Problem Set #7

Submit your answers on a separate sheet on the day assigned by your instructor. You must explain all answers. You may (and, in fact, are encouraged) to work with your peers on problem sets, but your final answers must be your own. Please write an acknowledgement for anyone who provides you help.

1. (a) Find the third degree Taylor polynomial for each function $f(x)$ below, and use it to estimate $f(0.5)$.
   i. $\ln(1+x)$
   ii. $\ln(1-x)$
   iii. $\ln(1-x^2)$

   (b) Notice that $\ln(1-x^2) = \ln(1+x) + \ln(1-x)$. Show similarly that the Taylor polynomials that you computed in parts (i) and (ii) above also sum to the polynomial you computed in part (iii).

   (c) Explain why the Taylor polynomial for $(f(x) + g(x))$ is equal to the sum of the Taylor polynomials for $f(x)$ and $g(x)$.

2. Find the radius of convergence and interval of convergence of the following series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

   (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n3^n}$

   (d) If k is a positive integer, find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$.

3. (a) Suppose that $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?
   i. $\displaystyle\sum_{n=0}^{\infty} c_n$
   ii. $\displaystyle\sum_{n=0}^{\infty} c_n 8^n$
   iii. $\displaystyle\sum_{n=0}^{\infty} c_n (-3)^n$
   iv. $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

   (b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

   (c) Find a power series that has interval of convergence:
   i. $(2, 6)$
   ii. $[2, 6)$
   iii. $(2, 6]$
   iv. $[2, 6]$

4. Define function $f(x)$ on $\mathbb{R}$ by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$.

   (a) Show that $f(x)$ is continuous and differentiable. Find $f'(x)$.

(b) Show that $f(x)$ is infinitely times differentiable on $\mathbb{R}$ (i.e. for any integer k and $x \in \mathbb{R}$, $f^{(k)}(x)$ exists). (Hint: you only need to check the infinite differentiability at $x = 0$ since e^{-1/x^2} is smooth for $x \neq 0$.)

(c) Find the Taylor series of $f(x)$ at $x = 0$.

5. (a) Compute the 5th degree Taylor Polynomials of $\sin(x)$

   (b) Compute the 4th degree Taylor Polynomials of $\cos(x)$

   (c) Use the 4th degree Taylor Polynomials of $\cos(x)$ to estimate $\int_0^t \cos(x)dx$, and compare the answer to the results of problem (a) and (b).