Supplementary Appendix for:

Behavioral Characterizations of Naivete for Time-Inconsistent Preferences

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Abstract

Theorem A.1 from Appendix A of the main paper (Ahn, Iijima, Le Yaouanq, and Sarver (2018)) is an extension of the characterization of comparative temptation aversion from Dekel and Lipman (2012): While their result required a finite consumption space, our extension applies to any random Strotz representation defined on any compact and metrizable consumption space C, provided the measure in the representation has finite-dimensional support. As discussed in the paper, this extension is important for a number of applications, including dynamic consumption decisions where C is a set of infinite consumption streams. In this supplement, we provide a proof of Theorem A.1.

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S.1 Proof of Theorem A.1 in the Main Paper

S.1.1 Sufficiency: more temptation averse $\Longrightarrow$ less u-aligned

The following is the relevant result from Dekel and Lipman (2012), which they proved for the case of finite C.

Theorem S.1 (Dekel and Lipman (2012)). *Suppose C has finite cardinality. Suppose $\succsim_1$ and $\succsim_2$ have random Strotz representations (u, μ_1) and (u, μ_2). Then $\succsim_2$ is more temptation averse than $\succsim_1$ if and only if $\mu_1 \gg_u \mu_2$.*

Proof. Theorem 4 in Dekel and Lipman (2012) establishes the equivalence of $\succsim_2$ being more temptation averse than $\succsim_1$ and another condition on the representations that they refer to as conditional dominance. However, they also establish that $\mu_1 \gg_u \mu_2$ as an intermediate step in their proof.[1] The equivalence asserted in Theorem S.1 is also stated explicitly in Theorem 4 of their working paper, Dekel and Lipman (2010).[2] ∎

To prove the sufficiency part of Theorem A.1, we now show that the sufficiency direction in Theorem S.1 can be extended to any compact and metrizable space C and any random Strotz representations (u, μ_1) and (u, μ_2) defined on that space, subject to our restriction that each μ_i has finite-dimensional support. Our approach is to show that the relationship between μ_1 and μ_2, specifically $\mu_1 \gg_u \mu_2$, can be inferred from looking at the restriction of the representations and preferences to a carefully chosen finite consumption space $C^* \subset C$.

The following preliminary result will be useful in the sequel. Recall that $\mathcal{V}$ denotes the set of all continuous functions $v : C \to \mathbb{R}$, i.e., the set of all expected-utility functions.

Lemma S.1. *Suppose the set $\{v_1, \ldots, v_n\} \subset \mathcal{V}$ is linearly independent. Then there exists a finite subset $C^* \subset C$ such that the set $\{v_1^*, \ldots, v_n^*\}$ is linearly independent, where $v_i^* = v_i|_{C^*}$ is the restriction of the function v_i to C^*.*

[1] To show that $\succsim_2$ being more temptation averse that $\succsim_1$ implies $\mu_1 \gg_u \mu_2$, the relevant results in Dekel and Lipman (2012) are the following: Lemma 3 shows that a partial order vC_uv' used in their paper is equivalent to our order $v \gg_u v'$ (ignoring their normalization of utility functions). Lemmas 4, 5, and 6 and the arguments on page 1296 show that for any set W that is closed under C_u (is a u-upper set in our terminology), $\mu_1(W) \geq \mu_2(W)$.

[2] Dekel and Lipman (2010) impose a normalization on the set of utility functions used in their result. However, by the uniqueness properties of the random Strotz representation established in Theorem 3 of Dekel and Lipman (2012), the probability of any u-upper set is the same for any random Strotz representation of the same preference. Therefore, their normalization of utilities is inconsequential for the result.

Proof. Suppose to the contrary that for every finite $B \subset C$, the collection $\{v_1|_B, \dots, v_n|_B\}$ is linearly dependent. Then for any finite $B \subset C$, the set $A_B \subset \mathbb{R}^n$ defined by

$$A_B = \{\alpha \in \mathbb{R}^n : \|\alpha\| = 1 \text{ and } \alpha_1 v_1(c) + \cdots + \alpha_n v_n(c) = 0 \ \forall c \in B\}$$

is nonempty. Note that A_B is also a closed subset of the unit ball in $\mathbb{R}^n$, which is itself compact because n is finite. Let $\mathcal{B}$ denote the set of all nonempty finite subsets of C. For any $B_1, \dots, B_k \in \mathcal{B}$, we have

