# Problem Set #8

*Submit your answers on a separate sheet on the day assigned by your instructor. You must explain all answers. You may (and, in fact, are encouraged) to work with your peers on problem sets, but your final answers must be your own. Please write an acknowledgement for anyone who provides you help.*

1. (a) We define the hyperbolic sine function to be

$$\sinh(x) := \frac{e^x - e^{-x}}{2}.$$

   i. Find the Taylor series for $\sinh(x)$ by computing the Taylor coefficients directly.
   ii. Find the Taylor series for $\sinh(x)$ using the Taylor series for $e^x$.

   (b) Recall that $\int e^{-x^2}\,dx$ cannot be written in terms of elementary functions.

   i. Find the Taylor series for $\int_0^x e^{-t^2}\,dt$. *Hint:* How can you modify the Taylor series for $e^x$ to get the desired series?
   ii. Use the 5th Taylor polynomial (using part i.) to estimate $\int_0^1 e^{-t^2}\,dt$.

2. (a) For the following questions, approximate $f$ by a Taylor polynomial with degree $n$ at the number $a$. Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when $x$ lies in the given interval.

   i. $f(x) = \sqrt{x}, a = 4, n = 2, 4 \le x \le 4.2$
   ii. $f(x) = \ln(1 + 2x), a = 1, n = 3, 0.5 \le x \le 1.5$
   iii. $f(x) = x \sin x, a = 0, n = 4, -1 \le x \le 1$

   (b) Let $T_n$ be the $n$th degree Taylor Polynomial for $e^x$. What $n$ should be used to estimate $e^{0.1}$ to within 0.00001? How many terms are there in the $T_n$ of your choice?

3. Taylor series is helpful when computing limits, approximating definite integrals and summing series.

   (a) Find $\lim_{x\to 0} \dfrac{1 - \cos x}{1 + x - e^x}$.

   (b) Approximate $\int_0^{0.5} x^2 e^{-x^2} dx$ so that the error of your approximationg is less than 0.001.

   (c) Find $1 - \ln 2 + \dfrac{(\ln 2)^2}{2!} - \dfrac{(\ln 2)^3}{3!} + \dots.$

4. Find a power series representation, centered at $x = 0$, for each of the following:

   (a) $\dfrac{x^2}{(1 - x)^2}$

   (b) $x \arctan(x^3)$

5. Newton's method for approximating a root, say $r$, of the equation $f(x) = 0$ is an iterative method that generates a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to r$ as $n \to \infty$ under certain assumptions on $f$. The algorithm is designed as following: given $x_n$, $x_{n+1}$ is computed by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

   (a) **Intuitively understand why the algorithm works**. Let $f(x) = x^2 - 2$. Obviously we know the two roots of $f(x) = 0$ are $\sqrt{2}$ and $-\sqrt{2}$. We start our algorithm at $x_1 = 1$.

   i. Rewrite the algorithm in a form of fixed point problem, i.e. $x_{n+1} = g(x_n)$ given that $f(x) = x^2 - 2$. What is $g$?
   ii. Compute $x_2, x_3, x_4$.
   iii. How many decimal places is $x_4$ accurate to?

(b) **Tentatively prove that the algorithm works**. From (a) we see that even with only 3 iterations, one can have a fairly accurate approximation of the root. Now we work on general equation $f(x) = 0$, not necessarily $x^2 - 2 = 0$. Suppose that there exists $K > 0$ such that $|f'(x)| \geq K$ for any $x$, also given that the sequence $\{x_n\}$ produced by this method has a limit $r$. Prove that $f(r) = 0$ by passing the limit on both sides of (1).

(c) **Rigorously complete the proof by the theory of Taylor series**. (b) tells you that under some hypothesis, the limit of $x_n$ has to be a root of $f(x) = 0$. Now we will prove this convergence does hold and it is also a rapid one. Recall that $r$ is a root of $f(x) = 0$, i.e. $f(r) = 0$.

i. Prove that the algorithm can be written in the following way:

$$f'(x_n)(x_{n+1} - r) = f'(x_n)(x_n - r) - f(x_n). \tag{2}$$

ii. Suppose that $f''(x)$ exists on an interval $I$ containing $r, x_n, x_{n+1}$ and $|f''(x)| \leq M$, $|f'(x)| \geq K$ for all $x \in I$. Use the Taylor Remainder theorem (for function $f$ at $a = x_n$) to prove that the R.H.S. (right hand side) of (2) can be bounded by $|x_n - r|^2$, or more precisely,

$$|f'(x_n)(x_n - r) - f(x_n)| \leq \frac{M}{2}|x_n - r|^2. \tag{3}$$

iii. Notice that under the same assumption of (ii), we know that the L.H.S. (left hand side) of (2) is lower bounded by $K|x_{n+1} - r|$, i.e.

$$|f'(x_n)(x_{n+1} - r)| \geq K|x_{n+1} - r| \tag{4}$$

due to the lower bound of $f'$. Prove that

$$|x_{n+1} - r| \leq \frac{M}{2K}|x_n - r|^2$$

by plugging (3) and (4) into (2). This estimate shows you the speed of convergence of Newton method: if $x_n$ is accurate to $d$ decimal places, i.e. $|x_n - r| \leq 10^{-d}$, then $x_{n+1}$ is accurate to $2d$ decimal places. Thus this algorithm is fairly effcient.