Problem Set #8 Solutions

1. (a) We define the hyperbolic sine function to be

$$\sinh(x) := \frac{e^x - e^{-x}}{2}$$

i. Find the Taylor series for $\sinh(x)$ by computing the Taylor coefficients directly.

Solution: We have that $\sinh(0) = 0$, while the first derivative is $\dfrac{e^x + e^{-x}}{2}$, which is 1 at 0. Taking another derivative returns us to $\sinh(x)$. Therefore, the k-th derivative of $\sinh(x)$ at 0 is 0 if k is even, and 1 if k is odd. Therefore, the Taylor series is

$$T(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

ii. Find the Taylor series for $\sinh(x)$ using the Taylor series for e^x.
Solution: Recall that the Taylor series for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

and so

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}.$$

Therefore

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{1-(-1)^k}{2k!} x^k.$$

Note that $1 - (-1)^k$ is 0 when k is even and 2 when k is odd. So the above sum is simply

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

(b) Recall that $\int e^{-x^2}\,dx$ cannot be written in terms of elementary functions.

i. Find the Taylor series for $\int_0^x e^{-t^2}\,dt$. *Hint:* How can you modify the Taylor series for e^x to get the desired series?
Solution: Since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

we have

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}.$$

Then integrating the power series gives

$$\int_0^x e^{-t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

ii. Use the 5th Taylor polynomial (using part i.) to estimate $\int_0^1 e^{-t^2}\,dt$.
Solution: The first few terms of the above series are

$$x - \frac{x^3}{3} + \frac{x^5}{10} + \ldots$$

Therefore, evaluating $T_5(1)$ gives

$$1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30} = 0.76666\ldots$$

2. (a) For the following questions, approximate f by a Taylor polynomial with degree n at the number a. Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

i. $f(x) = \sqrt{x}, a = 4, n = 2, 4 \le x \le 4.2$

Solution: $f'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}}, f''(x) = -\frac{1}{4} \cdot x^{-\frac{3}{2}}$. Thus $T_2 = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$. In order to find R_2, we need $|f^{(3)}(x)| = \frac{3}{8} \cdot x^{-\frac{5}{2}} \le \frac{3}{8} \cdot \frac{1}{32} = \frac{3}{256}$ when $4 \le x \le 4.2$. Thus $|R_2| \le \frac{\frac{3}{256}}{3!} \cdot |0.2|^3 \approx 1.5625 * 10^{-5}$.

ii. $f(x) = \ln(1+2x), a = 1, n = 3, 0.5 \le x \le 1.5$

Solution: $f'(x) = \frac{2}{1+2x}, f''(x) = \frac{-4}{(1+2x)^2}, f^{(3)}(x) = \frac{16}{(1+2x)^3}$. Thus $T_3 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}}{3!}(x-1)^3 = \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3$. In order to find R_3, we need $|f^{(4)}| = |\frac{-96}{(1+2x)^4}| \le \frac{96}{2^4} = 6$ when $0.5 \le x \le 1.5$. Thus $|R_3| \le \frac{6}{4!} \cdot |0.5|^4 = 1.5625 * 10^{-2}$.

iii. $f(x) = x \sin x, a = 0, n = 4, -1 \le x \le 1$

Solution: $f'(x) = x \sin x, f''(x) = 2\cos x - x \sin x, f^{(3)}(x) = -3 \sin x - x \cos x, f^{(4)}(x) = -4 \cos x + x \sin x$. Thus $T_4 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = 0 + 0 + x^2 + 0 - \frac{1}{6}x^4 = x^2 - \frac{1}{6}x^4$. In order to find R_4, We need $f^{(5)} = 5 \sin x + x \cos x \le 5 \sin 1 + \cos 1$ when $-1 \le x \le 1$. Thus $|R_4| \le \frac{(5 \sin 1 + \cos 1)}{5!} \cdot |1|^5 \approx 0.03956$

(b) Let T_n be the nth degree Taylor Polynomial for e^x. What n should be used to estimate $e^{0.1}$ to within 0.00001? How many terms are there in the T_n of your choice?

Solution: $f^{(n)}(x) = e^x$ for all n. Thus for $0 \le x \le 0.1$ we have $|R_n| \le \frac{e^{0.1}}{n!} \cdot |0.1|^{n+1} < 0.00001$. Try plugging in different values of n we get $|R_2| \le 0.0001842$ and $|R_3| \le 4.605 * 10^{-6}$. Thus $n = 3$. Since it's a degree 3 Taylor Polynomial, there are 4 terms including the constant term.

3. (a) By the Taylor series of $\cos x, e^x$, we know that

$$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - ...;$$
$$1 + x - e^x = -\frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - ....$$

Thus we know that

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - ...}{-\frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - ...} = \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - ...}{-\frac{1}{2!} - \frac{x}{3!} - \frac{x^2}{4!} - ...} = -1.$$

(b) By the Taylor series of e^x, we know that

$$x^2 e^{-x^2} = x^2 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...\right)$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{k!}.$$

So the antiderivative of $x^2 e^{-x^2}$ has a power series expression reading as

$$\int x^2 e^{-x^2} \mathrm{d}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{(2k+3)k!} + C.$$

By FTC, we have

$$\int_0^{0.5} x^2 e^{-x^2} \mathrm{d}x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{(2k+3)k!}\Big|_{x=0.5} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)\cdot k! \cdot 2^{2k+3}}.$$

Denote the sum of the series as S. Notice that this is an alternating series with decreasing terms having limit 0, we know that

$$|S - S_n| \le b_{n+1} = \frac{1}{(2n+3)\cdot n! \cdot 2^{2n+3}}.$$

We need to find the least n such that the term on the R.H.S is less than 0.001. One can tell that $n = 2$ is good enough. So

$$S \approx S_2 = \frac{1}{24} - \frac{1}{160} = \frac{17}{480} = 0.0354167.$$

(c) Notice that $e^{-x} = 1 - x + x^2/2 - x^3/3! + ...$, so

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + ... = e^{-\ln 2} = \frac{1}{2}.$$

4. (a) We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, taking the derivatives and we get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$,and then we get $\frac{x^2}{(1-x)^2} = x^2 * \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n+1}$.

(b) We know $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, so $x\arctan(x^3) = x(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{2n+1}$

5.