$$A_{B_1} \cap \cdots \cap A_{B_k} = A_{B_1 \cup \cdots \cup B_k} \neq \emptyset,$$

since $B_1 \cup \cdots \cup B_k$ is finite and hence also in $\mathcal{B}$. Thus the collection $\{A_B\}_{B \in \mathcal{B}}$ has the finite intersection property. Since these sets are closed subsets of a compact set, this implies $\bigcap_{B \in \mathcal{B}} A_B \neq \emptyset$. However, since

$$\bigcap_{B \in \mathcal{B}} A_B = \{\alpha \in \mathbb{R}^n : \|\alpha\| = 1 \text{ and } \alpha_1 v_1(c) + \cdots + \alpha_n v_n(c) = 0 \ \forall c \in C\},$$

this implies the set $\{v_1, \dots, v_n\}$ is linearly dependent, a contradiction. ∎

Since μ_1 and μ_2 have finite-dimensional support, there exists a finite set of expected-utility functions $\{v_1, \dots, v_n\} \subset \mathcal{V}$ such that $\text{supp}(\mu_i) \subset \text{span}(\{v_1, \dots, v_n\})$ for $i = 1, 2$. Consider the set of function $\{u, \mathbf{1}, v_1, \dots, v_n\}$, where $\mathbf{1}$ denotes the constant function with $\mathbf{1}(c) = 1$ for all $c \in C$. Without loss of generality, assume that this set of functions is linearly independent. Otherwise, we can sequentially remove the functions v_i until we obtain a linearly independent set.[3] To simplify notation in what follows, let $\mathcal{V}_s \equiv \text{span}(\{u, \mathbf{1}, v_1, \dots, v_n\}) \subset \mathcal{V}$. Thus $\mu_1(\mathcal{V}_s) = \mu_2(\mathcal{V}_s) = 1$.

Take C^* as in Lemma S.1 for the set $\{u, \mathbf{1}, v_1, \dots, v_n\}$. Let $\mathcal{V}^*$ denote the set of all continuous real-valued functions on C^* and let $\mathcal{V}_s^* \equiv \text{span}(\{u^*, \mathbf{1}^*, v_1^*, \dots, v_n^*\}) \subset \mathcal{V}^*$, where $u^* = u|_{C^*}$, $\mathbf{1}^* = \mathbf{1}|_{C^*}$, and $v_i^* = v_i|_{C^*}$. Note that each of the functions $u^*, v_1^*, \dots, v_n^*$ must be nontrivial (i.e., not constant) since function $\mathbf{1}^*$ together with these functions forms a linearly independent set.

Lemma S.2. *Define a function $g : \mathcal{V}_s \to \mathcal{V}_s^*$ by $g(v) = v|_{C^*}$, and define a measure μ_i^* on $\mathcal{V}^*$ by $\mu_i^*(E) = \mu_i(g^{-1}(E))$ for any measurable set $E \subset \mathcal{V}^*$ for $i = 1, 2$.*[4]

[3] Note that the set $\{u, \mathbf{1}\}$ must be linearly independent since u assumed to be nontrivial (i.e., not constant). Moreover, if $\text{span}\{u, \mathbf{1}\} = \text{span}\{u, \mathbf{1}, v_1, \dots, v_n\}$, then the support of the measures in the random Strotz representations (u, μ_i) must assign all probability to the set of affine transformations of u. In this case, the representations reduce to time-consistent expected-utility maximization, and we have $\mu_1 \approx \mu_2$. Except in this trivial case, the linearly independent set of expected-utility functions whose span contains the support of μ_i must contain u, $\mathbf{1}$, and at least some of the v_i functions.

[4] In the definition of μ_i^*, we are implicitly treating g as a function from $\mathcal{V}_s$ into $\mathcal{V}^*$. We could equivalently define μ_i^* by $\mu_i^*(E) = \mu_i(g^{-1}(E \cap \mathcal{V}_s^*))$.

