Responses to Puzzles: Prospect Theory and Non-Expected-Utility Theory

Todd Sarver

Econ 885 – Duke University

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Early Models of Reference Dependence

- Markowitz (1952) first proposed defining utility in terms of gains and losses relative to current wealth to explain why at all wealth levels individuals might both purchase insurance and take gambles.
- Kahneman and Tversky (1979) use this idea, together with probability weighting, to explain Allais and the other puzzles for expected-utility documented in the last lecture.
- Prospect theory takes the reference point as given. We will discuss some forward-looking models of endogenous reference point formation later.

Prospect Theory

Kahneman and Tversky (1979) describe a formula for evaluating simple prospects (gambles) of the form $(x, p; y, q)$ where $p + q \leq 1$. If $p + q < 1$, the remaining probability is understood to be assigned to zero.

Two ingredients in the model:

1. Value function $v(x)$, which satisfies $v(0) = 0$.
2. Weighting function $\pi(p)$, which satisfies $\pi(0) = 0$ and $\pi(1) = 1$.

Evaluation of prospects (assume $x \geq y$):

$$
\begin{aligned}
V(x, 1) &= V(x) = v(x) \\
V(x, p; y, q) &= \pi(p)v(x) + \pi(q)v(y) && \text{if } p + q < 1 \text{ or } x \geq 0 \geq y \\
V(x, p; y, q) &= v(y) + \pi(p)[v(x) - v(y)] && \text{if } p + q = 1 \text{ and } x > y > 0
\end{aligned}
$$

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Shape of the Value Function

Aversion to symmetric bets of the form $(x, .5; -x, .5)$ also implies

$$0 = v(0) > \pi(.5)v(x) + \pi(.5)v(-x) \implies v(x) < -v(-x)$$

- Significant concavity around the reference point 0 (e.g., kinks) can help explain scale-of-risk related paradoxes for EU (Rabin paradox, equity premium puzzle, etc.).

Problem 13 in KT: Choose between

$$A : \begin{cases} 6,000 & \text{with prob } .25 \\ 0 & \text{with prob } .75 \end{cases} \qquad B : \begin{cases} 4,000 & \text{with prob } .25 \\ 2,000 & \text{with prob } .25 \\ 0 & \text{with prob } .5 \end{cases}$$

Problem 13 in KT: Choose between

$$A : \begin{cases} 6,000 & \text{with prob } .25 \\ 0 & \text{with prob } .75 \end{cases} \qquad B : \begin{cases} 4,000 & \text{with prob } .25 \\ 2,000 & \text{with prob } .25 \\ 0 & \text{with prob } .5 \end{cases}$$

$$\begin{aligned} B \succ A &\implies \pi(.25)v(6,000) < \pi(.25)v(4,000) + \pi(.25)v(2,000) \\ &\implies v(6,000) < v(4,000) + v(2,000) \implies \text{concave for gains} \end{aligned}$$

Problem 13 in KT: Choose between

$$A: \begin{cases} 6,000 & \text{with prob } .25 \\ 0 & \text{with prob } .75 \end{cases} \qquad B: \begin{cases} 4,000 & \text{with prob } .25 \\ 2,000 & \text{with prob } .25 \\ 0 & \text{with prob } .5 \end{cases}$$

$$B \succ A \implies \pi(.25)v(6,000) < \pi(.25)v(4,000) + \pi(.25)v(2,000)$$
$$\implies v(6,000) < v(4,000) + v(2,000) \implies \text{concave for gains}$$

Problem 13' in KT: Choose between

$$C: \begin{cases} -6,000 & \text{with prob } .25 \\ 0 & \text{with prob } .75 \end{cases} \qquad D: \begin{cases} -4,000 & \text{with prob } .25 \\ -2,000 & \text{with prob } .25 \\ 0 & \text{with prob } .5 \end{cases}$$

Problem 13 in KT: Choose between

$$A : \begin{cases} 6,000 & \text{with prob } .25 \\ 0 & \text{with prob } .75 \end{cases} \qquad B : \begin{cases} 4,000 & \text{with prob } .25 \\ 2,000 & \text{with prob } .25 \\ 0 & \text{with prob } .5 \end{cases}$$

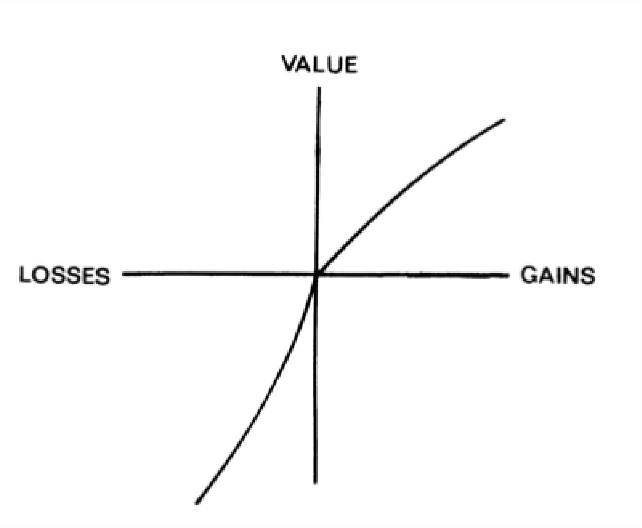
$$\begin{aligned} B \succ A &\implies \pi(.25)v(6,000) < \pi(.25)v(4,000) + \pi(.25)v(2,000) \\ &\implies v(6,000) < v(4,000) + v(2,000) \implies \text{concave for gains} \end{aligned}$$

