Problem Set #9 Solutions

1. A function f is given on the interval $[-\pi, \pi]$ and f is periodic with period 2π. For each of the function below -

   (a) Find the Fourier coefficients of f.

   (b) Find the Fourier series of f.

   (c) For what values of x is $f(x)$ equal to its Fourier series?

   (1) $f(x) = x^2$

   Solution: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\cdot\frac{x^3}{\pi}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3}$. For $n \geq 1, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$, by integration by parts, this integral is $\frac{1}{\pi}\cdot\frac{4\pi\cos(n\pi)}{n^2} = \begin{cases} \frac{4}{n^2} & \text{if } n \text{ even} \\ -\frac{4}{n^2} & \text{if } n \text{ odd} \end{cases}$. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx = 0$ since x^2 is an even function. Therefore $f(x) = \frac{\pi^2}{3} - 4\cos(x) + \cos(2x) - \frac{4}{9}\cos(3) + \cdots = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}4\cos(kx)}{n^2}$. $f(x)$ is continuous for all x, thus $f(x)$ is equal to its Fourier series for all x.

   (2) $f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ \cos x & \text{if } 0 \leq x < \pi \end{cases}$

   Solution: $a_0 = \frac{1}{2\pi}\cdot 0 + \frac{1}{2\pi}\int_0^{\pi}\cos x\,dx = 0$. For $n \geq 1$, $a_n = 0 + \frac{1}{\pi}\int_0^{\pi}\cos x\cos nx\,dx = \begin{cases} 0 & \text{if } n > 1 \\ \frac{1}{2} & \text{if } n = 1 \end{cases}$. $b_n = 0 + \frac{1}{\pi}\int_0^{\pi}\cos x\sin nx\;dx = \begin{cases} 0 & \text{if } n = 1 \\ \frac{-n\cos(n-1)\pi + n}{n^2-1} & \text{if } n > 1 \end{cases} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{2n}{n^2-1} & \text{if } n \text{ even} \end{cases}$. There $f(x) = \frac{1}{2}\cos(x) + \frac{4}{3}\sin(2x) + \frac{8}{15}\sin(4x) + \cdots = \frac{1}{2}\cos(x) + \sum_{k=1}^{\infty}\frac{4k}{4k^2-1}\sin(2kx)$. $f(x)$ is equal to its Fourier series except at points $\{\cdots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \cdots\}$.

   (3) $f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} \leq x < 0 \\ 0 & \text{if } 0 \leq x < \pi \end{cases}$

   Solution: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{-\frac{\pi}{2}} -1\;dx + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{0} 1\;dx + 0 = 0$. For $n \geq 1$, $a_n = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} -\cos nx\;dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0}\cos nx\,dx + 0 = \frac{1}{\pi}\cdot\frac{\sin\frac{n\pi}{2}}{n} + \frac{1}{\pi}\cdot\frac{\sin\frac{n\pi}{2}}{n} = \frac{2\sin\frac{n\pi}{2}}{n\pi} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2}{n\pi} & \text{if } n = 4k-3 \\ -\frac{2}{n\pi} & \text{if } n = 4k-1 \end{cases}$ for $k = 1, 2, 3\cdots$. $b_n = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} -\sin nx\;dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{0}\sin nx\,dx + 0 = \frac{1}{\pi}\cdot\frac{\cos\frac{n\pi}{2} - \cos n\pi}{n} + \frac{1}{\pi}\cdot\frac{\cos\frac{n\pi}{2} - 1}{n} = \frac{2\cos\frac{n\pi}{2} - \cos n\pi - 1}{n\pi} = \begin{cases} 0 & \text{if } n \text{ odd} \\ 0 & \text{if } n = 4k \\ -\frac{4}{n\pi} & \text{if } n = 4k-2 \end{cases}$ for $k = 1, 2, 3\cdots$. Therefore $f(x) = \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \frac{2}{5\pi}\cos 5x - \frac{2}{7\pi}\cos 7x + \cdots - \frac{4}{2\pi}\sin 2x - \frac{4}{6\pi}\sin 6x - \frac{4}{10\pi}\sin 10x - \cdots = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}2}{(2k-1)\pi}\cos(2k-1)x - \frac{4}{(4k-2)\pi}\sin(4k-2)x$. $f(x)$ is equal to its Fourier series except at points $\{-\pi \pm 2k\pi, -\frac{\pi}{2} \pm 2k\pi, \pm 2k\pi\}$ where $k = 0, 1, 2, \cdots$ are the integers.