1. *The function g is a homeomorphism. That is, g is bijection and both g and its inverse function g^{-1} are continuous.*

2. *For any measurable set $E \subset \mathcal{V}$, $\mu_i(E) = \mu_i^*(g(E \cap \mathcal{V}_s))$.*

3. *For any proper u-upper set $\mathcal{U}$ in $\mathcal{V}$ (i.e., $\mathcal{U} \subsetneq \mathcal{V}$), the set $\mathcal{U}^* = g(\mathcal{U} \cap \mathcal{V}_s)$ is a u^*-upper set in $\mathcal{V}^*$.*

4. *Let $\succsim_i^*$ denote the restriction of $\succsim_i$ to sets of lotteries with support in C^*, which we can identify with the set $\mathcal{K}(\Delta(C^*))$. Then (u^*, μ_i^*) is a random Strotz representation for $\succsim_i^*$ for $i = 1, 2$.*

Proof. (1): This is a standard application of the fundamental theorem of linear algebra for finite-dimensional vector spaces. Note that g is a linear function from the linear space $\mathcal{V}_s$ with basis vectors $\{u, \mathbf{1}, v_1, \ldots, v_n\}$ to the linear space $\mathcal{V}_s^*$ with basis vectors $\{u^*, \mathbf{1}^*, v_1^*, \ldots, v_n^*\}$. Since g maps each basis vector for $\mathcal{V}_s$ to the corresponding basis vector for $\mathcal{V}_s^*$ and the number of basis vectors is the same for each space, g is a bijection. Since any linear function between finite-dimensional spaces is continuous, both g and g^{-1} are continuous.[5]

(2): Fix any measurable set $E \subset \mathcal{V}$. Then

$$\mu_i(E) = \mu_i(E \cap \mathcal{V}_s) = \mu_i(g^{-1}(g(E \cap \mathcal{V}_s))) = \mu_i^*(g(E \cap \mathcal{V}_s)),$$

where the first equality follows from $\mu_i(\mathcal{V}_s) = 1$, the second follows from $g^{-1}(g(E \cap \mathcal{V}_s)) = E \cap \mathcal{V}_s$ (which holds because g is a bijection), and the third follows from the definition of μ_i^*.

(3): First observe that for any $v, v' \in \mathcal{V}_s$,

$$\begin{aligned} v \approx v' &\iff v = av' + b\mathbf{1} \text{ for some } a > 0, b \in \mathbb{R} \\ &\iff g(v) = ag(v') + b\mathbf{1} \text{ for some } a > 0, b \in \mathbb{R} \\ &\iff g(v) \approx g(v'). \end{aligned} \tag{S.1}$$

Now fix any proper u-upper set $\mathcal{U}$ in $\mathcal{V}$, and let $\mathcal{U}^* = g(\mathcal{U} \cap \mathcal{V}_s)$. To see that $\mathcal{U}^*$ is a u^*-upper set, fix any $v^* \in \mathcal{U}^*$ and $v^{*\prime} \in \mathcal{V}^*$ with $v^{*\prime} \gg_{u^*} v^*$. We need to show that $v^{*\prime} \in \mathcal{U}^*$. Let $v = g^{-1}(v^*) \in \mathcal{U} \cap \mathcal{V}_s$. Note that we cannot have $v^* \approx -u^*$, as this would imply by Equation (S.1) that $v \approx g^{-1}(-u^*) = -u$, which would in turn imply by the

[5] A more detailed argument is as follows: Define $h : \mathbb{R}^{n+2} \to \mathcal{V}_s$ by $h(\alpha) = \alpha_1 v_1 + \cdots + \alpha_n v_n + \alpha_{n+1} u + \alpha_{n+2}\mathbf{1}$ and define $h^* : \mathbb{R}^{n+2} \to \mathcal{V}_s^*$ by $h^*(\alpha) = \alpha_1 v_1^* + \cdots + \alpha_n v_n^* + \alpha_{n+1} u^* + \alpha_{n+2}\mathbf{1}^*$. By the linear independence of these sets of functions, both h and h^* are bijections. It is trivial that both functions are continuous, and by Aliprantis and Border (2006, Corollary 5.24) both h^{-1} and h^{*-1} are also continuous. Note that $g = h^* \circ h^{-1}$ and $g^{-1} = h \circ h^{*-1}$, and hence these functions are continuous.

definition of a u-upper set that $\mathcal{U} = \mathcal{V}$, contradicting our assumption that $\mathcal{U}$ is a proper subset of $\mathcal{V}$. Therefore, there exists some $\alpha \in [0,1]$ such that

$$v^{*\prime} \approx \alpha u^* + (1-\alpha)v^*.$$

Thus there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$v^{*\prime} = a\alpha u^* + a(1-\alpha)v^* + b\mathbf{1}^*.$$

Let

$$v' = a\alpha u + a(1-\alpha)v + b\mathbf{1}.$$

Clearly $v' \in \mathcal{V}_s$. Moreover, since $v' \gg_u v$ we have $v' \in \mathcal{U}$. Thus $v' \in \mathcal{U} \cap \mathcal{V}_s$, which implies $v^{*\prime} = g(v') \in \mathcal{U}^*$.