Problem 13' in KT: Choose between

$$C : \begin{cases} -6,000 & \text{with prob } .25 \\ 0 & \text{with prob } .75 \end{cases} \qquad D : \begin{cases} -4,000 & \text{with prob } .25 \\ -2,000 & \text{with prob } .25 \\ 0 & \text{with prob } .5 \end{cases}$$

$$C \succ D \implies \text{convex for losses}$$

Kahneman and Tversky (1979) Value Function



Kahneman and Tversky (1979)

Recap: What Can Be Explained So Far

- We observed that significant concavity in the value function around the reference point might help to explain the scale of risk puzzles for expected utility:
  - Rabin paradox
  - Equity premium puzzle (see, e.g., Benartzi and Thaler (1995))
  - Kinks at 0 generate first-order risk aversion (as opposed to second-order risk aversion for expected utility)

Recap: What Can Be Explained So Far

- We observed that significant concavity in the value function around the reference point might help to explain the scale of risk puzzles for expected utility:
  - Rabin paradox
  - Equity premium puzzle (see, e.g., Benartzi and Thaler (1995))
  - Kinks at 0 generate first-order risk aversion (as opposed to second-order risk aversion for expected utility)
- But we cannot yet explain the Allais paradox: Fixing the reference point at 0, the value function v is just an expected utility function and hence is subject to the paradox — unless we introduce probability weighting.

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

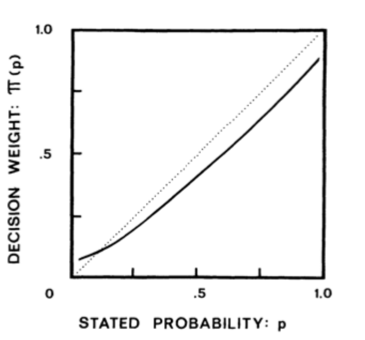
- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Probability Weighting and Allais

- Kahneman and Tversky (1979) show that explaining the Allais common consequence effect requires $\pi(.66) + \pi(.34) < 1$.
- The also argue that risk taking for low-probability gambles implies overweighting of low probability events.



Kahneman and Tversky (1979)

Issues with KT Weighting Function: Framing

One problem for the KT weighting function is sensitivity to framing.

- For example suppose $2\pi(p) > \pi(2p)$ for some $p < 1/2$ (as must be the case for the shape they proposed).
- Then, for $x > 0$,

$$\begin{aligned} V(x, p; x, p; 0, 1-2p) &= 2\pi(p)v(x) \\ &> \pi(2p)v(x) = V(x, 2p; 0, 1-2p). \end{aligned}$$

- With examples like this in mind, KT proposed an "editing phase" where the framing of gambles is set: $(x, p; x, p; 0, 1-2p)$ must be treated like $(x, 2p; 0, 1-2p)$.

Issues with KT Weighting Function: Stochastic Dominance

Unfortunately, imposing a framing requirement does not resolve the issue.

- Suppose again that $2\pi(p) > \pi(2p)$, but alter the previous example slightly.
- Suppose $x > 0$. Then, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} V(x, p; x - \varepsilon, p; 0, 1 - 2p) &= \pi(p)v(x) + \pi(p)v(x - \varepsilon) \\ &> \pi(2p)v(x) = V(x, 2p; 0, 1 - 2p) \end{aligned}$$

- This is a violation of stochastic dominance.
- In light of such examples, KT proposed that, in addition, stochastically dominated alternatives are dropped during the "editing phase" prior to applying the function V.

There is a cleaner alternative approach (cumulative prospect theory) that we will explore next.

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Rank-Dependent Utility (for simple lotteries)

Definition

A rank-dependent utility (RDU) representation consists of:

1. a utility function $u : X \to \mathbb{R}$ for outcomes; and
2. a function $\varphi : [0,1] \to [0,1]$ (the probability distortion function) that is continuous, nondecreasing, and onto.

The utility of a simple lottery of the form $(x_1, p_1; \ldots; x_n, p_n)$ with outcomes ordered $x_1 < x_2 < \cdots < x_n$ is

$$U(x_1, p_1; \ldots; x_n, p_n) = \sum_{i=1}^{n} \pi_i u(x_i),$$

where

$$\pi_i = \varphi(p_1 + \cdots + p_i) - \varphi(p_1 + \cdots + p_{i-1})$$

for $i > 1$ and $\pi_1 = \varphi(p_1)$.

