Nov 6.

Using Taylor Series:

$$\int f(x) = \sum_{n=1}^{\infty} C_n (x-a)^n + C_0.$$

We can use left the side to do inty. try. Compute the limit, do the integrate, different:

Ex: $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$.

① You can use L'hospital Rule.

②: $\cos(x) = \sum_{n=0}^{\infty} \frac{}{n!} + 1$

the second $\cos(x) = 1 - x^2 + R_2(x)$

$R_2(x) \leq \left|\frac{M}{3!}(x-a)^3\right| = \left|\frac{M}{3!}x^3\right| = \left|\frac{1}{3!}x^3\right|$

$$\Rightarrow \lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \lim_{x \to 0} \frac{1 - 1 + x^2 - R_2(x)}{x^2} = 1 + \left|\frac{R_2(x)}{x^2}\right|$$

$$\frac{\frac{1}{3!}x^3}{x^2} \leq \left|\frac{x}{6}\right| \to 0.$$

$\therefore \frac{1-\cos(x)}{x^2} = 1$, we can discard those higher term.

Ex: Approximate $\int_0^{0.5} \frac{1}{1+x^7} dx$

$$= \int_0^{0.5} (1 - x^7 + x^{14} \cdots) dx$$

By Alterny, $\left|\frac{1}{1+x^7}-1+x^7\right| = |R_1(x)| \le x^{14} = \left(\frac{1}{2}\right)^{14}$, not long

$\left(\frac{1}{2}\right)^{14}\cdot\frac{1}{2}$ $\left(\frac{1}{2}\right)^{15} = \left(\frac{1}{8}\right)^{5}$, not long

$\left|\frac{1}{1+x^7}-1+x^7-x^{14}\right| = |R_2(x)| \le x^{21} = \left(\frac{1}{2}\right)^{21}$.

$\left(\frac{1}{2}\right)^{21}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{22}$

$= \left(\frac{1}{4}\right)^{11}$

$= \left(\frac{1}{16}\right)^{\frac{11}{2}}$ (✓)

Chinese good enough, but x^{21} is sufficient.

Ex 5: Use the 3rd degree Taylor polynomial centered at $x=0$ to approximate $\sin\left(\frac{1}{2}\right)$.

$\sin(x) = 1 - x + \frac{x^3}{3!} + \underline{O(x^4)}$

$\sin\frac{1}{2} \approx 1-\frac{1}{2}+\frac{\frac{1}{8}}{6}$

$= \frac{1}{2}+\frac{1}{48}$

$= \underline{\frac{25}{48}}$

Ex 4: The 11th degree polynomial of $f(x)$ centered at $x=0$ is

$T_{11}(x) = x + \frac{x^3}{2} - \frac{x^5}{4} + \frac{x^7}{8} - \frac{x^9}{16} + \frac{x^{11}}{32}$

Find the 9th deg of $f(x)$ at $x=0$?

$$f^{(9)}(0) = 9!\, c_9 = -\frac{9!}{16}$$

10th? $f^{(10)}(0) = 10!\, c_{10} = 0$

Ex I: Suppose $f(x)$ is a function with Taylor Series $\sum_{n=0}^{\infty} \frac{(-1)^n (n+1) x^{2n}}{2^n}$

(1) what is $f(0)$?

$f(0) = 1$

(2) Is $f(x)$ increasing or decr at $x=0$?

$\cancel{x=0} \Rightarrow \frac{(-1)(2)x^2}{2} = -2x < 0$

$\therefore$ decr

What is $f^{(100)}(0)$? $\Longrightarrow$ $c_{100} = \frac{(-1)^{100}(100+1)}{2^{100}} = \frac{101}{2^{100}}$

$\therefore f^{(100)}(0) = (100!)\, c_{100}$