2. Let $f(x)$ be a function defined on $[-\pi, \pi]$. We define the *even part* of f to be the function

$$f_e(x) := \frac{f(x) + f(-x)}{2}.$$

Similarly, we define the *odd part* of f to be the function

$$f_o(x) := \frac{f(x) - f(-x)}{2}$$

(a) Show that $f(x) = f_o(x) + f_e(x)$.
Solution: We have

$$f_o(x) + f_e(x) = \frac{f(x) - f(-x)}{2} + \frac{f(x) + f(-x)}{2} = f(x).$$

(b) Show that $f_e(x)$ is an even function, and that $f_o(x)$ is an odd function.
Solution: We have

$$f_e(-x) = \frac{f(-x) + f(-(-x))}{2} = f_e(x)$$

so $f_e(x)$ is even. On the other hand,

$$f_o(-x) = \frac{f(-x) - f(-(-x))}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x),$$

and so $f_o(x)$ is odd.

(c) Use substitution to show that

$$\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = -\int_{-\pi}^{\pi} f(-x)\sin(kx)\,dx.$$

Solution: Apply substitution $u = -x$ to get

$$\int_{-\pi}^{\pi} f(-x)\sin(kx)\,dx = \int_{\pi}^{-\pi} f(u)\sin(-ku)(-du) = -\int_{-\pi}^{\pi} f(u)\sin(ku)\,du.$$

Since the above are all definite integrals, the variable u on the righthand side is a dummy variable, and we can replace it with x to get the desired equality.

(d) Suppose the Fourier series for $f(x)$ is $a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$. Show that the Fourier series for $f_o(x)$ is

$$\sum_{k=1}^{\infty} b_k \sin(kx).$$

That is, if $\tilde{a}_k, \tilde{b}_k$ denote the Fourier coefficients of $f_o(x)$, show that $\tilde{b}_k = b_k$, and that $\tilde{a}_k = 0$.
Solution: Since $f_o(x)$ is odd, we have that $\tilde{a}_k = 0$ for all k. On the other hand, we have that the coefficients $\tilde{b}_k$ are given by

$$\begin{aligned}
\tilde{b}_k &= \int_{-\pi}^{\pi} f_o(x)\sin(kx)\,dx \\
&= \int_{-\pi}^{\pi} \frac{f(x) - f(-x)}{2}\sin(kx)\,dx \\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx - \int_{-\pi}^{\pi} f(-x)\sin(kx)\,dx\right) \\
&= \int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \\
&= b_k.
\end{aligned}$$

As an exercise (not to be handed in), you can also show that the Fourier series of $f_a(x)$ is $a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$.

3. (a) $y = \sin(x^2)$, $y' = 2x\cos(x^2)$. Thus

$$\begin{aligned}
(y')^2 + 4x^2y^2 &= (2x\cos(x^2))^2 + 4x^2(\sin(x^2))^2 \\
&= 4x^2(\cos^2(x^2) + \sin^2(x^2)) \\
&= 4x^2.
\end{aligned}$$

(b) i. $\sin(\pi y) = 0$ implies $y = k\pi, k \in \mathbb{Z}$.

ii. $y^3 - 3y^2 + 3y - 1 = 0$ implies $(y-1)^3 = 0$. So the equilibrium solutions are $y = 1$.

iii. $e^y - 1 = 0$ implies $y = \ln 1 = 0$.

iv. No equilibrium solutions.

(c) Plug in $y = e^{kx}$, we have $2k^2e^{kx} + 3ke^{kx} + e^{kx} = 0$, i.e. $(2k^2 + 3k + 1)e^{kx} = 0$. Thus $2k^2 + 3k + 1 = 0$, $k = -1/2$ or -1.

