_0 = \mathcal{A}^\circ \cap \mathcal{S}$ , take any  $A \in \mathcal{A}_0$  and recall that  $A = \{p\}$  for some  $p \in \Delta(Z)$ . Then take  $\bar{\alpha} > 0$  such that Equation (S.14) holds. Then for all  $\alpha \in [0, \bar{\alpha}]$ , since  $p + \alpha\theta \in \Delta(Z)$ , we have  $A + \alpha\theta \in \mathcal{A}_0$ .

We now prove that if the property holds for  $\mathcal{A}_i$ , then it also holds for  $\mathcal{A}_{i+1}$ . Take any  $A \in \mathcal{A}_{i+1}$  and  $\theta \in \Theta$ . Then,  $A = B + \beta\theta^*$  for some  $B \in \mathcal{A}'_i$  and  $\beta \in \mathbb{R}$ , and hence  $B \sim C$  for some  $C \in \mathcal{A}_i$ . Choose  $\bar{\alpha} > 0$  to be the minimum of that required to satisfy Equation (S.14) for  $A$  and  $B$  and to satisfy Equation (S.13) for  $C$ . Fix any  $\alpha \in [0, \bar{\alpha}]$ . Then  $C + \alpha\theta \in \mathcal{A}_i$ . Since  $B + \alpha\theta \sim C + \alpha\theta$  by  $B \sim C$  and TI, this implies  $B + \alpha\theta \in \mathcal{A}'_i$ . Thus,  $B + \alpha\theta + \beta\theta^* = A + \alpha\theta \in \mathcal{A}_{i+1}$ .

(ii) We again prove by induction. To prove the result for  $\mathcal{A}'_0$ , suppose  $A, B \in \mathcal{A}'_0$  and  $A \succsim C \succsim B$  for some  $C \in \mathcal{A}^\circ$ . Since  $A, B \in \mathcal{A}'_0$ , there exist  $\{p\}, \{q\} \in \mathcal{A}_0$  such that  $\{p\} \sim A \succsim C \succsim B \sim \{q\}$ . Continuity implies there exists a  $\lambda \in [0, 1]$  such that  $\{\lambda p + (1 - \lambda)q\} \sim C$ . By the convexity of  $\mathcal{A}_0 = \mathcal{A}^\circ \cap \mathcal{S}$  and the definition of  $\mathcal{A}'_0$ , this implies that  $C \in \mathcal{A}'_0$ .

We now show that if  $\mathcal{A}'_i$  satisfies the desired condition, then  $\mathcal{A}'_{i+1}$  does also. Suppose  $A, B \in \mathcal{A}'_{i+1}$  and  $A \succsim C \succsim B$  for some  $C \in \mathcal{A}^\circ$ . If there exist  $A', B' \in \mathcal{A}'_i$  such that  $A' \succsim C \succsim B'$ , then  $C \in \mathcal{A}'_i \subset \mathcal{A}'_{i+1}$  by the induction assumption. Thus

<sup>51</sup>As the proof of this lemma will illustrate, the same property holds for  $\mathcal{A}'_i$ .

without loss of generality, suppose  $C \succ A'$  for all  $A' \in \mathcal{A}'_i$ . Since  $A \in \mathcal{A}'_{i+1}$ , there exists a  $A' \in \mathcal{A}'_{i+1}$  such that  $A' \sim A \succsim C$ . Since  $A' \in \mathcal{A}'_{i+1}$ , there exists a  $A'' \in \mathcal{A}'_i$  and  $\alpha \in \mathbb{R}$  such that  $A' = A'' + \alpha\theta^*$ . Since  $A'' \in \mathcal{A}'_i$  implies  $C \succ A''$ , this implies  $A'' + \alpha\theta^* \succsim C \succ A''$  and, therefore,  $\alpha > 0$  by Lemma S.6. By continuity, there exists a  $\alpha' \in [0, \alpha]$  such that  $A'' + \alpha'\theta^* \sim C$ . But  $A'' + \alpha'\theta^* \in \mathcal{A}'_{i+1}$ , so it must be that  $C \in \mathcal{A}'_{i+1}$ . Q.E.D.

The following lemmas allow us to prove the desired properties of each  $V_i$  and  $V'_i$  by induction.

LEMMA S.8: *For all  $i \geq 0$ , if  $V_i$  is well defined, translation linear, and represents  $\succsim$  on  $\mathcal{A}_i$ , then  $V'_i$  is also well defined, translation linear, and represents  $\succsim$  on  $\mathcal{A}'_i$ .*

PROOF: *Well Defined:* Suppose  $A \in \mathcal{A}'_i$  and  $B, B' \in \mathcal{A}_i$  are such that  $A \sim B$  and  $A \sim B'$ . Since  $V_i$  represents  $\succsim$  on  $\mathcal{A}_i$  and  $\succsim$  is transitive,  $V_i(B) = V_i(B')$ , and hence  $V'_i(A)$  is uniquely defined.

*Represents  $\succsim$ :* If  $A, A' \in \mathcal{A}'_i$ , then there exist  $B, B' \in \mathcal{A}_i$  such that  $A \sim B$  and  $A' \sim B'$ . Therefore,  $V'_i(A) = V_i(B) \geq V_i(B') = V'_i(A')$  if and only if  $B \succsim B'$  if and only if  $A \succsim A'$ , so  $V'_i$  represents  $\succsim$  on  $\mathcal{A}'_i$ .

*Translation Linear:* Throughout we will use the fact if  $\theta \in \Theta$  and  $A, A + \theta \in \mathcal{A}'_i$ , then  $A + \alpha\theta \in \mathcal{A}'_i$  for all  $\alpha \in [0, 1]$ . This follows by part (ii) of Lemma S.7 because by Lemma S.4, either  $A + \theta \succsim A + \alpha\theta \succsim A$  or  $A \succsim A + \alpha\theta \succsim A + \theta$ .

