Introduction to Fourier Series

We've seen one example so far of series of functions. The Taylor Series of a function is a series of polynomials and can be used to approximate a function at a point.

Another kind of series of functions are Fourier Series. Rather than using polynomials to approximate a function at a point we can use trigonometric functions to approximate periodic functions over the entire period. We will assume for this introduction that we are interested in approximating periodic functions of period 2π.

1. The Taylor Series Revisited

The idea for both Taylor and Fourier Series is that we have some basic functions and we want to express an arbitrary function in terms of our basic functions. In general this will require us to use an infinite series of basic functions. For Taylor Series the basic functions were powers of x. To express a function in terms of powers of x we need a way to determine the "x^n part" of a function. If we call the x^n part of $f(x)$ a_n then we express $f(x)$ as the series $\sum a_n x^n$.

For this to be reasonable, our list of basic functions must satisfy some properties:

(1) Independence: The x^m part of x^n is 0 if $m \neq n$.
(2) Uniformity: The x^n part of x^n is 1.
(3) Completeness: The various powers of x form a complete list in that our series of functions can be written entirely in terms of powers of x.

Given a function $f(x)$ we have a way to "filter out" the x^n part. For a polynomial, the n-th derivative of the polynomial at 0 is exactly the n-th coefficient times $n!$. For example:

$$\begin{aligned}
p(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \text{higher order terms} \\
p'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \text{HOT} \\
p''(x) &= 2a_2 + 3 \cdot 2 \cdot a_3 x + \text{HOT} \\
p'''(x) &= 3 \cdot 2 \cdot a_3 + \text{HOT}
\end{aligned}$$

So $p'''(0) = 3! \cdot a_3$ since the higher order terms all still contain powers of x, so they vanish when we evaluate at $x = 0$.

We use this method to determine the x^n part of any function for which the n-th derivative is defined: the x^n part of $f(x)$ is:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Once we know the x^n part for each n we can reassemble our function as $\sum a_n x^n$.

We can verify the first two properties in the above list. Suppose $m < n$. The x^n part of x^m is $\frac{d^n(x^m)}{dx^n} = 0$, since the m-th derivative of x^m is the constant $m!$, and higher derivatives are all 0. Conversely, the m-th derivative of x^n is a multiple of x^{n-m}. When we evaluate this at $x = 0$ we get 0. The x^m part of x^m is $\frac{\frac{d^m(x^m)}{dx^m}}{m!} = \frac{m!}{m!} = 1$.

Verifying the third property is harder. This was the content of Taylor's Theorem, that if we want to know that the series we compute represents the original function we must check to see that the remainder term limits to 0.

2. Fourier Series

The idea for the Fourier Series is similar to what we did for Taylor Series. Instead of using powers of x as our basic functions we use $\sin(kx)$ and $\cos(kx)$ for $k = 0, 1, 2, 3, \ldots$.

We would like to have some method of "filtering out" the $\sin(kx)$ and $\cos(kx)$ parts of a function like we had for the Taylor Series. In the Fourier Series case we do this filtering by multiplying by the basic function and integrating the result. In the Taylor Series case we also had to correct by a factor of $n!$, and we get a correction factor in the Fourier Series case as well.

Definition 2.1. The *Fourier Series* for a function $f(x)$ with period 2π is given by:

$$\sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(0x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
b_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin(0x)dx = 0 \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx \quad \text{for } k > 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \quad \text{for } k > 0
\end{aligned}$$

Analogous to the Taylor Series, we define the *Fourier Polynomials* to be the finite sum:

$$F_n = \sum_{k=0}^{n} a_k \sin(kx) + b_k \cos(kx)$$

Note: The reason the $k = 0$ terms are treated separately is that $\sin(0x) = 0$ and $\cos(0x) = 1$.

We check that these basic functions and our method of determining coefficients satisfy the same properties as in the Taylor Series:

Fist we check that $\cos(0x) = 1$ is independent from all the other $\sin(kx)$ and $\cos(kx)$.

$$\int_{-\pi}^{\pi} \cos(0x)\cos(kx)dx = \int_{-\pi}^{\pi} \cos(kx)dx = 0$$

$$\int_{-\pi}^{\pi} \cos(0x)\sin(kx)dx = \int_{-\pi}^{\pi} \sin(kx)dx = 0$$

These results occur because $\sin(kx)$ and $\cos(kx)$ are periodic with period $\frac{2\pi}{k}$, so if we integrate them over the interval $[-\pi, \pi]$ we are integrating k complete cycles, and the negative areas cancel out the positive areas.

