Problem Set #9

Submit your answers on a separate sheet on the day assigned by your instructor. You must explain all answers. You may (and, in fact, are encouraged) to work with your peers on problem sets, but your final answers must be your own. Please write an acknowledgement for anyone who provides you help.

1. A function f is given on the interval $[-\pi, \pi]$ and f is periodic with period 2π. For each of the function below -
   (a) Find the Fourier coefficients of f.
   (b) Find the Fourier series of f.
   (c) For what values of x is $f(x)$ equal to its Fourier series?
   (1) $f(x) = x^2$
   (2) $f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ \cos x & \text{if } 0 \le x < \pi \end{cases}$
   (3) $f(x) = \begin{cases} -1 & \text{if } -\pi \le x < -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} \le x < 0 \\ 0 & \text{if } 0 \le x < \pi \end{cases}$

2. Let $f(x)$ be a function defined on $[-\pi, \pi]$. We define the *even part* of f to be the function

$$f_e(x) := \frac{f(x) + f(-x)}{2}.$$

   Similarly, we define the *odd part* of f to be the function

$$f_o(x) := \frac{f(x) - f(-x)}{2}$$

   (a) Show that $f(x) = f_o(x) + f_e(x)$.
   (b) Show that $f_e(x)$ is an even function, and that $f_o(x)$ is an odd function.
   (c) Use substitution to show that

$$\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = -\int_{-\pi}^{\pi} f(-x)\sin(kx)\,dx.$$

   (d) Suppose the Fourier series for $f(x)$ is $a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$. Show that the Fourier series for $f_o(x)$ is

$$\sum_{k=1}^{\infty} b_k \sin(kx).$$

   That is, if $\tilde{a}_k, \tilde{b}_k$ denote the Fourier coefficients of $f_o(x)$, show that $\tilde{b}_k = b_k$, and that $\tilde{a}_k = 0$.

   As an exercise (not to be handed in), you can also show that the Fourier series of $f_a(x)$ is $a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$.

3. Before proceeding to learn more about differential equations, a review of basic concepts is necessary.
   (a) Prove that $y(x) = \sin(x^2)$ is a solution to the differential equation $(y')^2 + 4x^2y^2 = 4x^2$.
   (b) Find **all** equilibrium solutions to following equations (you need to explain why you have exhausted all of them):
      i. $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(\pi y)$;
      ii. $\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - 3y^2 + 3y - 1$;

iii. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^y - 1$;

iv. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 1$.

(c) Suppose that $y(x) = e^{kx}$ is a solution to equation $2y'' + 3y' + y = 0$. Find k.

4. Consider the $\dfrac{dy}{dx} = \epsilon y$ $(*)$

(a) Check that $y = Ce^{\epsilon x}$ is the solution to this differential equation.

(b) Find the series solution to $(*)$ and compare it with the solution in (a). Are they the same?

(c) (extra credit) Let ϵ go to 0, check that the solution of $(*)$ becomes the solution of $\dfrac{dy}{dx} = 0$.

5. (a) Show that

$$\int_{-\pi}^{\pi} e^x \sin(kx)\, dx = (-1)^{k+1} \frac{k}{1+k^2}(e^{\pi} - e^{-\pi})$$

(b) We define the *hyperbolic sine* function to be the odd part of e^x, as in Question 2:

$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

Using your answers for Question 2, and for part (a) of this question, write down the Fourier series $F(x)$ for $\sinh x$.

(c) Evaluate the following:

i. $F(200\pi)$

ii. $F(201\pi)$

iii. $F(200\pi + 1)$

(Hint: Use the Fourier Convergence Theorem *and* 2π-periodicity of the Fourier series.)