We first show that  $V'_i$  satisfies the following local version of translation linearity: For all  $A \in \mathcal{A}'_i$  and  $\theta \in \Theta$  with  $A + \theta \in \mathcal{A}'_i$ , there exist  $\bar{\alpha} > 0$  such that for all  $\alpha \in [0, \bar{\alpha}]$ ,

$$V'_i(A + \alpha\theta) = V'_i(A) + \alpha(v \cdot \theta).$$

To see this property holds, suppose  $\theta \in \Theta$  and  $A, A + \theta \in \mathcal{A}'_i$ . By the definition of  $\mathcal{A}'_i$  there exists  $B \in \mathcal{A}_i$  such that  $A \sim B$ . By part (i) of Lemma S.7, there exists  $\bar{\alpha} \in (0, 1]$  such that  $B + \alpha\theta \in \mathcal{A}_i$  for all  $\alpha \in [0, \bar{\alpha}]$ . Fix any  $\alpha \in [0, \bar{\alpha}]$ , and  $A \sim B$  implies  $A + \alpha\theta \sim B + \alpha\theta$  by TI. Therefore, using the translation linearity of  $V_i$  on  $\mathcal{A}_i$ ,

$$V'_i(A + \alpha\theta) = V_i(B + \alpha\theta) = V_i(B) + \alpha(v \cdot \theta) = V'_i(A) + \alpha(v \cdot \theta).$$

We now show that this local version of translation linearity implies translation linearity. Fix any  $A \in \mathcal{A}'_i$  and  $\theta \in \Theta$  with  $A + \theta \in \mathcal{A}'_i$ , and let

$$\alpha^* \equiv \sup\{\bar{\alpha} \in [0, 1] : V'_i(A + \alpha\theta) = V'_i(A) + \alpha(v \cdot \theta) \forall \alpha \in [0, \bar{\alpha}]\}.$$

Note that  $V'_i(A + \alpha^*\theta) = V'_i(A) + \alpha^*(v \cdot \theta)$ . If  $\alpha^* = 0$ , this is obvious. If  $\alpha^* > 0$ , then local translation linearity applied to  $A' = A + \alpha^*\theta \in \mathcal{A}'_i$  and  $\theta' = -\alpha^*\theta$

implies there exists  $\bar{\alpha} > 0$  such that  $V'_i(A + \alpha^*\theta - \bar{\alpha}\theta) = V'_i(A + \alpha^*\theta) - \bar{\alpha}(v \cdot \theta)$ . Therefore,

$$\begin{aligned} V'_i(A + \alpha^*\theta) &= V'_i(A + (\alpha^* - \bar{\alpha})\theta) + \bar{\alpha}(v \cdot \theta) \\ &= V'_i(A) + (\alpha^* - \bar{\alpha})(v \cdot \theta) + \bar{\alpha}(v \cdot \theta) \\ &= V'_i(A) + \alpha^*(v \cdot \theta), \end{aligned}$$

where the second equality follows by the definition of  $\alpha^*$  since  $0 < \alpha^* - \bar{\alpha} < \alpha^*$ . It remains only to show that  $\alpha^* = 1$ . If not, then local translation linearity applied to  $A' = A + \alpha^*\theta \in \mathcal{A}'_i$  and  $\theta' = (1 - \alpha^*)\theta$  implies there exists  $\bar{\alpha} > 0$  such that for all  $\alpha \in [0, \bar{\alpha}]$ ,

$$\begin{aligned} V'_i(A + \alpha^*\theta + \alpha\theta) &= V'_i(A + \alpha^*\theta) + \alpha(v \cdot \theta) \\ &= V'_i(A) + (\alpha^* + \alpha)(v \cdot \theta). \end{aligned}$$

This would imply  $\alpha^* \geq \alpha^* + \bar{\alpha}$ , a contradiction. Thus  $\alpha^* = 1$ . *Q.E.D.*

**LEMMA S.9:** *For all  $i \geq 1$ , if  $V'_{i-1}$  is well defined, translation linear, and represents  $\succsim$  on  $\mathcal{A}'_{i-1}$ , then  $V_i$  is also well defined, translation linear, and represents  $\succsim$  on  $\mathcal{A}_i$ .*

**PROOF:** *Well Defined:* Suppose  $A \in \mathcal{A}_i$  and  $A = B + \alpha\theta^* = B' + \alpha'\theta^*$  for  $B, B' \in \mathcal{A}'_{i-1}$  and  $\alpha, \alpha' \in \mathbb{R}$ . Then  $B = B' + (\alpha' - \alpha)\theta^*$ , so the translation linearity of  $V'_{i-1}$  implies  $V'_{i-1}(B) = V'_{i-1}(B') + (\alpha' - \alpha)(v \cdot \theta^*)$ . Therefore,  $V'_{i-1}(B) + \alpha(v \cdot \theta^*) = V'_{i-1}(B') + \alpha'(v \cdot \theta^*)$ , and hence  $V_i(A)$  is uniquely defined.

