Problem Set #6 Solutions

1. Use Integral Test to determine whether the following series are convergent or divergent.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

   Solution: Let $f(x) = \dfrac{x}{x^2+1}$. Note that $f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$, so $f(x)$ is decreasing for $x > 1$. Since $f(x)$ is also positive and continuous, we can apply Integral Test.
   We have that

   $$\int_1^{\infty} \frac{x}{1+x^2}\, dx = \frac{1}{2}\int_2^{\infty} \frac{1}{u}\, du.$$

   Since this integral diverges, the series diverges.

   (b) $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k^3}$

   Solution: Let $f(x) = x^2 e^{-x^3}$. Note that $f'(x) = (2x - 3x^4)e^{-x^3}$, so $f(x)$ is decreasing for $x > \left(\dfrac{2}{3}\right)^{\frac{1}{3}}$.
   Since $f(x)$ is also positive and decreasing, we can apply the Integral Test.
   We have that

   $$\int_1^{\infty} x^2 e^{-x^3}\, dx = \left[\frac{-e^{-x^3}}{3}\right]_1^{\infty}$$
   $$= \frac{e}{3}.$$

   Since the integral converges, the series converges.

   (c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\ln k}$

   Solution: Let $f(x) = \dfrac{1}{x\ln x}$. Note that $\dfrac{d}{dx} x\ln x = \ln x + 1$, and so the denominator of $f(x)$ is increasing for $x > 1$, and thus $f(x)$ is decreasing for $x > 1$. Note that $f(x)$ is also positive and continuous for $x > 1$. Therefore, we can apply Integral Test.
   We have that

   $$\int_2^{\infty} \frac{1}{x\ln x}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u}\, du.$$

   Since the integral diverges, the series diverges also.

2. (a) Use Comparison Test to determine whether the series is convergent or divergent.

   i. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4+1}$

   Solution: $\dfrac{n^3}{n^4+1} < \dfrac{n^3}{n^4} = \dfrac{1}{n}$. Thus we need to use limit comparison test - We know that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n}$ is convergent by p-test. Let $a_n = \dfrac{n^3}{n^4+1}, b_n = \dfrac{1}{n}$. Then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n^3}{n^4+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^4}{n^4+1} = 1.$
   Thus by limit comparison test the series is divergent.

   ii. $\displaystyle\sum_{n=0}^{\infty} \frac{1+\cos n}{10^n}$

   Solution: $\cos n \le 1$, thus $\dfrac{1+\cos n}{10^n} \le \dfrac{2}{10^n}$. $\displaystyle\sum_{n=0}^{\infty}\frac{2}{10^n} = \sum_{n=0}^{\infty} 2\left(\frac{1}{10}\right)^n$ is a geometric series and converges since $\left|\dfrac{1}{10}\right| < 1$. Thus our series converges by limit comparison test.

iii. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Solution: Let $a_n = \sin\left(\frac{1}{n}\right), b_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n}$. This is an indeterminate form of $\frac{0}{0}$. Converting it to a function and using L'Hopital's rule we have $\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{(-1/x^2)} = \lim_{x\to\infty} \cos(1/x) = 1$. Thus $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, our series diverges by limit comparison test.

(b) Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3+1}$.

i. Use the Comparison Test to show that this series converges.

Solution: $\sum_{k=1}^{\infty} \frac{1}{k^3+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^3}$ which converges by p-test. Thus the series converges by comparison test.

ii. Note that this series satisfies the conditions of the Integral Test. Thus, we can use the associated error bounds to say that if we approximate $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$ by its 10th partial sum, the resulting error is bounded above by $\int_{10}^{\infty} \frac{1}{x^3+1}\, dx$. The value of this integral is difficult to find but we know that it is bounded above by $\int_{10}^{\infty} \frac{1}{x^3}\, dx$. Use this to find an upper bound of the error.

Solution: $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3+1}\, dx \leq \int_{10}^{\infty} \frac{1}{x^3}\, dx = \lim_{t\to\infty} \frac{-x^{-2}}{2}\Big|_{10}^{t} = \frac{1}{200}$

(c) Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1+a_n)$ is convergent.