Illustration of Probability Distortions

$$U(x_1, p_1; \dots; x_n, p_n) = \sum_{i=1}^{n} \pi_i u(x_i)$$

$$\pi_i = \varphi(p_1 + \cdots + p_i) - \varphi(p_1 + \cdots + p_{i-1}).$$

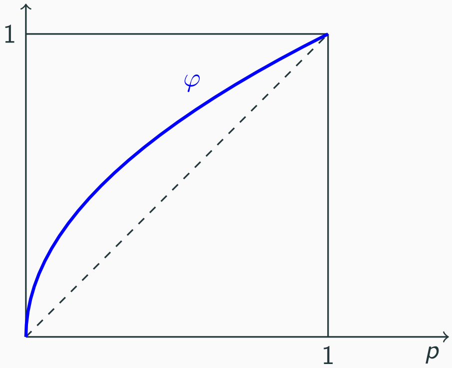


Illustration of Probability Distortions

$$U(x_1, p_1; \ldots; x_n, p_n) = \sum_{i=1}^{n} \pi_i u(x_i)$$
$$\pi_i = \varphi(p_1 + \cdots + p_i) - \varphi(p_1 + \cdots + p_{i-1}).$$

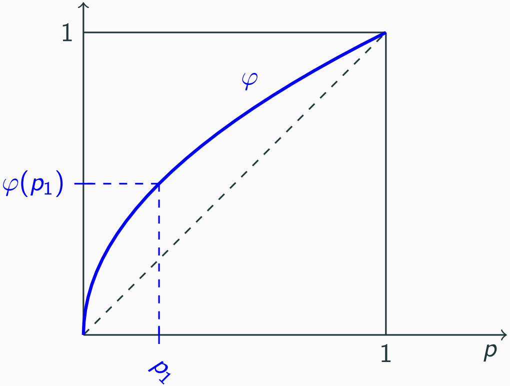


Illustration of Probability Distortions

$$U(x_1, p_1; \dots; x_n, p_n) = \sum_{i=1}^{n} \pi_i u(x_i)$$

$$\pi_i = \varphi(p_1 + \cdots + p_i) - \varphi(p_1 + \cdots + p_{i-1}).$$

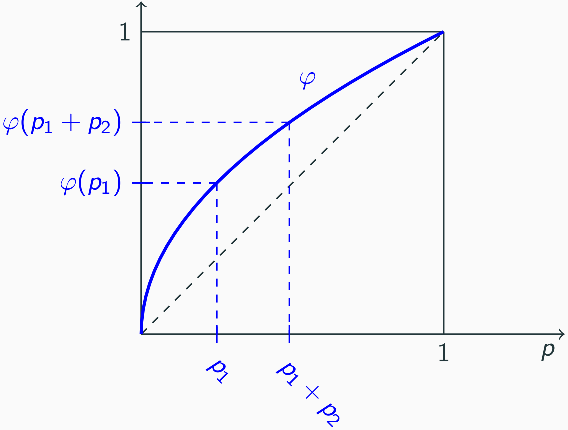


Illustration of Probability Distortions

$$U(x_1, p_1; \ldots; x_n, p_n) = \sum_{i=1}^{n} \pi_i u(x_i)$$

$$\pi_i = \varphi(p_1 + \cdots + p_i) - \varphi(p_1 + \cdots + p_{i-1}).$$

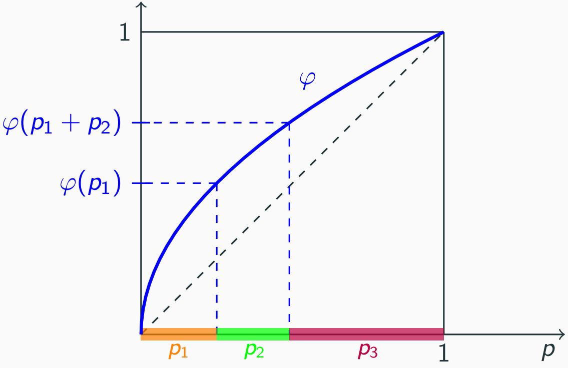
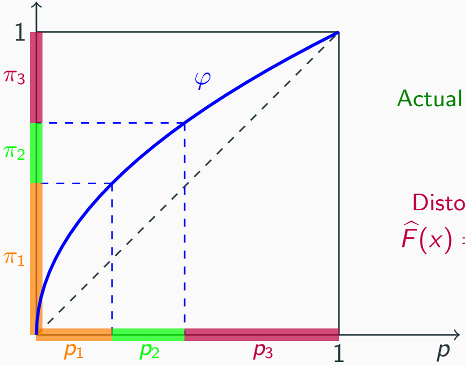


Illustration of Probability Distortions

$$U(x_1, p_1; \dots; x_n, p_n) = \sum_{i=1}^{n} \pi_i u(x_i)$$

$$\pi_i = \varphi(p_1 + \cdots + p_i) - \varphi(p_1 + \cdots + p_{i-1}).$$



Actual CDF: $F(x)$

↓

Distorted CDF:
$\widehat{F}(x) = \varphi(F(x))$

Special Cases and Properties

- No probability distortions ($\varphi(p) = p$) is expected utility.
- Linear utility ($u(x) = x$) is the dual theory of Yaari (1987)

Concavity of u and φ both play a role in risk aversion.

Special Cases and Properties

- No probability distortions ($\varphi(p) = p$) is expected utility.
- Linear utility ($u(x) = x$) is the dual theory of Yaari (1987)

Concavity of u and φ both play a role in risk aversion.

Example (binary lotteries, $u(x) = x$)

$$\begin{aligned} U(x_1, p_1; x_2, p_2) &= \varphi(p_1)x_1 + (1 - \varphi(p_1))x_2 \\ &< p_1 x_1 + (1 - p_1)x_2 && \text{(if } \varphi(p_1) > p_1\text{)} \\ &= U(p_1 x_1 + p_2 x_2, 1) \end{aligned}$$

Special Cases and Properties

- No probability distortions ($\varphi(p) = p$) is expected utility.
- Linear utility ($u(x) = x$) is the dual theory of Yaari (1987)

Concavity of u and φ both play a role in risk aversion.