*Translation Linear:* Suppose  $\theta \in \Theta$  and  $A, A + \theta \in \mathcal{A}_i$ . Then there exist  $B, B' \in \mathcal{A}'_{i-1}$  and  $\alpha, \alpha' \in \mathbb{R}$  such that  $A = B + \alpha\theta^*$  and  $A + \theta = B' + \alpha'\theta^*$ . Then  $B' = B + (\alpha - \alpha')\theta^* + \theta$ , so the translation linearity of  $V'_{i-1}$  implies  $V'_{i-1}(B') = V'_{i-1}(B) + v \cdot [(\alpha - \alpha')\theta^* + \theta]$ . By the definition of  $V_i$ , we therefore have

$$\begin{aligned} V_i(A + \theta) &= V'_{i-1}(B') + \alpha'(v \cdot \theta^*) \\ &= V'_{i-1}(B) + \alpha(v \cdot \theta^*) + v \cdot \theta = V_i(A) + v \cdot \theta. \end{aligned}$$

*Represents  $\succsim$ :* Suppose  $A, A' \in \mathcal{A}_i$ , so that  $A = B + \alpha\theta^*$  and  $A' = B' + \alpha'\theta^*$  for some  $B, B' \in \mathcal{A}'_{i-1}$  and  $\alpha, \alpha' \in \mathbb{R}$ . There are several different cases to consider, but in the interest of brevity, we only work through one of them here:  $A, A' \succsim B' \succsim B$ . Thus  $B + \alpha\theta^* \succsim B' \succsim B$ , which implies  $\alpha \geq 0$  by Lemma S.6. Continuity implies there exists  $\alpha'' \in [0, \alpha]$  such that  $B + \alpha''\theta^* \sim B'$ , which therefore implies  $B + \alpha''\theta^* \in \mathcal{A}'_{i-1}$ . Thus by Lemma S.6 and the definition of  $V_i$ , we have  $A \succsim A'$  if and only if  $\alpha - \alpha'' \geq \alpha'$  if and only if

$$\begin{aligned} V_i(A) &= V'_{i-1}(B + \alpha''\theta^*) + (\alpha - \alpha'')(v \cdot \theta^*) \\ &= V'_{i-1}(B') + (\alpha - \alpha'')(v \cdot \theta^*) \\ &\geq V'_{i-1}(B') + \alpha'(v \cdot \theta^*) = V_i(A'). \end{aligned}$$

The other cases are similar.<sup>52</sup>

*Q.E.D.*

Using induction and the results of Lemmas S.8 and S.9, we have proved that for all  $i \geq 0$ ,  $V_i: \mathcal{A}_i \rightarrow \mathbb{R}$  is well defined, translation linear, and represents  $\succsim$  on  $\mathcal{A}_i$ . We now define a function  $\hat{V}: \bigcup_i \mathcal{A}_i \rightarrow \mathbb{R}$  by  $\hat{V}(A) \equiv V_i(A)$  if  $A \in \mathcal{A}_i$ . This is well defined because if  $A \in \mathcal{A}_i$  and  $A \in \mathcal{A}_j$ , then without loss of generality suppose  $j \geq i$ , so  $\mathcal{A}_i \subset \mathcal{A}_j$ . Then  $V_j(B) = V_i(B)$  for all  $B \in \mathcal{A}_i$ , and hence  $V_j(A) = V_i(A)$ . Note that  $\hat{V}$  represents  $\succsim$  on  $\bigcup_i \mathcal{A}_i$  and is translation linear.

By the following lemma, we have now established a translation-linear representation for  $\succsim$  on all of  $\mathcal{A}^\circ$ .

LEMMA S.10:  $\mathcal{A}^\circ = \bigcup_i \mathcal{A}_i$ .

PROOF: The inclusion  $\bigcup_i \mathcal{A}_i \subset \mathcal{A}^\circ$  follows immediately from the definition of  $\mathcal{A}_i$ , so it remains only to prove that  $\mathcal{A}^\circ \subset \bigcup_i \mathcal{A}_i$ . Consider any set  $A \in \mathcal{A}^\circ$ . By the definition of  $\mathcal{A}^\circ$ , there exists some  $\alpha > 0$  such that  $A + \alpha\theta^*$ ,  $A - \alpha\theta^* \in \mathcal{A}^\circ$ . Fix any  $p \in A$ , and we therefore have  $\{p\} + \alpha\theta^*$ ,  $\{p\} - \alpha\theta^* \in \mathcal{A}_0 \subset \mathcal{A}^\circ$ . For every  $\lambda \in [0, 1]$ , define  $A(\lambda) \equiv \lambda A + (1 - \lambda)\{p\}$ . Note that  $A(\lambda) + \alpha\theta^*$ ,  $A(\lambda) - \alpha\theta^* \in \mathcal{A}^\circ$ , which follows from the convexity of  $\mathcal{A}^\circ$  since

$$\begin{aligned} A(\lambda) + \alpha\theta^* &= \lambda A + (1 - \lambda)\{p\} + \alpha\theta^* \\ &= \lambda(A + \alpha\theta^*) + (1 - \lambda)(\{p\} + \alpha\theta^*), \end{aligned}$$

and similarly for  $A(\lambda) - \alpha\theta^*$ . By Lemma S.6, for all  $\lambda \in [0, 1]$ ,  $A(\lambda) + \alpha\theta^* \succ A(\lambda) \succ A(\lambda) - \alpha\theta^*$ . By continuity, for each  $\lambda$  there exists an open (relative to  $[0, 1]$ ) interval  $e(\lambda)$  such that  $\lambda \in e(\lambda)$  and for all  $\lambda' \in e(\lambda)$ ,

$$A(\lambda) + \alpha\theta^* \succ A(\lambda') \succ A(\lambda) - \alpha\theta^*.$$

