Problem Set #7 Solutions

1. (a) Find the third degree Taylor polynomial for each function $f(x)$ below, and use it to estimate $f(0.5)$.

i. $\ln(1+x)$
Solution: Letting $f(x) = \ln(1+x)$ we have

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= \left. \frac{1}{1+x} \right|_{x=0} = 1, \\ f^{(2)}(0) &= -\left. \frac{1}{(1+x)^2} \right|_{x=0} = -1, \\ f^{(3)}(0) &= \left. \frac{2}{(1+x)^3} \right|_{x=0} = 2. \end{aligned}$$

Therefore the third Taylor polynomial is

$$T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

ii. $\ln(1-x)$
Solution: Letting $f(x) = \ln(1-x)$ we have

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= -1, \\ f^{(2)}(0) &= -1, \\ f^{(3)}(0) &= -2, \end{aligned}$$

and so

$$T_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

iii. $\ln(1-x^2)$
Solution: Letting $f(x) = \ln(1-x^2)$,

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= \left. \frac{-2x}{1-x^2} \right|_{x=0} = 0 \\ f^{(2)}(0) &= \left. \frac{-2}{1-x^2} + \frac{4x^2}{(1-x^2)^2} \right|_{x=0} = \left. \frac{-2+6x^2}{(1-x^2)^2} \right|_{x=0} = -2 \\ f^{(3)}(0) &= \left. \frac{12x}{(1-x^2)^2} + \frac{4x(-2+6x^2)}{(1-x^2)^3} \right|_{x=0} = 0, \end{aligned}$$

and so the Taylor polynomial is

$$T_3(x) = -x^2.$$

(b) Notice that $\ln(1-x^2) = \ln(1+x) + \ln(1-x)$. Show similarly that the Taylor polynomials that you computed in parts (i) and (ii) above also sum to the polynomial you computed in part (iii).
Solution: Adding the two polynomials for i. and ii. together gives

$$\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right) = -x^2,$$

as desired.

(c) Explain why the Taylor polynomial for $(f(x) + g(x))$ is equal to the sum of the Taylor polynomials for $f(x)$ and $g(x)$.

Let a_k be the Taylor coefficients for $f(x)$, let b_k be the Taylor coefficients for $g(x)$, and let c_k be the Taylor coefficients for $f(x)+g(x)$. Then

$$\begin{aligned} c_k &= \frac{1}{k!}\left(\frac{d^k}{dx^k}(f(x)+g(x))\Big|_{x=0}\right) \\ &= \frac{f^{(k)}(0)}{k!} + \frac{g^{(k)}(0)}{k!} \\ &= a_k + b_k. \end{aligned}$$

Since the coefficients sum together, so do the polynomials.

2. Find the radius of convergence and interval of convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}(n)!}{(n+1)!x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$. Thus the series is convergent for all x. $R=\infty$ and I.o.C is $(-\infty,\infty)$.

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{n^2x^n}{2^n}$

Solution: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}2^n}{2^{n+1}n^2x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2}\cdot\frac{x}{2}\right| = \frac{|x|}{2} < 1$. Thus $|x|<2$. $R=2$. When $x=-2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}(-1)^{2n}n^2$ which diverges by Test for Divergence. When $x=2$, the series becomes $\displaystyle\sum_{n=0}^{\infty}(-1)^n n^2$ which also diverges by Test for Divergence. Thus the I.o.C is $(-2,2)$.

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(3x-2)^n}{n3^n}$

Solution: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}n3^n}{(n+1)3^{n+1}(3x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{3x-2}{3}\right| = \frac{|3x-2|}{3} < 1. \implies$ $|3x-2|<3 \implies |x-\frac{2}{3}|<1$. Thus $R=1$ and $-\frac{1}{3}<x<\frac{5}{3}$. When $x=-\frac{1}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ which converges by AST. When $x=\frac{5}{3}$, the series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n}$ which diverges by p-test. Thus the I.o.C is $\left[-\frac{1}{3},\frac{5}{3}\right)$.

(d) If k is a positive integer, find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^k}{(kn)!}x^n$.

Solution: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)!)^kx^{n+1}(kn)!}{(kn+k)!(n!)^kx^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^kx}{(kn+1)(kn+2)\cdots(kn+k)}\right| = \left|\frac{x}{k^k}\right| <$ 1. Thus $|x|<k^k \implies R=k^k$.

3. (a) Suppose that $\displaystyle\sum_{n=0}^{\infty}c_nx^n$ converges when $x=-4$ and diverges when $x=6$. What can be said about the convergence or divergence of the following series?

i. $\displaystyle\sum_{n=0}^{\infty}c_n$

ii. $\displaystyle\sum_{n=0}^{\infty}c_n8^n$

iii. $\displaystyle\sum_{n=0}^{\infty}c_n(-3)^n$

iv. $\sum_{n=0}^{\infty}(-1)^n c_n 9^n$

Solution: i. converges ii. diverges iii. converges iv. diverges

(b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

Solution: $\lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\cdot|x| < 1 \Longrightarrow |x| < \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = R$. Thus using Ratio Test on the new series we get $\lim_{n\to\infty}\left|\frac{c_{n+1}x^{2n+2}}{c_n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\cdot|x^2| < 1$. This implies $|x|^2 < \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = R \Longrightarrow |x| < \sqrt{R}$. The radius of convergence is $\sqrt{R}$.

(c) Find a power series that has interval of convergence:

i. $(2, 6)$
ii. $[2, 6)$
iii. $(2, 6]$
iv. $[2, 6]$

Solution: i. $\sum_{k=0}^{\infty}\frac{(x-4)^k}{2^k}$ ii. $\sum_{k=0}^{\infty}\frac{(x-4)^k}{k2^k}$ iii. $\sum_{k=0}^{\infty}\frac{(-1)^k(x-4)^k}{k2^k}$ iv. $\sum_{k=0}^{\infty}\frac{(x-4)^k}{k^2 2^k}$

4. (a) Because $\lim_{x\to 0} e^{-1/x^2} = 0$, so $f(x)$ is continuous at $x = 0$. To prove differentiability, by definition, we have

$$\lim_{x\to 0}\frac{e^{-1/x^2}}{x} = \lim_{t\to\infty}\frac{t}{e^{t^2}} = 0.$$

So the derivative exists and $f'(0) = 0$.

(b) By induction, one can prove that for any order derivative of $f(x)$, it has the form $P(t)e^{-t^2}$ where $t = \frac{1}{x}$ and P is a polynomial. By dominance of e^{t^2} over any polynomial of t as $t \to \infty$, we know that the limit of $tP(t)e^{-t^2}$ is 0 as $t \to \infty$ which indicates that $f^{(k)}(0) = 0$ for any k.

(c) By definition, we know that the Taylor series of $f(x)$ at $x = 0$ should be

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)x^k}{k!} = 0.$$

5. (a) $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

(b) $T_4'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

(c) We can compute

$$\int_0^t T_4'(x)dx = \int_0^t 1 - \frac{x^2}{2!} + \frac{x^4}{4!}dx = t - \frac{t^3}{3!} + \frac{t^5}{5!} = T_5(t).$$

We can see that the integral of the Taylor Polynomials of $\cos(x)$ is the Taylor Polynomials of $\sin(x)$, the antiderivatives(with degree minus 1).

One remark is that same truth holds for the function f and its derivaties f'.