Example (binary lotteries, $u(x) = x$)

$$\begin{aligned} U(x_1, p_1; x_2, p_2) &= \varphi(p_1)x_1 + (1 - \varphi(p_1))x_2 \\ &< p_1 x_1 + (1 - p_1)x_2 && (\text{if } \varphi(p_1) > p_1) \\ &= U(p_1 x_1 + p_2 x_2, 1) \end{aligned}$$

Theorem (Chew, Karni, and Safra (1987))

An RDU preference is risk averse (in the sense of SOSD monotonicity) if and only if both u and φ are concave.

Rank-Dependent Utility (for any lottery)

Definition (RDU for general distributions)

For non-simple lotteries, rank-dependent utility simply transforms the cdf F using the distortion function φ:

$$U(F) = \int u(x)\, d(\varphi \circ F)(x).$$

For the cdf of a simple lottery, this reduces to the previous formula.

- RDU has been studied in many papers: Quiggin (1982), Chew, Karni, and Safra (1987), Schmeidler (1989),. . . , Abdellaoui (2002)
- The formula above is the "cumulative" version of RDU (transformation of the cumulative distribution).
  - Some papers instead use the "decumulative" formula for RDA (transformation of the decumulative distribution).
  - The cumulative and decumulative versions of RDU are equivalent after suitable transformation of the probability distortion function.

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

RDU and the Allais Paradox (Common Consequence Effect)

Let $\varphi(p) = \sqrt{p}$ and $u(x) = x$.

$$A : \begin{cases} 2,500 & \text{with prob } .33 \;\rightarrow\; \sqrt{1} - \sqrt{.67} \approx 0.181 \\ 2,400 & \text{with prob } .66 \;\rightarrow\; \sqrt{.67} - \sqrt{.01} \approx 0.719 \\ 0 & \text{with prob } .01 \;\rightarrow\; \sqrt{0.01} = 0.1 \end{cases}$$

$$\begin{aligned} U(A) = (2,500)(0.181) + (2,400)(0.719) + (0)(0.1) &\approx 2,178 \\ &< U(B) = 2,400 \end{aligned}$$

RDU and the Allais Paradox (Common Consequence Effect)

Let $\varphi(p) = \sqrt{p}$ and $u(x) = x$.

$$A : \begin{cases} 2,500 & \text{with prob .33} \;\rightarrow\; \sqrt{1} - \sqrt{.67} \approx 0.181 \\ 2,400 & \text{with prob .66} \;\rightarrow\; \sqrt{.67} - \sqrt{.01} \approx 0.719 \\ 0 & \text{with prob .01} \;\rightarrow\; \sqrt{0.01} = 0.1 \end{cases}$$

$$U(A) = (2,500)(0.181) + (2,400)(0.719) + (0)(0.1) \approx 2,178$$
$$< U(B) = 2,400$$

$$C : \begin{cases} 2,500 & \text{with prob .33} \;\rightarrow\; \sqrt{1} - \sqrt{.67} \approx 0.181 \\ 0 & \text{with prob .67} \;\rightarrow\; \sqrt{.67} \approx 0.819 \end{cases}$$

$$D : \begin{cases} 2,400 & \text{with prob .34} \;\rightarrow\; \sqrt{1} - \sqrt{.66} \approx 0.188 \\ 0 & \text{with prob .66} \;\rightarrow\; \sqrt{.66} \approx 0.812 \end{cases}$$

$$U(C) = (2,500)(0.181) + (0)(0.819) \approx 454$$
$$> U(D) = (2,400)(0.188) + (0)(0.812) \approx 450$$

Allais Paradox: Expected Utility

Why does RDU behave differently than EU?

Recall the expected utility comparison of gambles:

$$U(A) = u(2,500)(0.33) + u(2,400)(0.66) + u(0)(0.01)$$

$$\downarrow$$

$$U(C) = u(2,500)(0.33) + \quad u(0) \quad (0.66) + u(0)(0.01)$$

$$U(B) = u(2,400)(0.33) + u(2,400)(0.66) + u(2,400)(0.01)$$

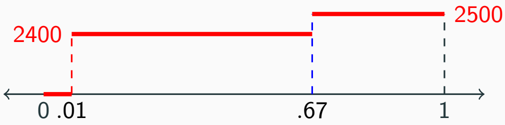
$$\downarrow$$

$$U(D) = u(2,400)(0.33) + \quad u(0) \quad (0.66) + u(2,400)(0.01)$$

Allais Paradox: Expected Utility

$$U(A) = u(2,500)(0.33) + u(2,400)(0.66) + u(0)(0.01)$$

$$\longrightarrow U(C) = u(2,500)(0.33) + \quad u(0) \quad (0.66) + u(0)(0.01)$$



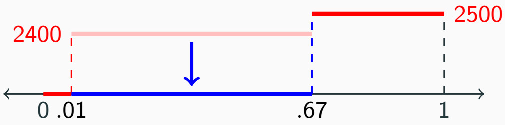
$$U(B) = u(2,400)(0.33) + u(2,400)(0.66) + u(2,400)(0.01)$$

$$\longrightarrow U(D) = u(2,400)(0.33) + \quad u(0) \quad (0.66) + u(2,400)(0.01)$$