4. (a) Directly check $\dfrac{dCe^{\epsilon x}}{dx} = \epsilon Ce^{\epsilon x}$

(b) Let $y = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, plug in the equation we will find $\displaystyle\sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty} \epsilon a_n x^n$, so $a_n = a_{n-1}\epsilon/n$, then $y = \displaystyle\sum_{n=0}^{\infty} a_0 \frac{(\epsilon x)^n}{n!} = a_0 e^{\epsilon t}$. Let $C = a_0$ and we get the same answer as (a).

(c) as $\epsilon \to 0$, the solution become $y = C$,which is the solution to $\dfrac{dy}{dx} = 0$

5. (a) Show that

$$\int_{-\pi}^{\pi} e^x \sin(kx)\, dx = (-1)^{k+1} \frac{k}{1+k^2}(e^{\pi} - e^{-\pi})$$

Solution: By integration by parts with $u = \sin(kx)$ and $dv = e^x\, dx$, we have

$$\int_{-\pi}^{\pi} e^x \sin(kx)\, dx = [e^x \sin(kx)]_{-\pi}^{\pi} - k \int_{-\pi}^{\pi} e^x \cos(kx)\, dx.$$

Note that $[e^x \sin(kx)]_{-\pi}^{\pi} = 0$. To evaluate the integral on the righthand side, we use integration by parts with $u = \cos(kx)$ and $dv = e^x\, dx$, we have

$$-k \int_{-\pi}^{\pi} e^x \cos(kx)\, dx = [-ke^x \cos(kx)]_{-\pi}^{\pi} - k^2 \int_{-\pi}^{\pi} e^k \sin(kx)\, dx.$$

Now,

$$[-ke^x \cos(kx)]_{-\pi}^{\pi} = -ke^{\pi} \cos(k\pi) + ke^{-\pi} \cos(-k\pi) = (-1)^{k+1} k(e^{\pi} - e^{-\pi})$$

Adding $k^2 \displaystyle\int_{-\pi}^{\pi} e^x \sin(kx)\, dx$ to both sides gives

$$(1+k^2) \int_{-\pi}^{\pi} e^x \sin(kx)\, dx = (-1)^{k+1} k(e^{\pi} - e^{-\pi}).$$

Dividing both sides by $(1+k^2)$ gives the result.

(b) We define the *hyperbolic sine* function to be the odd part of e^x, as in Question 2:

$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

Using your answers for Question 2, and for part (a) of this question, write down the Fourier series $F(x)$ for $\sinh x$.

Solution: From part (a), the coefficients b_k of the Fourier series for e^x is $(-1)^{k+1}\dfrac{k}{1+k^2}(e^{\pi} - e^{-\pi})$. Question 2 then gives the Fourier coefficients for the odd part of e^x as $a_k = 0$ and b_k as for e^x. That is,

$$F(x) = \sum_{k=1}^{\infty} \frac{k}{1+k^2}(e^{\pi} - e^{-\pi}) \sin(kx).$$

(c) Evaluate the following:

i. $F(200\pi)$
Solution: By 2π-periodicity,

$$F(200\pi) = F(0)$$

Since $\sinh(x)$ is continuous at 0, we have $F(x) = \sinh(0) = 0$ by the Fourier Convergence Theorem.

ii. $F(201\pi)$
Solution: By 2π-periodicity,

$$F(201\pi) = F(\pi).$$

At the end points of the interval $[-\pi, \pi]$, we have that $\sinh(x)$ (extended 2π-periodically) is discontinuous. So by the Fourier Convergence Theorem,

$$F(\pi) = \frac{1}{2}\left(\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right) = \frac{1}{2}\left(\frac{e^{\pi} - e^{-\pi}}{2} + \frac{e^{-\pi} - e^{\pi}}{2}\right) = 0.$$

iii. $F(200\pi + 1)$
Solution: By 2π-periodicity,

$$F(200\pi + 1) = F(1).$$

Since $f(x)$ is continuous at 1, we have that $F(1) = f(1) = \dfrac{e^1 - e^{-1}}{2}$.

(Hint: Use the Fourier Convergence Theorem *and* 2π-periodicity of the Fourier series.)