11.11 Fourier Series TWO.

Ex: Find the Fourier Series for the 2π-periodic function:

$$f(x) = \begin{cases} -x & -\pi \le x < 0 \\ 0 & 0 \le x < \pi \end{cases}$$

(half of last Ex.)

$$a_0 = \int f(x)\,dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{0} f(x)\cdot \sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cdot \sin(nx)\,dx$$

$$= -\frac{1}{n\pi}\cos(nx)\cdot x\Big|_{-\pi}^{0} + \frac{1}{n\pi}\int_0^{\pi} \cos(nx)\,dx \xrightarrow{} 0$$

$$= \boxed{\frac{(-1)^n}{n}}$$

$b_n = \frac{1}{2}b_n$ we compute yet $= \frac{-2}{\pi n^2}$ when n is odd.

$$\therefore F(x) = \underbrace{\frac{\pi}{4}}_{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\sin(nx) + \sum_{m=1}^{\infty} \frac{-4}{(2m-1)^2\pi}\cos((2m-1)x)$$

See the Graph of Fourier Series of f.

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 < x \le \pi \end{cases}$$

$$f(x) = \begin{cases} -x & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$

In the convergence.

(don't write the detail.

Fourier Convergence Theorem: If f is periodic function with period 2π, and f, and f' are piecewise continuous $(-\pi, \pi)$. Then the Fourier Series is convergent.

$$\boxed{F(x) = f(x)}$$ both 2π-periodic

Some can use Fourier Series to do the computation

$$f(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x < \pi. \end{cases}$$

But also: $f(x) = F(x) = \frac{\pi}{2} + \sum \frac{-4}{(2k-1)^2} \cos((2k-1)x)$

$$f(\pi) = \pi = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi}$$

$$\Longrightarrow \frac{\pi}{2} = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^2\pi}$$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2\pi}$$

We can compute $\sum \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}$.

Learn $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, very similar, try $[f(x) = x^2]$

Also, use Fourier Series to find some Integral directly:

Suppose $f(x)$ has Fourier Series: $F(x) = \pi + \sum \frac{-4}{\pi(2k-1)^2} \cos((2k-1)x)$

(a) $\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \quad = a_0 = \pi \times 2 = 2\pi.$

(b.) $\int_{-\pi}^{\pi} f(x)\cdot \sin(3x)\,dx \Rightarrow \quad \underline{a_3 = 0}$

(c): $\int_{-\pi}^{\pi} f(x)\cos(5x)\,dx \Rightarrow b_5 = \frac{4}{\pi(6-1)^2} = \frac{4}{25\pi}.$

(d): $\int_{-\pi}^{\pi} f(x)\cos(16x)\,dx \Rightarrow b_8 = \frac{4}{\pi(16-1)^2} = \frac{4}{225\pi}$

Additional Notes on AHP HW.

$$\int_{-\pi}^{\pi} a\cdot dx = a\,2\pi$$

$$\int_{-\pi}^{\pi} \sin(mx)\,dx = -\frac{1}{m}\cos(m\quad)\Big|_{-\pi}^{\pi} = 0.$$

$$\int_{-\pi}^{\pi} \cos(mx)\,dx = \frac{1}{m}\sin(mx)\Big|_{-\pi}^{\pi} = 0.$$

$\sin^2(mx) = \frac{1-\cos(2mx)}{2}$

$$\int_{-\pi}^{\pi} \sin^2(mx)\,dx = \int_{-\pi}^{\pi} \frac{1-\cos(2mx)}{2}\,dx = \pi$$

$\cos^2(mx) = \frac{1+\cos(2mx)}{2}$

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \int_{-\pi}^{\pi} \frac{1+\cos(2mx)}{2}\,dx = \pi$$

$$\int_{-\pi}^{\pi} \cos(mx)\cdot\sin(mx)\,dx = \frac{1}{2}\sin^2(mx)\Big|_{-\pi}^{\pi} = 0.$$

$m \neq n.$

$$\int_{-\pi}^{\pi} \sin(mx)\,\sin(nx)\,dx = -\frac{1}{n}\int \cos(nx)\,d\sin(mx)$$

$$= -\frac{m}{n}\int \cos(nx)\,\cos(mx)\,dx$$

$$\int_{-\pi}^{\pi} \cos(mx)\cdot\cos(nx)\,dx = \frac{m}{n}\int \sin(mx)\,\sin(nx) = -\frac{n}{m}\int \cos(nx)\,\cos(mx)\,dx$$

$$= \frac{n}{m}\int \sin(nx)\,\sin(mx)\,dx.$$