Allais Paradox: Expected Utility

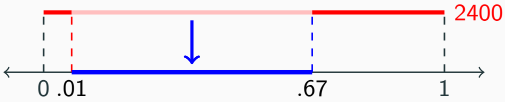
$$U(A) = u(2,500)(0.33) + u(2,400)(0.66) + u(0)(0.01)$$

$$\longrightarrow U(C) = u(2,500)(0.33) + \quad u(0) \quad (0.66) + u(0)(0.01)$$



$$U(B) = u(2,400)(0.33) + u(2,400)(0.66) + u(2,400)(0.01)$$

$$\longrightarrow U(D) = u(2,400)(0.33) + \quad u(0) \quad (0.66) + u(2,400)(0.01)$$



Allais Paradox: Rank-Dependent Utility

RDU comparison of gambles:

$$U(A) = u(2,500)(0.181) + u(2,400)(0.719) + u(0)(0.1)$$

$$\downarrow$$

$$U(C) = u(2,500)(0.181) + \quad u(0) \quad (0.719) + u(0)(0.1)$$

$$U(B) = u(2,400)(0.181) + u(2,400)(0.719) + u(2,400)(0.1)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$U(D) = u(2,400)(0.181) + \quad u(0) \quad (0.812) + u(2,400)(0.007)$$

since the weights in D can be decomposed as

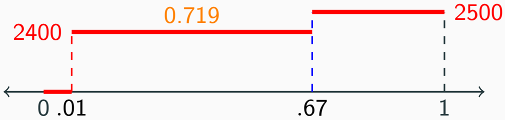
$$\text{weight on } 2,400 = \begin{cases} \sqrt{1} - \sqrt{.67} \approx 0.181 \\ \sqrt{.67} - \sqrt{.66} \approx 0.007 \end{cases}$$

$$\text{weight on } 0 = \left\{ \sqrt{.66} \approx 0.812 \right.$$

Allais Paradox: Rank-Dependent Utility

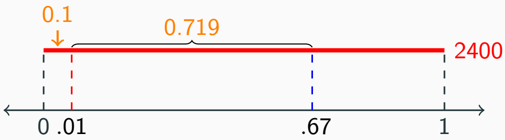
$$U(A) = u(2,500)(0.181) + u(2,400)(0.719) + u(0)(0.1)$$

$$\longrightarrow U(C) = u(2,500)(0.181) + \quad u(0) \quad (0.719) + u(0)(0.1)$$



$$U(B) = u(2,400)(0.181) + u(2,400)(0.719) + u(2,400)(0.1)$$

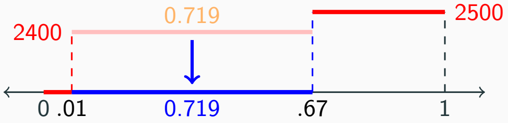
$$\longrightarrow U(D) = u(2,400)(0.181) + \quad u(0) \quad (0.812) + u(2,400)(0.007)$$



Allais Paradox: Rank-Dependent Utility

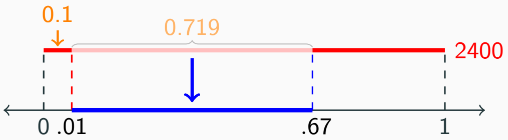
$$U(A) = u(2,500)(0.181) + u(2,400)(0.719) + u(0)(0.1)$$

$$\longrightarrow U(C) = u(2,500)(0.181) + \quad u(0) \quad (0.719) + u(0)(0.1)$$



$$U(B) = u(2,400)(0.181) + u(2,400)(0.719) + u(2,400)(0.1)$$

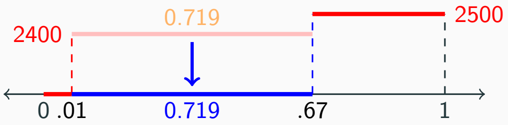
$$\longrightarrow U(D) = u(2,400)(0.181) + \quad u(0) \quad (0.812) + u(2,400)(0.007)$$



Allais Paradox: Rank-Dependent Utility

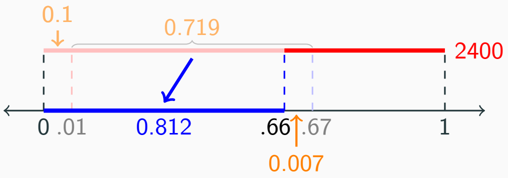
$$U(A) = u(2,500)(0.181) + u(2,400)(0.719) + u(0)(0.1)$$

$$\longrightarrow U(C) = u(2,500)(0.181) + \quad u(0) \quad (0.719) + u(0)(0.1)$$



$$U(B) = u(2,400)(0.181) + u(2,400)(0.719) + u(2,400)(0.1)$$

$$\longrightarrow U(D) = u(2,400)(0.181) + \quad u(0) \quad (0.812) + u(2,400)(0.007)$$



Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

RDU and the Rabin Paradox

Recall the Rabin (2000) calibration result for expected utility:

TABLE I

IF AVERSE TO 50-50 LOSE \$100 / GAIN g BETS FOR ALL WEALTH LEVELS, WILL TURN DOWN 50-50 LOSE L / GAIN G BETS; G'S ENTERED IN TABLE.