Thus  $\{e(\lambda) : \lambda \in [0, 1]\}$  is an open cover of  $[0, 1]$ . Since  $[0, 1]$  is compact, there exists a finite subcover,  $\{e(\lambda_1), \dots, e(\lambda_n)\}$ . Assume the  $\lambda_i$ 's are ordered so that  $e(\lambda_i) \cap e(\lambda_{i+1}) \neq \emptyset$ ,  $0 \in e(\lambda_1)$ , and  $1 \in e(\lambda_n)$ . We can prove that  $A(\lambda_1) \in \mathcal{A}_1$  by first observing that

$$A(\lambda_1) + \alpha\theta^* \succ A(0) = \{p\} \succ A(\lambda_1) - \alpha\theta^*,$$

which by continuity implies there exists  $\alpha' \in (-\alpha, \alpha)$  such that  $A(\lambda_1) + \alpha'\theta^* \sim \{p\}$ . This implies  $A(\lambda_1) + \alpha'\theta^* \in \mathcal{A}'_0$ , which implies that  $A(\lambda_1) \in \mathcal{A}_1$ . We now show that  $A(\lambda_i) \in \mathcal{A}_i$  implies  $A(\lambda_{i+1}) \in \mathcal{A}_{i+1}$ . If  $A(\lambda_i) \in \mathcal{A}_i$ , then we also have

<sup>52</sup>The only substantively different cases are the variations of  $B' \succsim A$ ,  $A' \succsim B$ . However, in this case we can apply Lemma S.7, which implies  $A, A' \in \mathcal{A}'_{i-1}$ , and hence the result is obtained by assumption.

$A(\lambda_i) + \alpha'\theta^* \in \mathcal{A}_i$  for all  $\alpha' \in (-\alpha, \alpha)$ . Since  $e(\lambda_i) \cap e(\lambda_{i+1}) \neq \emptyset$ , choose any  $\lambda \in e(\lambda_i) \cap e(\lambda_{i+1})$ . Then

$$\begin{aligned} A(\lambda_i) + \alpha\theta^* &> A(\lambda) > A(\lambda_i) - \alpha\theta^*, \\ A(\lambda_{i+1}) + \alpha\theta^* &> A(\lambda) > A(\lambda_{i+1}) - \alpha\theta^*. \end{aligned}$$

By continuity, there exist  $\alpha', \alpha'' \in (-\alpha, \alpha)$  such that  $A(\lambda_i) + \alpha'\theta^* \sim A(\lambda) \sim A(\lambda_{i+1}) + \alpha''\theta^*$ , which implies  $A(\lambda_{i+1}) + \alpha''\theta^* \in \mathcal{A}'_i$ . Hence,  $A(\lambda_{i+1}) \in \mathcal{A}_{i+1}$ . By induction, we conclude that  $A(\lambda_i) \in \mathcal{A}_i$  for  $i = 1, \dots, n$  and also that  $A \in \mathcal{A}'_n \subset \mathcal{A}_{n+1} \subset \bigcup_i \mathcal{A}_i$ . *Q.E.D.*

We have now proved that  $\hat{V}$  is translation linear and represents  $\succsim$  on  $\mathcal{A}^\circ$ . Before extending  $\hat{V}$  to  $\mathcal{A}^c$ , we first establish that  $\hat{V}$  is Lipschitz continuous and convex.

LEMMA S.11:  $\hat{V}$  is Lipschitz continuous.

PROOF: For all  $\delta \in (0, 1)$ , define

$$\mathcal{A}_\delta^\circ \equiv \{A \in \mathcal{A}^c : \forall p \in A, \forall z \in Z : p_z \geq \delta\}.$$

We next summarize some straightforward facts about  $\mathcal{A}_\delta^\circ$  whose proofs we omit:

- (i)  $\mathcal{A}_\delta^\circ$  is a convex subset of  $\mathcal{A}^\circ$ .
- (ii) For all  $A \in \mathcal{A}_\delta^\circ$  and  $\alpha \in (0, \delta)$  there exists a unique menu  $A^\alpha \in \mathcal{A}^\circ$  such that  $A = (1 - \alpha)A^\alpha + \alpha\{p_*\}$ .<sup>53</sup>
- (iii) For all  $A, B \in \mathcal{A}_\delta^\circ$ ,  $\alpha \in (0, \delta)$ :  $(1 - \alpha)d_h(A^\alpha, B^\alpha) = d_h(A, B)$ .
- (iv) For all  $A \in \mathcal{A}_\delta^\circ$ ,  $\alpha \in (0, \delta)$ :  $A + \alpha\theta^* \in \mathcal{A}^\circ$ .

Let  $K \equiv 2M(v \cdot \theta^*) > 0$  and  $\delta \in (0, 1/2)$ . We first show that

$$(S.15) \quad A, B \in \mathcal{A}_\delta^\circ \text{ \& } d_h(A, B) < \frac{\delta}{2M} \implies |\hat{V}(A) - \hat{V}(B)| \leq Kd_h(A, B).$$

Suppose that  $A, B$  are as in the left-hand side of Equation (S.15). Let  $\alpha \in (2Md_h(A, B), \delta)$ . Then

$$d_h(A^\alpha, B^\alpha) = \frac{1}{1 - \alpha}d_h(A, B) \leq 2d_h(A, B) < \frac{\alpha}{M},$$

where the weak inequality follows from  $\alpha < \delta < 1/2$ . Applying L-continuity, we have

$$A + \alpha\theta^* = (1 - \alpha)A^\alpha + \alpha\{p_*\} > (1 - \alpha)B^\alpha + \alpha\{p_*\} = B.$$

<sup>53</sup>The menu  $A^\alpha$  is given by  $A^\alpha = \{q \in \mathbb{R}^Z : q = \frac{1}{1-\alpha}(p - \alpha p_*) \text{ for some } p \in A\}$ .

