11.05.

# Taylor Remainder Theorem

Def: $R_n(x) = f(x) - T_n(x)$, where $T_n(x) = \sum_{n=1}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n + f(a)$.

注: The Remainder term (difference of Taylor Polynomials and function f.) depends on both x and a.

If we can show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$\underline{f(x) = \lim T_n(x)}$

In this case, we say that f(x) is the sum of its Taylor.

Theorem: If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth degree Taylor Polynomials of f at a and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$, where R is the Radius of Convergence.

Taylor Inequality: If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

Fix $n=1$ case: an Mult PGT.

The idea is to use the middle value Theorem.

Optional: [ $|f''(x)| \le M$. $\quad a \le x \le a+d$

$$\int_a^x f''(t)\,dt \le \int_a^x M\,dt.$$

An antiderivative of $f''$ is $f'$, so by the Evaluation Theorem, we have

$$f'(x) - f'(a) = M(x-a) \quad \text{or} \quad f'(x) \le f'(a) + M(x-a)$$

Thus $\int_a^x f'(t)\,dt \le \int_a^x [f'(a) + M(t-a)]\,dt$

$\therefore f(x) - f(a) \le f'(a)(x-a) + M\frac{(x-a)^2}{2}$

$$\underbrace{f(x) - f(a) - f'(a)(x-a)}_{T_1(x)} \le M\frac{(x-a)^2}{2}$$

$$R_1(x) \le M\frac{(x-a)^2}{2}$$

Similarly, $f''(x) \ge -M \implies R_1(x) \ge \frac{-M(x-a)^2}{2}$.

$$\therefore |R_1(x)| \le \frac{M}{2}(x-a)^2.$$

Ex: We show certainly: $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} + 1$ for any $x_0$.

Consider: $|R_n(x)| \le \frac{e^{x_0}}{(n+1)!} x_0^{n+1}$

$|f^{(n)}(x)| = |e^x|$ and $= |e^{x_0}|$

$$|f(x)-T_n(x)| = \frac{e^{(c)} x_0^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$

$\therefore$ abs conv

Ex: $\ln(1+x) \overset{?}{=} \sum \frac{(-1)^{n+1} x^n}{n}$ $\quad 0 < x \le 1.$

$$f^{(n)}(x) = (-1)^{n-1} \frac{n!}{(x+1)^n}$$

$$|R_n(x)| \le \frac{M^{(n+1)}}{(n+1)!} x^{n+1}$$
$$\le \left| \frac{x^{n+1}}{(x+1)^{n+1}} \right|$$

$\lim n \to \infty \implies R_n(x) \to 0.$

Ex:

Let $T_4(x)$ be fourth degree Taylor Polynomial for $f(x) = e^x$ centered at $x=1$. Now suppose we use $T_4(1.1)$ to estimate $e^{1.1}$, use Taylor's Theorem to bound the resulting error above:

$$|R_4(x)| \le \frac{M}{(n+1)!} (c-5)^5$$
$$= \frac{e^{1.1}}{(n+1)!} (1.1-1)^5$$
$$= \frac{(0.1)^5 e^{1.1}}{5!}$$

Ex: $T_n(x)$ for $e^{2x}$ center at $x=0$. Use Taylor Theorem to find the smallest value of $n$ so that the error in approximating $e^{2x}$ by the $T_n(x)$ on the interval $[-1, 1]$ is smaller than $10^{-6}$.

$$R_n(x) = \frac{M}{(n+1)!}\underline{(x-a)^{n+1}} = ?$$

$$= \frac{2^{n+1}e^{2}}{(n+1)!} \le 10^{-6}.$$

Compute the smallest $n$.