		g		
L	\$101	\$105	\$110	\$125
\$400	400	420	550	1,250
\$600	600	730	990	∞
\$800	800	1,050	2,090	∞
\$1,000	1,010	1,570	∞	∞
\$2,000	2,320	∞	∞	∞
\$4,000	5,750	∞	∞	∞
\$6,000	11,810	∞	∞	∞
\$8,000	34,940	∞	∞	∞
\$10,000	∞	∞	∞	∞
\$20,000	∞	∞	∞	∞

Now, consider instead the RDU utility from a 50-50 gamble:

$$U(w - L, 0.5; w + G, 0.5) = \varphi(0.5)u(w - L) + (1 - \varphi(0.5))u(w + G)$$

RDU and the Rabin Paradox

Take $u(x) = x$ for simplicity (i.e., "dual theory" of Yaari (1987)), and

$$\begin{aligned} & U(w-L, 0.5; w+G, 0.5) \geq U(w) \\ \iff & \varphi(0.5)(w-L) + (1-\varphi(0.5))(w+G) \geq w \\ \iff & \frac{G}{L} \geq \frac{\varphi(0.5)}{1-\varphi(0.5)} \end{aligned}$$

For example, if $\varphi(0.5) = 3/5$, then accept gamble iff $G/L \geq 1.5$.

L	G	accept/reject
100	125	reject
400	500	reject
400	610	accept
10,000	14,000	reject
10,000	16,000	accept

Thus RDU resolves the Rabin Paradox. The same is true for the other non-expected-utility theories that we'll discuss later. However...

Is that it for the Rabin Paradox?

Most people do not consider the Rabin Paradox fully resolved because of the issue of "background risk":

- Suppose I offer you a gain 110 or lose 100 gamble.
- This is probably not the only risk you face today, and certainly not the only risk you face over a longer period such as a year.
- If we integrate this background risk ($\tilde{y}$) into your decision about the gamble ($\tilde{x}$), then you are really choosing between $\tilde{x} + \tilde{y} + w$ and $\tilde{y} + w$, rather than $\tilde{x} + w$ and w.

Is that it for the Rabin Paradox?

Most people do not consider the Rabin Paradox fully resolved because of the issue of "background risk":

- Suppose I offer you a gain 110 or lose 100 gamble.
- This is probably not the only risk you face today, and certainly not the only risk you face over a longer period such as a year.
- If we integrate this background risk ($\tilde{y}$) into your decision about the gamble ($\tilde{x}$), then you are really choosing between $\tilde{x} + \tilde{y} + w$ and $\tilde{y} + w$, rather than $\tilde{x} + w$ and w.
- It turns out that sufficiently large $\tilde{y}$ makes it difficult to generate aversion to the small-stakes Rabin gambles, so the paradox remains (see Safra and Segal (2008)).
- One possible solution is narrow framing (see, e.g., Barberis, Huang, and Thaler (2006)), but we will also discuss others.

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Cumulative Prospect Theory

Tversky and Kahneman (1992) incorporate rank-dependent probability weighting into their model of prospect theory.

- They suggest decomposing any lottery into gains and losses and applying the rank-dependent formula to cumulative gain and loss probabilities separately.

- They suggest that a concave-convex shape for the weighting functions best fits the evidence (decumulative formula):

$$\varphi(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$$

Cumulative Prospect Theory

Tversky and Kahneman (1992) incorporate rank-dependent probability weighting into their model of prospect theory.

- They suggest decomposing any lottery into gains and losses and applying the rank-dependent formula to cumulative gain and loss probabilities separately.

- They suggest that a concave-convex shape for the weighting functions best fits the evidence (decumulative formula):

$$\varphi(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$$

- Overweights both low probability bad outcomes (certainty effect) and low probability good outcomes (possibility effect).

- This shape might also help to explain gambling and preferences for skewed asset returns (e.g., Barberis and Huang (2008)).

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Disappointment Aversion (Gul (1991))

Consider simple lotteries over a bounded interval $X \subset \mathbb{R}$.

Definition

An elation/disappointment decomposition (EDD) of a lottery p is (α, q, r) where $q \in B(p)$, $r \in W(p)$ and $\alpha q + (1-\alpha)r = p$:

$$B(p) = \{q : q(x) > 0 \Longrightarrow x \succsim p\}$$
$$W(p) = \{r : r(x) > 0 \Longrightarrow p \succsim x\}$$

In other words, letting $CE(p)$ denote the certainty equivalent of p,

- $q \in B(p)$ means that $x \geq CE(p)$ for all $x \in \mathrm{supp}(q)$.
- $r \in W(p)$ means that $x \leq CE(p)$ for all $x \in \mathrm{supp}(r)$.
- $\alpha = p(\{x : x > CE(p)\})$ is the probability of being elated (assuming $q(CE(p)) = 0$), and $1 - \alpha$ is the probability of being disappointed.

Disappointment Aversion

Gul (1991) proposed weak independence-like conditions that holds whenever mixtures of lotteries do not alter the EDD, and showed that these axioms imply the following representation:

Definition

A disappointment aversion representation $V(p)$ of $\succsim$ consists of a function $u(x)$ and a scalar $\beta \in (-1, \infty)$ such that

$$V(p) = \gamma(\alpha) \sum_x u(x)q(x) + (1 - \gamma(\alpha)) \sum_x u(x)r(x)$$

where (α, q, r) is an EDD of p and

$$\gamma(\alpha) = \frac{\alpha}{1 + (1 - \alpha)\beta}.$$

- $\beta = 0 \implies \gamma(\alpha) = \alpha, \ \forall \alpha \in (0, 1)$ (expected utility)
- $\beta > 0 \implies \gamma(\alpha) < \alpha, \ \forall \alpha \in (0, 1)$ (more risk averse than EU)

A More Direct (and Useful) Formula

Recall that $\alpha = p(\{x : x > CE(p)\}) = p(\{x : u(x) > V(p)\})$.