Since  $\hat{V}$  represents  $\succsim$  and is translation linear on  $\mathcal{A}^\circ$ , we have  $\hat{V}(A) + \alpha(v \cdot \theta^*) > \hat{V}(B)$ , implying

$$\alpha(v \cdot \theta^*) > \hat{V}(B) - \hat{V}(A).$$

Since the above inequality holds for any  $\alpha \in (2Md_h(A, B), \delta)$ , we conclude that

$$\hat{V}(B) - \hat{V}(A) \leq 2Md_h(A, B)(v \cdot \theta^*) = Kd_h(A, B).$$

Interchanging the roles of  $A$  and  $B$  above, we also have that  $\hat{V}(A) - \hat{V}(B) \leq Kd_h(A, B)$ , proving Equation (S.15).

Next, we use the argument in the proof of Lemma 8 in the Supplemental Material of [Dekel, Lipman, Rustichini, and Sarver \(2007\)](#) to show that

$$(S.16) \quad A, B \in \mathcal{A}_\delta^\circ \implies |\hat{V}(A) - \hat{V}(B)| \leq Kd_h(A, B),$$

that is, the requirement  $d_h(A, B) < \frac{\delta}{2M}$  in Equation (S.15) is not necessary. To see this, take any sequence  $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n < \lambda_{n+1} = 1$  such that  $(\lambda_{i+1} - \lambda_i)d_h(A, B) < \frac{\delta}{2M}$ . Let  $A_i = \lambda_i A + (1 - \lambda_i)B$ . It is straightforward to verify that

$$d_h(A_{i+1}, A_i) = (\lambda_{i+1} - \lambda_i)d_h(A, B) < \frac{\delta}{2M}.$$

Combining this with the triangular inequality and Equation (S.15), we obtain

$$\begin{aligned} |\hat{V}(A) - \hat{V}(B)| &\leq \sum_{i=0}^n |\hat{V}(A_{i+1}) - \hat{V}(A_i)| \\ &\leq K \sum_{i=0}^n d_h(A_{i+1}, A_i) = K \sum_{i=0}^n (\lambda_{i+1} - \lambda_i)d_h(A, B) \\ &= Kd_h(A, B). \end{aligned}$$

To conclude the proof, note that by part (i) of Lemma 4, for any  $A, B \in \mathcal{A}^\circ$ , there exists a small enough  $\delta \in (0, 1/2)$  such that  $A, B \in \mathcal{A}_\delta^\circ$ . Hence by Equation (S.16),  $\hat{V}$  is Lipschitz continuous on  $\mathcal{A}^\circ$  with the Lipschitz constant  $K$ . *Q.E.D.*

LEMMA S.12:  $\hat{V}$  is convex.

PROOF: The argument given here is similar to a result contained in a working-paper version of [Maccheroni, Marinacci, and Rustichini \(2006\)](#). We will show that every  $A_0 \in \mathcal{A}^\circ$  has a convex and open neighborhood in  $\mathcal{A}^\circ$  on

which  $\hat{V}$  is convex. By a standard result from convex analysis, this implies that  $\hat{V}$  is convex on  $\mathcal{A}^\circ$ .

Let  $A_0 \in \mathcal{A}^\circ$ . Define  $\mathcal{C}$  to be the collection of all closed and bounded non-empty convex subsets of  $\{p \in \mathbb{R}^Z : \sum_{z \in Z} p_z = 1\}$ , endowed with the Hausdorff metric topology. It follows from part (i) of Lemma 4 that there exists an  $\varepsilon > 0$  such that  $B_\varepsilon(A_0) \subset \mathcal{A}^\circ$ , where we define

$$B_\varepsilon(A_0) \equiv \{A \in \mathcal{C} : d_h(A, A_0) < \varepsilon\}.$$

Note that  $d_h(\cdot, \cdot)$  indicates the Hausdorff metric. For any  $\theta \in \Theta$  and  $A \in \mathcal{C}$ , we have  $A + \theta \in \mathcal{C}$  and  $d_h(A, A + \theta) = \|\theta\|$ , where  $\|\cdot\|$  indicates the Euclidean norm. There exists  $\theta \in \Theta$  such that  $\|\theta\| < \varepsilon$  and  $v \cdot \theta > 0$ .<sup>54</sup> This implies that  $A_0 + \theta \in B_\varepsilon(A_0)$  and  $A_0 + \theta \succ A_0$ . By continuity, there exists  $\rho \in (0, \frac{1}{3})$  such that for all  $A \in B_{\rho\varepsilon}(A_0)$ ,  $|\hat{V}(A) - \hat{V}(A_0)| < \frac{1}{3}(v \cdot \theta)$ . Therefore, if  $A, B \in B_{\rho\varepsilon}(A_0)$ , then

$$|\hat{V}(A) - \hat{V}(B)| \leq |\hat{V}(A) - \hat{V}(A_0)| + |\hat{V}(A_0) - \hat{V}(B)| < \frac{2}{3}(v \cdot \theta).$$

Let  $\alpha \equiv \frac{\hat{V}(A) - \hat{V}(B)}{v \cdot \theta}$ , which implies  $|\alpha| < \frac{2}{3}$ . Then we have

$$\begin{aligned} d_h(A_0, B + \alpha\theta) &\leq d_h(A_0, B) + d_h(B, B + \alpha\theta) \\ &< \rho\varepsilon + \|\alpha\theta\| \\ &< \frac{1}{3}\varepsilon + \frac{2}{3}\varepsilon = \varepsilon, \end{aligned}$$

so  $B + \alpha\theta \in B_\varepsilon(A_0) \subset \mathcal{A}^\circ$ . Thus  $\hat{V}$  is defined at  $B + \alpha\theta$ . Note that  $\alpha(v \cdot \theta) = \hat{V}(A) - \hat{V}(B)$ , so that  $\hat{V}(B + \alpha\theta) = \hat{V}(B) + \alpha(v \cdot \theta) = \hat{V}(A)$ . Since  $\succsim$  satisfies ACP, for any  $\lambda \in [0, 1]$ ,

$$\hat{V}(A) \geq \hat{V}(\lambda A + (1 - \lambda)(B + \alpha\theta)).$$

Therefore,

$$\begin{aligned} \hat{V}(A) &\geq \hat{V}(\lambda A + (1 - \lambda)B) + (1 - \lambda)\alpha(v \cdot \theta) \\ &= \hat{V}(\lambda A + (1 - \lambda)B) + (1 - \lambda)(\hat{V}(A) - \hat{V}(B)), \end{aligned}$$

so we have

$$\lambda\hat{V}(A) + (1 - \lambda)\hat{V}(B) \geq \hat{V}(\lambda A + (1 - \lambda)B).$$

<sup>54</sup>For instance,  $\theta = \alpha\theta^*$  for any  $\alpha \in (0, \varepsilon/\|\theta^*\|)$ , where  $\theta^* = p^* - p_*$ .