Solution: Use limit comparison test. Let $x_n = \ln(1+a_n), y_n = a_n$, $\lim_{n\to\infty} \frac{x_n}{y_x} = \lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n}$. Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Thus the limit is an indeterminate form of $\frac{0}{0}$. Converting it to a function where $a_n \to 0$ as $n\to\infty$ is equivalent as $x \to 0$ and using L'Hopital's rule we have $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$. Since $\sum a_n$ converges, $\sum \ln(1+a_n)$ also converges by limit comparison test.

3. (a) It is conditionally convergent. It is not absolutely convergent since $\sum_{k=1}^{\infty} \left|\frac{(-1)^{k-1}}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ which is divergent. However, it is convergent by AST: the series is alternating , $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n+1} \leq \frac{1}{n}$.

(b) It is absolutely convergent. Notice that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, and $\frac{|\cos k \sin k|}{k^2} \leq \frac{1}{k^2}$, so by comparison test, series $\sum_{k=1}^{\infty} \frac{\sin k \cos k}{k^2}$ is absolutely convergent.

(c) It is divergent. Notice that

$$\sum_{k=1}^{\infty} \frac{\pi + 3\sin k}{3\sqrt{k}} = \sum_{k=1}^{\infty} \frac{\frac{\pi}{3} + \sin k}{\sqrt{k}} \geq \sum_{k=1}^{\infty} \frac{\frac{\pi}{3} - 1}{\sqrt{k}}$$

and $\pi/3 > 1$, so by comparison test, this series is divergent.

(d) It is conditionally convergent. It is not absolutely convergent since $\ln k/k \geq 1/k$ for $k \geq 3$, so by comparison test, it is not absolutely convergent. However, it is convergent by AST: the series is alternating, $\lim \frac{\ln k}{k} = 0$ and $f(x) = \ln x/x$ is decreasing after $x > e$. So it is convergent.

4. (a) Use ratio test we know that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^4\cdot\frac{1}{4}\right| = \frac{1}{4} < 1.$$

Thus it's convergent.

(b)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.$$

So we know it's divergent.

(c)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^n\right| = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e} < 1.$$

Thus it's convergent.

(d)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(kn+k)(kn+k-1)\cdots(kn+1)} < 1.$$

when $k \geq 2$. So we know that it's convergent.

5. Consider the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{1+k^2}$. We will approximate the value of this series using partial sums $s_n = \displaystyle\sum_{k=1}^{n}\frac{1}{1+k^2}$.

(a) Use Integral Test to show the series converges.

Solution: Let $f(x) = \dfrac{1}{1+x^2}$. Note that $f(x)$ is positive, continuous, and decreasing, so we can apply Integral Test. We have

$$\int_1^\infty \frac{1}{1+x^2}\,dx = [\arctan x]_1^\infty = \frac{\pi}{4}.$$

Since the integral converges, so does the series.

(b) Compute s_{15}. (You may want to use a computer or a calculator for this and the following steps)

Solution: $s_{15} = 1.0122693692050062$

(c) Find an upper bound on the error for your approximation in part b).

Solution: We have that

$$\begin{aligned}\text{Error}(15) &< \int_{15}^\infty \frac{1}{1+x^2}\,dx \\ &= [\arctan x]_{15}^\infty \\ &= \frac{\pi}{2} - \arctan(15) \\ &= 0.06656816377582375\end{aligned}$$

(d) Find a lower bound on the error for your approximation in part b).

Solution: We have that

$$\begin{aligned}\text{Error}(15) &> \int_{16}^\infty \frac{1}{1+x^2}\,dx \\ &= [\arctan x]_{16}^\infty \\ &= \frac{\pi}{2} - \arctan(16) \\ &= 0.06241880999595728\end{aligned}$$

(e) Use parts c) and d) to write the value of the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{1+k^2}$, accurate to two decimal places.

Solution: We have from the above that

$$s_{15} + \mathrm{Error}(15) < 1.0122693692050062 + 0.06656816377582375 = 1.07883753298083,$$

and

$$s_{15} + \mathrm{Error}(15) > 1.0122693692050062 + 0.06241880999595728 = 1.0746881792009635.$$

Since the actual value of the series is equal to $s_{15} + \mathrm{Error}(15)$, we see from the two inequalities above that up to two decimal places, the series is equal to 1.07.