Thus $V(p) = v$ where v is the solution to:

$$
\begin{aligned}
v &= \frac{\alpha}{1+(1-\alpha)\beta}\sum_x u(x)q(x) + \left(1 - \frac{\alpha}{1+(1-\alpha)\beta}\right)\sum_x u(x)r(x) \\
&= \frac{\alpha}{1+(1-\alpha)\beta}\sum_x u(x)q(x) + \frac{(1-\alpha)(1+\beta)}{1+(1-\alpha)\beta}\sum_x u(x)r(x) \\
&= \frac{1}{1+(1-\alpha)\beta}\sum_{u(x)>v} u(x)p(x) + \frac{(1+\beta)}{1+(1-\alpha)\beta}\sum_{u(x)\leq v} u(x)p(x) \\
&= \frac{\sum_{u(x)>v} u(x)p(x) + (1+\beta)\sum_{u(x)\leq v} u(x)p(x)}{1+\beta\sum_{u(x)\leq v} p(x)}
\end{aligned}
$$

DA Satisfies Betweenness

DA can be written a third way that makes the "disappointment" interpretation more obvious.

$$v = \frac{\sum_{u(x)>v} u(x)p(x) + (1+\beta)\sum_{u(x)\leq v} u(x)p(x)}{1+\beta\sum_{u(x)\leq v} p(x)}$$

DA Satisfies Betweenness

DA can be written a third way that makes the "disappointment" interpretation more obvious.

$$v = \frac{\sum_{u(x)>v} u(x)p(x) + (1+\beta)\sum_{u(x)\le v} u(x)p(x)}{1+\beta\sum_{u(x)\le v} p(x)}$$

$$\iff v + \beta v \sum_{u(x)\le v} p(x) = \sum_{x} u(x)p(x) + \beta \sum_{u(x)\le v} u(x)p(x)$$

$$\iff v = \sum_{x} u(x)p(x) + \underbrace{\beta \sum_{u(x)\le v} \big(u(x) - v\big)p(x)}_{\text{Disappointment utility loss}}$$

DA Satisfies Betweenness

DA can be written a third way that makes the "disappointment" interpretation more obvious.

$$v = \frac{\sum_{u(x)>v} u(x)p(x) + (1+\beta)\sum_{u(x)\leq v} u(x)p(x)}{1+\beta\sum_{u(x)\leq v} p(x)}$$

$$\iff v + \beta v \sum_{u(x)\leq v} p(x) = \sum_{x} u(x)p(x) + \beta \sum_{u(x)\leq v} u(x)p(x)$$

$$\iff v = \sum_{x} u(x)p(x) + \underbrace{\beta \sum_{u(x)\leq v} \big(u(x)-v\big)p(x)}_{\text{Disappointment utility loss}}$$

This is a special case of the "betweenness" preferences (Chew (1983) and Dekel (1986)) representation $v = \sum_x \phi(x,v)p(x)$, where

$$\phi(x,v) = \begin{cases} u(x) + \beta(u(x)-v) & \text{if } u(x) \leq v \\ u(x) & \text{if } u(x) > v. \end{cases}$$

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Negative Certainty Independence

- Recall that Machina (1982) introduced Hypothesis II as a way to impose some discipline (weaker than independence) that is consistent with the Allais paradox.
- Many other non-expected-utility models are (at least in part) motivated by the same paradox.
    - This includes RDU and DA that we just studied.
    - But the connection is not always obvious from the axioms.

Negative Certainty Independence

- Recall that Machina (1982) introduced Hypothesis II as a way to impose some discipline (weaker than independence) that is consistent with the Allais paradox.
- Many other non-expected-utility models are (at least in part) motivated by the same paradox.
  - This includes RDU and DA that we just studied.
  - But the connection is not always obvious from the axioms.
- Dillenberger (2010) and Cerreia-Vioglio, Dillenberger, and Ortoleva (2015) studied an axiom that is more directly tied to the Allais common ratio effect: Negative Certainty Independence.

Negative Certainty Independence

Axiom

Negative Certainty Independence (NCI):

$$p \succsim \delta_x \implies \alpha p + (1-\alpha)q \succsim \alpha\delta_x + (1-\alpha)q.$$

Interpretation:

- Suppose p (which may involve risk) is preferred to x (*despite* it being certain).
- When these are each mixed with q then x no longer has the benefit of certainty, so the mixture with p should be favored all the more.

NCI and the Common Ratio Effect

Axiom

Negative Certainty Independence (NCI):

$$p \succsim \delta_x \implies \alpha p + (1-\alpha) q \succsim \alpha \delta_x + (1-\alpha) q.$$

Only reversals in one direction are permitted by NCI:

$$\begin{pmatrix} 4000 & .8 \\ 0 & .2 \end{pmatrix} \succsim \begin{pmatrix} 3000 \end{pmatrix}$$

$$\implies \frac{1}{4}\begin{pmatrix} 4000 & .8 \\ 0 & .2 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 \end{pmatrix} \succsim \frac{1}{4}\begin{pmatrix} 3000 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 4000 & .2 \\ 0 & .8 \end{pmatrix} \succsim \begin{pmatrix} 3000 & .25 \\ 0 & .75 \end{pmatrix}$$

Thus NCI is compatible with either no reversals or the Allais reversals, but not reversals in the “wrong” direction.