Therefore,  $\hat{V}$  is convex on the convex and open neighborhood  $B_{\rho\varepsilon}(A_0)$  of  $A_0$  in  $\mathcal{A}^\circ$ . *Q.E.D.*

Since  $\mathcal{A}^\circ$  is dense in  $\mathcal{A}^c$  (see Lemma 4), we can extend  $\hat{V}$  to  $\mathcal{A}^c$  by continuity. That is, define a function  $V: \mathcal{A}^c \rightarrow \mathbb{R}$  as follows: For any  $A \in \mathcal{A}^c$ , there exists a sequence  $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}^\circ$  such that  $A_n \rightarrow A$ , so define  $V(A) \equiv \lim_{n \rightarrow \infty} \hat{V}(A_n)$ . Since  $\hat{V}$  is Lipschitz continuous, the following lemma establishes that  $V$  is well defined and also Lipschitz continuous. Furthermore, this extension  $V$  of  $\hat{V}$  represents  $\succsim$  on  $\mathcal{A}^c$  and preserves the translation linearity and convexity of  $\hat{V}$ .

LEMMA S.13: *The function  $V: \mathcal{A}^c \rightarrow \mathbb{R}$  is well defined and it satisfies properties (i)–(iii) from Proposition 1.*

PROOF: By Lemma 4,  $\mathcal{A}^\circ$  is dense in  $\mathcal{A}^c$ . Since  $\mathcal{A}^c$  is a compact metric space, it is complete. Since  $\hat{V}$  is Lipschitz continuous, it is uniformly continuous (see Aliprantis and Border (1999, p. 76)). Therefore, by Lemma 3.8 in Aliprantis and Border (1999, p. 77),  $V$  is well defined and it is the unique continuous extension of  $\hat{V}$  to  $\mathcal{A}^c$ . To see that  $V$  is Lipschitz continuous, let  $K > 0$  be a Lipschitz constant for  $\hat{V}$  on  $\mathcal{A}^\circ$  and let  $A, B \in \mathcal{A}^c$ . Take sequences  $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}} \subset \mathcal{A}^\circ$  such that  $A_n \rightarrow A$  and  $B_n \rightarrow B$ . Then

$$\begin{aligned} |V(A) - V(B)| &= \lim_{n \rightarrow \infty} |\hat{V}(A_n) - \hat{V}(B_n)| \\ &\leq \lim_{n \rightarrow \infty} K d_h(A_n, B_n) = K d_h(A, B). \end{aligned}$$

Hence,  $V$  is Lipschitz continuous with the same constant  $K$ .

To see that  $V$  is translation linear, let  $A, A + \theta \in \mathcal{A}^c$  for some  $\theta \in \Theta$ . Fix any  $p \in \Delta(Z)$  such that  $p_z > 0$  for all  $z \in Z$ . For all  $n \in \mathbb{N}$ , define  $A_n \equiv (1 - 1/n)A + (1/n)\{p\}$  and  $\theta_n \equiv (1 - 1/n)\theta$ . By Lemma 4, for all  $n \in \mathbb{N}$ ,  $A_n \in \mathcal{A}^\circ$  and  $A_n + \theta_n = (1 - 1/n)(A + \theta) + (1/n)\{p\} \in \mathcal{A}^\circ$ . Moreover,  $A_n \rightarrow A$  and  $A_n + \theta_n \rightarrow A + \theta$  as  $n \rightarrow \infty$ . Therefore,

$$\begin{aligned} V(A + \theta) - V(A) &= \lim_{n \rightarrow \infty} [\hat{V}(A_n + \theta_n) - \hat{V}(A_n)] \\ &= \lim_{n \rightarrow \infty} v \cdot \theta_n = v \cdot \theta. \end{aligned}$$

Thus, we see that  $V$  is translation linear on all of  $\mathcal{A}^c$ . The proof that  $V$  is convex is straightforward and follows from a similar line of reasoning; it is therefore omitted.

To show that  $V$  represents  $\succsim$  on  $\mathcal{A}^c$ , we prove  $A \succ B \iff V(A) > V(B)$ . Let  $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}} \subset \mathcal{A}^\circ$  be such that  $A_n \rightarrow A$  and  $B_n \rightarrow B$  as  $n \rightarrow \infty$ .