(4): We can treat a lottery $p \in \Delta(C^*)$ as a measure defined only on the space C^*, or we treat this as a lottery in $\Delta(C)$ that assigns probability zero to the set $C \setminus C^*$. Thus we will abuse notation slightly and evaluate the lotteries $p \in \Delta(C^*)$ using both functions in $\mathcal{V}^*$ and functions in $\mathcal{V}$. Note that for any $v \in \mathcal{V}_s$, $v(p) = v^*(p)$ for $v^* = g(v) \in \mathcal{V}_s^*$. Therefore, for any $x \in \mathcal{K}(\Delta(C^*))$,

$$\begin{aligned}
U_i^*(x) &= \int_{\mathcal{V}^*} \max_{p \in B_{v^*}(x)} u^*(p)\, d\mu_i^*(v^*) \\
&= \int_{\mathcal{V}_s^*} \max_{p \in B_{v^*}(x)} u^*(p)\, d(\mu_i \circ g^{-1})(v^*) && \text{(definition of } \mu_i^*\text{)} \\
&= \int_{\mathcal{V}_s} \max_{p \in B_{g(v)}(x)} u^*(p)\, d\mu_i(v) && \text{(change of variables)} \\
&= \int_{\mathcal{V}_s} \max_{p \in B_v(x)} u(p)\, d\mu_i(v) \\
&= U_i(x).
\end{aligned}$$

Thus U_i^* is the restriction of U_i to $\mathcal{K}(\Delta(C^*))$. Also, note that μ_i^* is nontrivial (i.e., assigns probability zero to the set of constant functions) since

$$\mu_i^*(\{\alpha\mathbf{1}^* : \alpha \in \mathbb{R}\}) = \mu_i(g^{-1}(\{\alpha\mathbf{1}^* : \alpha \in \mathbb{R}\})) = \mu_i(\{\alpha\mathbf{1} : \alpha \in \mathbb{R}\}) = 0,$$

by the nontriviality of μ_i. Hence (u^*, μ_i^*) is a random Strotz representation of $\succsim_i^*$. ■

We now prove that $\mu_1 \gg_u \mu_2$. By assumption, $\succsim_2$ is more temptation averse than $\succsim_1$. Thus for any menu x and lottery p, $\{p\} \succ_1 x$ implies $\{p\} \succ_2 x$. This implies a fortiori that the same condition must hold for lotteries and menus of lotteries with support in C^*, and hence $\succsim_2^*$ is more temptation averse than $\succsim_1^*$, where $\succsim_i^*$ is defined as in part 4 of Lemma S.2. Since C^* is finite and (u^*, μ_i^*) represents $\succsim_i^*$ for $i = 1, 2$, Theorem S.1 implies that $\mu_1^* \gg_{u^*} \mu_2^*$.

Now fix any u-upper set $\mathcal{U}$ in $\mathcal{V}$. If $\mathcal{U} = \mathcal{V}$, then trivially $\mu_1(\mathcal{U}) = \mu_2(\mathcal{U}) = 1$. Otherwise, by part 3 of Lemma S.2, $g(\mathcal{U} \cap \mathcal{V}_s)$ is a u^*-upper set in $\mathcal{V}^*$ and therefore

$$\mu_1(\mathcal{U}) = \mu_1^*(g(\mathcal{U} \cap \mathcal{V}_s)) \geq \mu_2^*(g(\mathcal{U} \cap \mathcal{V}_s)) = \mu_2(\mathcal{U}),$$

where the equalities follow from part 2 of Lemma S.2 and the inequality follows from $\mu_1^* \gg_{u^*} \mu_2^*$. Since this is true for any u-upper set $\mathcal{U}$, conclude that $\mu_1 \gg_u \mu_2$.