Cautious Expected Utility

Theorem (Cerreia-Vioglio, Dillenberger, and Ortoleva (2015))

A relation $\succsim$ satisfies weak order, continuity, monotonicity, and NCI if and only if it has an cautious expected utility representation: There exists a set $\mathcal{W}$ of Bernoulli utility functions such that

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}\big(\mathbb{E}_p[v]\big)$$

Comments:

- Disappointment Averse preferences (with $\beta \geq 0$) satisfy NCI and thus can be written as a CEU representation (see Cerreia-Vioglio, Dillenberger, and Ortoleva (2020))
- In earlier work, Dillenberger (2010) showed that NCI applied recursively in an intertemporal setting corresponds to a preference for one-shot resolution of uncertainty.

Outline

Prospect Theory

- Value Function
- Probability Weighting

Rank-Dependent Utility and Cumulative Prospect Theory

- Rank-Dependent Utility
- RDU and the Allais Paradox
- RDU and the Rabin Paradox
- Cumulative Prospect Theory

Other Non-Expected-Utility Theories

- Disappointment Aversion
- Cautious Expected Utility
- Mixture-Averse Preferences

Mixture-Averse Preferences

- Mixture-averse recursive preferences are studied in Sarver (2018).
- In the next lecture we will discuss some basic properties this model and applications to portfolio choice and the Rabin paradox.

Abdellaoui, M. (2002): "A Genuine Rank-Dependent Generalization of the Von Neumann-Morgenstern Expected Utility Theorem", *Econometrica*, 70, 717–736 (page 29).

Barberis, N. and M. Huang (2008): "Stocks as Lotteries: The Implications of Probability Weighting for Security Prices", *American Economic Review*, 98, 2066–2100 (pages 46, 47).

Barberis, N., M. Huang, and R. H. Thaler (2006): "Individual Preferences, Monetary Gambles, and Stock Market Participation: A Case for Narrow Framing", *American Economic Review*, 96, 1069–1090 (pages 43, 44).

Benartzi, S. and R. H. Thaler (1995): "Myopic Loss Aversion and the Equity Premium Puzzle", *Quarterly Journal of Economics*, 110, 73–92 (pages 12, 13).

Cerreia-Vioglio, S., D. Dillenberger, and P. Ortoleva (2015): "Cautious Expected Utility and the Certainty Effect", *Econometrica*, 83, 693–728 (pages 57, 58, 61).

Cerreia-Vioglio, S., D. Dillenberger, and P. Ortoleva (2020): "An Explicit Representation for Disappointment Aversion and Other Betweenness Preferences", *Theoretical Economics*, 15, 1509–1546 (page 61).

Chew, S. H. (1983): "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox", *Econometrica*, 51, 1065–1092 (pages 53–55).

Chew, S. H., E. Karni, and Z. Safra (1987): "Risk Aversion in the Theory of Expected Utility with Rank Dependent Probabilities", *Journal of Economic Theory*, 42, 370–381 (pages 26–29).

Dekel, E. (1986): "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom", *Journal of Economic Theory*, 40, 304–318 (pages 53–55).

Dillenberger, D. (2010): "Preferences for One-Shot Resolution of Uncertainty and Allais-Type Behavior", *Econometrica*, 78, 1973–2004 (pages 57, 58, 61).

Gul, F. (1991): "A Theory of Disappointment Aversion", *Econometrica*, 59, 667–686 (pages 50, 51).

Kahneman, D. and A. Tversky (1979): "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, 47, 263–292 (pages 3, 4, 11, 15).

Machina, M. J. (1982): "'Expected Utility' Analysis without the Independence Axiom", *Econometrica*, 50, 277–323 (pages 57, 58).

Markowitz, H. M. (1952): "The Utility of Wealth", *Journal of Political Economy*, 60, 151–158 (page 3).

Quiggin, J. (1982): "A Theory of Anticipated Utility", *Journal of Economic Behavior and Organization*, 3, 323–343 (page 29).

Rabin, M. (2000): "Risk Aversion and Expected-Utility Theory: A Calibration Theorem", *Econometrica*, 68, 1281–1292 (page 41).

Safra, Z. and U. Segal (2008): "Calibration Results for Non-Expected Utility Theories", *Econometrica*, 76, 1143–1166 (pages 43, 44).

Sarver, T. (2018): "Dynamic Mixture-Averse Preferences", *Econometrica*, 86, 1347–1382 (page 63).

Schmeidler, D. (1989): "Subjective Probability and Expected Utility without Additivity", *Econometrica*, 57, 571–587 (page 29).

Tversky, A. and D. Kahneman (1992): "Advances in Prospect Theory: Cumulative Representation of Uncertainty", *Journal of Risk and Uncertainty*, 5, 297–323 (pages 46, 47).

Yaari, M. E. (1987): "The Dual Theory of Choice under Risk", *Econometrica*, 55, 95–115 (pages 26–28, 42).