To see the  $\Rightarrow$  direction, suppose  $A \succ B$ . By the continuity of  $\succsim$ ,  $\{C \in \mathcal{A}^c : A \succ C \succ B\}$  is nonempty and open.<sup>55</sup> Since  $\mathcal{A}^\circ$  is dense in  $\mathcal{A}^c$ , there exists  $\bar{A} \in \mathcal{A}^\circ$  such that  $A \succ \bar{A} \succ B$ . Repeating the same argument for  $\bar{A} \succ B$ , there exists  $\bar{B} \in \mathcal{A}^\circ$  such that  $\bar{A} \succ \bar{B} \succ B$ . By continuity,  $\{C \in \mathcal{A}^c : C \succ \bar{A}\}$  is a neighborhood of  $A$ , so there exists  $N \in \mathbb{N}$  such that  $A_n \succ \bar{A}$  for all  $n \geq N$ . A similar argument implies there exists  $N' \in \mathbb{N}$  such that  $\bar{B} \succ B_n$  for all  $n \geq N'$ . Therefore,

$$V(A) = \lim_{n \rightarrow \infty} \hat{V}(A_n) \geq \hat{V}(\bar{A}) > \hat{V}(\bar{B}) \geq \lim_{n \rightarrow \infty} \hat{V}(B_n) = V(B).$$

To show the  $\Leftarrow$  direction, we will apply a similar argument using the continuity of  $V$ . Suppose  $V(A) > V(B)$ . By continuity of  $V$ ,  $\{C \in \mathcal{A}^c : V(A) > V(C) > V(B)\}$  is nonempty and open. Since  $\mathcal{A}^\circ$  is dense in  $\mathcal{A}^c$ , there exists  $\bar{A} \in \mathcal{A}^\circ$  such that  $V(A) > V(\bar{A}) > V(B)$ . Repeating the same argument for  $V(\bar{A}) > V(B)$ , there exists  $\bar{B} \in \mathcal{A}^\circ$  such that  $V(\bar{A}) > V(\bar{B}) > V(B)$ . By continuity,  $\{C \in \mathcal{A}^c : V(C) > V(\bar{A})\}$  is a neighborhood of  $A$ , so there exists  $N \in \mathbb{N}$  such that  $\hat{V}(A_n) = V(A_n) > V(\bar{A}) = \hat{V}(\bar{A})$  for all  $n \geq N$ . A similar argument implies there exists  $N' \in \mathbb{N}$  such that  $\hat{V}(\bar{B}) > \hat{V}(B_n)$  for all  $n \geq N'$ . Therefore, by continuity of  $\succsim$ ,

$$A = \lim_{n \rightarrow \infty} A_n \succ \bar{A} \succ \bar{B} \succ \lim_{n \rightarrow \infty} B_n = B.$$

Finally, since  $V$  represents  $\succsim$  on  $\mathcal{A}^c$  and  $p^* \succ p_*$ , we also have that  $V(\{p^*\}) > V(\{p_*\})$ . *Q.E.D.*

The following lemma establishes uniqueness of the representation, completing the proof of Proposition 1.

LEMMA S.14: *If  $V : \mathcal{A}^c \rightarrow \mathbb{R}$  and  $V' : \mathcal{A}^c \rightarrow \mathbb{R}$  are two functions that satisfy (ii) and (iii) from Proposition 1 and are ordinally equivalent in the sense that for any  $A, B \in \mathcal{A}^c$ ,  $V(A) \geq V(B) \iff V'(A) \geq V'(B)$ , then there exist  $\alpha > 0$  and  $\beta \in \mathbb{R}$  such that  $V' = \alpha V + \beta$ .*

PROOF: We first extend  $V$  and  $V'$  to  $\mathcal{A}$  using indifference to randomization: Define  $\bar{V} : \mathcal{A} \rightarrow \mathbb{R}$  and  $\bar{V}' : \mathcal{A} \rightarrow \mathbb{R}$  by  $\bar{V}(A) = V(\text{co}(A))$  and  $\bar{V}'(A) = V'(\text{co}(A))$  for  $A \in \mathcal{A}$ . Note that  $\bar{V}$  and  $\bar{V}'$  satisfy (ii) and (iii) from Proposition 1. The convexity, translation linearity, and singleton nontriviality of these functions follow immediately from the properties of  $V$  and  $V'$ . The Lipschitz continuity of  $\bar{V}$  follows from the Lipschitz continuity of  $V$  since for any

<sup>55</sup>Note that the sets  $\{\lambda \in [0, 1] : \lambda A + (1 - \lambda)B \succ B\}$  and  $\{\lambda \in [0, 1] : A \succ \lambda A + (1 - \lambda)B\}$  are nonempty and open relative to  $[0, 1]$  (by continuity of  $\succsim$  and continuity of convex combinations), and their union is  $[0, 1]$ . Since  $[0, 1]$  is connected, their intersection must be nonempty. Hence the set  $\{C \in \mathcal{A}^c : A \succ C \succ B\}$  is also nonempty.

$A, B \in \mathcal{A}$ ,

$$\begin{aligned} |\bar{V}(A) - \bar{V}(B)| &= |V(\text{co}(A)) - V(\text{co}(B))| \\ &\leq K d_h(\text{co}(A), \text{co}(B)) \leq K d_h(A, B), \end{aligned}$$

where  $K > 0$  is any Lipschitz constant of  $V$ . Note that the last inequality follows from the fact that  $d_h(\text{co}(A), \text{co}(B)) \leq d_h(A, B)$  for any  $A, B \in \mathcal{A}$ . Similarly,  $\bar{V}'$  is Lipschitz continuous.

Define a preference  $\bar{\succsim}$  on  $\mathcal{A}$  by  $A \bar{\succsim} B \iff \bar{V}(A) \geq \bar{V}(B)$  for  $A, B \in \mathcal{A}$ . Define the sets  $\mathcal{A}_i$  and  $\mathcal{A}'_i$  for  $i \in \mathbb{N}$  as above for this preference  $\bar{\succsim}$ . Since  $\bar{V}$  satisfies (ii) and (iii) from Proposition 1, it is easy to see that  $\bar{\succsim}$  satisfies the axioms of Proposition 1: weak order, strong continuity, ACP, and IDD.<sup>56</sup> Therefore, we can appeal to Lemma S.10 to conclude that  $\mathcal{A}^\circ = \bigcup_i \mathcal{A}_i$ .