S.1.2 Necessity: less u-aligned $\Longrightarrow$ more temptation averse

In this section we prove that the more temptation averse comparative is implied by $\mu_1 \gg_u \mu_2$. It is worth noting that the proof of this direction does not rely on the assumption that these measures have finite-dimensional support.

The following preliminary result will be useful.

Lemma S.3. *Let u, v, v' be expected-utility functions defined on $\Delta(C)$, and suppose $v \gg_u v'$. Then for any menu x,*

$$\max_{p \in B_v(x)} u(p) \geq \max_{q \in B_{v'}(x)} u(q).$$

Proof. If $v' \approx -u$, then for any menu x,

$$\max_{q \in B_{v'}(x)} u(q) = \min_{q \in x} u(q) \leq u(p), \quad \forall p \in x.$$

In particular,

$$\max_{q \in B_{v'}(x)} u(q) \leq \max_{p \in B_v(x)} u(p).$$

If we do not have $v' \approx -u$, then $v \gg_u v'$ implies $v \approx \alpha u + (1-\alpha)v'$ for some $\alpha \in [0, 1]$. First, consider $\alpha = 0$. In this case, $v \approx v'$. Therefore $B_v(x) = B_{v'}(x)$, which implies

$$\max_{p \in B_v(x)} u(p) = \max_{q \in B_{v'}(x)} u(q).$$

Finally, consider the case of $\alpha > 0$. Note that for any menu x and any $p \in B_v(x)$ and $q \in B_{v'}(x)$,

$$\alpha u(p) + (1-\alpha)v'(p) \geq \alpha u(q) + (1-\alpha)v'(q) \quad \text{and} \quad v'(q) \geq v'(p).$$

Since $\alpha > 0$, these inequalities imply $u(p) \geq u(q)$. Therefore,

$$\max_{p \in B_v(x)} u(p) \geq \max_{q \in B_{v'}(x)} u(q),$$

as claimed. ■

Suppose (u, μ_1) and (u, μ_2) are random Strotz representations of $\succsim_1$ and $\succsim_2$, and suppose $\mu_1 \gg_u \mu_2$. Fix any menu x, and let $[a, b] = u(x)$. Define $f_x : \mathcal{V} \to [a, b]$ by

$$f_x(v) = \max_{p \in B_v(x)} u(p).$$

By Lemma S.3, $v \gg_u v'$ implies $f_x(v) \geq f_x(v')$. Therefore, for any $\alpha \in [a, b]$ and $v \gg_u v'$,

$$v' \in f_x^{-1}([\alpha, b]) \iff f_x(v') \geq \alpha \implies f_x(v) \geq \alpha \iff v \in f_x^{-1}([\alpha, b]).$$

Thus $f_x^{-1}([\alpha, b])$ is a u-upper set. Therefore,

$$\mu_1(f_x^{-1}([\alpha, b])) \geq \mu_2(f_x^{-1}([\alpha, b])).$$

Define distributions $\eta_i^x \equiv \mu_i \circ f_x^{-1}$ on $[a, b]$ for $i = 1, 2$. By the preceding arguments, η_1^x first-order stochastically dominates η_2^x. Therefore, by the change of variables formula,

$$U_1(x) = \int_{\mathcal{V}} f_x(v)\, d\mu_1(v) = \int_a^b \alpha\, d\eta_1^x(\alpha) \geq \int_a^b \alpha\, d\eta_2^x(\alpha) = \int_{\mathcal{V}} f_x(v)\, d\mu_2(v) = U_2(x).$$

Since this is true for every x, and using the fact that $U_1(\{p\}) = U_2(\{p\})$ for any lottery p, it follows immediately that $\succsim_2$ is more temptation averse than $\succsim_1$.

References

Ahn, D., R. Iijima, Y. Le Yaouanq, and T. Sarver (2018): "Behavioral Characterizations of Naivete for Time-Inconsistent Preferences," *Review of Economic Studies*, forthcoming.

Aliprantis, C., and K. Border (2006): *Infinite Dimensional Analysis*, 3rd edition. Berlin, Germany: Springer-Verlag.

Dekel, E., and B. L. Lipman (2010): "Costly Self-Control and Random Self-Indulgence," Working paper.

Dekel, E., and B. L. Lipman (2012): "Costly Self-Control and Random Self-Indulgence," *Econometrica*, 80, 1271–1302.