Translation linearity implies that  $\bar{V}$  and  $\bar{V}'$  are affine on singletons and, therefore, the standard von Neumann–Morgenstern uniqueness result implies  $\bar{V}'|_{\mathcal{S}} = \alpha \bar{V}|_{\mathcal{S}} + \beta$  for some  $\alpha > 0$ ,  $\beta \in \mathbb{R}$ . By translation linearity and that fact that  $\bar{V}'(A) \geq \bar{V}'(B) \iff \bar{V}(A) \geq \bar{V}(B) \iff A \bar{\succsim} B$ , a simple induction argument shows that  $\bar{V}'|_{\mathcal{A}_i} = \alpha \bar{V}|_{\mathcal{A}_i} + \beta$  for all  $i \in \mathbb{N}$ . Since  $\mathcal{A}^\circ = \bigcup_i \mathcal{A}_i$  by our previous arguments, this implies  $\bar{V}'|_{\mathcal{A}^\circ} = \alpha \bar{V}|_{\mathcal{A}^\circ} + \beta$ . Since  $\mathcal{A}^\circ$  is dense in  $\mathcal{A}^c$  (see Lemma 4), and the functions  $\bar{V}$  and  $\bar{V}'$  are continuous, we conclude that  $V' = \bar{V}'|_{\mathcal{A}^c} = \alpha \bar{V}|_{\mathcal{A}^c} + \beta = \alpha V + \beta$ . *Q.E.D.*

#### S.4. PROOF OF EQUATION (24)

LEMMA S.15: *If  $\succsim$  satisfies strong IDD, then for any  $A \in \mathcal{A}$ ,  $p \in \Delta(Z)$  and  $\alpha \in [0, 1]$ ,*

$$(S.17) \quad V(\alpha A + (1 - \alpha)\{p\}) = \alpha V(A) + (1 - \alpha)V(\{p\}).$$

PROOF: Take  $\theta^*$ ,  $\mathcal{A}^c$ ,  $\mathcal{A}^\circ$ ,  $\mathcal{A}_i$ , and  $\mathcal{A}'_i$  as defined in Appendix C. It easily is verified that for any signed RFCC representation, the consistency of the measures implies that  $V$  is affine on singleton menus:

$$\begin{aligned} V(\alpha\{q\} + (1 - \alpha)\{p\}) \\ = \alpha V(\{q\}) + (1 - \alpha)V(\{p\}) \quad \forall p, q \in \Delta(Z). \end{aligned}$$

Therefore, Equation (S.17) holds for all  $A \in \mathcal{A}_0$ . We prove by induction that Equation (S.17) holds on  $\mathcal{A}_i$  for all  $i \geq 0$ .

<sup>56</sup>The only axiom that is more difficult to verify is L-continuity, but this follows from Lemma S.3.

Fix any  $i \geq 0$ . Our first step is to establish that if Equation (S.17) holds for all  $A \in \mathcal{A}_i$ , then it must also hold for all  $A \in \mathcal{A}'_i$ . For suppose Equation (S.17) holds on  $\mathcal{A}_i$  and  $A \in \mathcal{A}'_i$ . Then  $A \sim B$  for some  $B \in \mathcal{A}_i$  and so, by strong IDD,

$$\begin{aligned} V(\alpha A + (1 - \alpha)\{p\}) &= V(\alpha B + (1 - \alpha)\{p\}) \quad (\text{by strong IDD}) \\ &= \alpha V(B) + (1 - \alpha)V(\{p\}) \quad (\text{by induction assumption}) \\ &= \alpha V(A) + (1 - \alpha)V(\{p\}). \end{aligned}$$

Next, we establish that if Equation (S.17) holds for all  $A \in \mathcal{A}'_i$ , then it must also hold for all  $A \in \mathcal{A}_{i+1}$ . Suppose Equation (S.17) holds on  $\mathcal{A}'_i$  and  $A \in \mathcal{A}_{i+1}$ . Then  $A = B + \bar{\alpha}\theta^*$  for some  $B \in \mathcal{A}'_i$  and  $\bar{\alpha} \in \mathbb{R}$ . By Lemma S.2,  $V$  is translation linear (as defined in Section C.1) and thus there exists  $v \in \mathbb{R}^Z$  such that

$$\begin{aligned} V(\alpha A + (1 - \alpha)\{p\}) &= V(\alpha B + (1 - \alpha)\{p\} + \alpha\bar{\alpha}\theta^*) \\ &= V(\alpha B + (1 - \alpha)\{p\}) + \alpha\bar{\alpha}(v \cdot \theta^*) \\ &= \alpha V(B) + (1 - \alpha)V(\{p\}) + \alpha\bar{\alpha}(v \cdot \theta^*) \\ &= \alpha V(B + \bar{\alpha}\theta^*) + (1 - \alpha)V(\{p\}) \\ &= \alpha V(A) + (1 - \alpha)V(\{p\}). \end{aligned}$$

By induction, we conclude that Equation (S.17) holds on  $\mathcal{A}_i$  for all  $i \geq 0$ , and hence by Lemma S.10, Equation (S.17) holds for all  $A \in \mathcal{A}^\circ$ . Since  $\mathcal{A}^\circ$  is dense in  $\mathcal{A}^c$  by Lemma 4, the continuity of  $V$  implies that the desired property also holds on  $\mathcal{A}^c$ . Finally, for any  $A \in \mathcal{A}$ ,  $\text{co}(A) \in \mathcal{A}^c$  and  $V(A) = V(\text{co}(A))$ . Therefore, by arguments identical to those used above, it follows that Equation (S.17) holds for all  $A \in \mathcal{A}$ . *Q.E.D.*

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