Also $\int_{-\pi}^{\pi} \cos(0x)\cos(0x)dx = \int_{-\pi}^{\pi} 1dx = 2\pi$, which gives us the correction factor of $\frac{1}{2\pi}$ in the definition of a_0.

For nonzero j, k we use the method of §7.2 for the Type 3 trigonometric integrals to show:

$$\int_{-\pi}^{\pi} \sin(jx)\sin(kx)dx = \begin{cases} \pi & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(jx)\cos(kx)dx = \begin{cases} \pi & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin(jx)\cos(kx)dx = 0$$

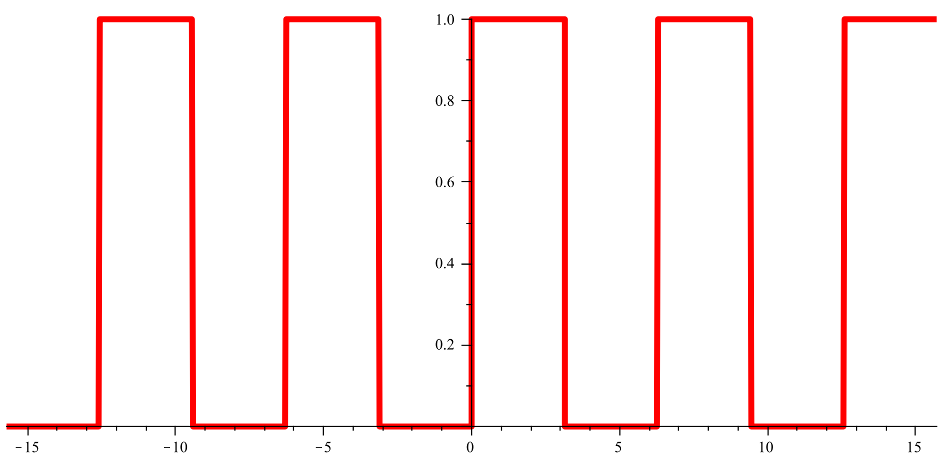
These computations show us the properties that we wanted: the $\cos(kx)$ part of $\cos(kx)$ is 1 (after taking into account the correction factor $\frac{1}{\pi}$), and the $\cos(kx)$ part of $\cos(jx)$ or of $\sin(jx)$ is 0, and similarly for $\sin(kx)$. We will take for granted the third property, that this list of basic function is enough to give us good approximations for functions with period 2π.

Notice that for Taylor Series we needed to know that the function $f(x)$ which we wanted to approximate was differentiable in order to compute the coefficients. For Fourier Series we only need the function to be integrable. We will see some examples where the functions don't even need to be continuous!

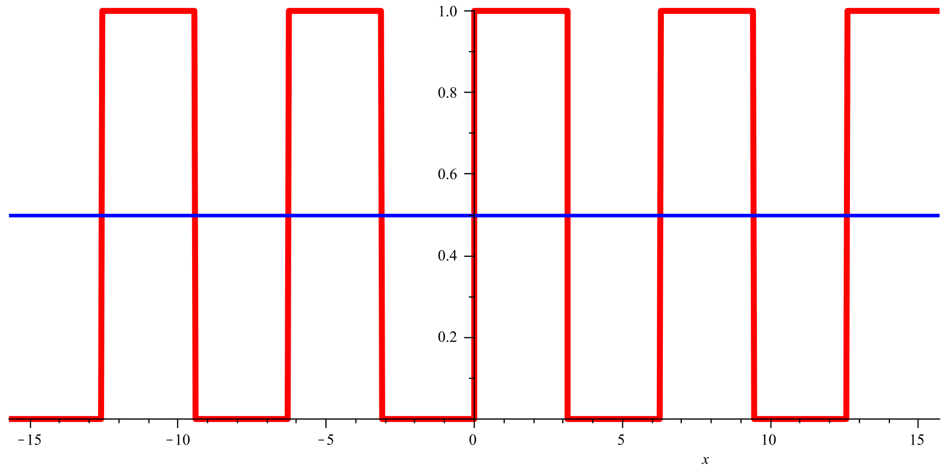
3. Examples

Example 3.1. Compute the Fourier Polynomials $F_0, \ldots, F_5$ for the 2π-periodic square wave given by:

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } -\pi < x < 0 \end{cases}$$



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} 1dx = \frac{1}{2}$$

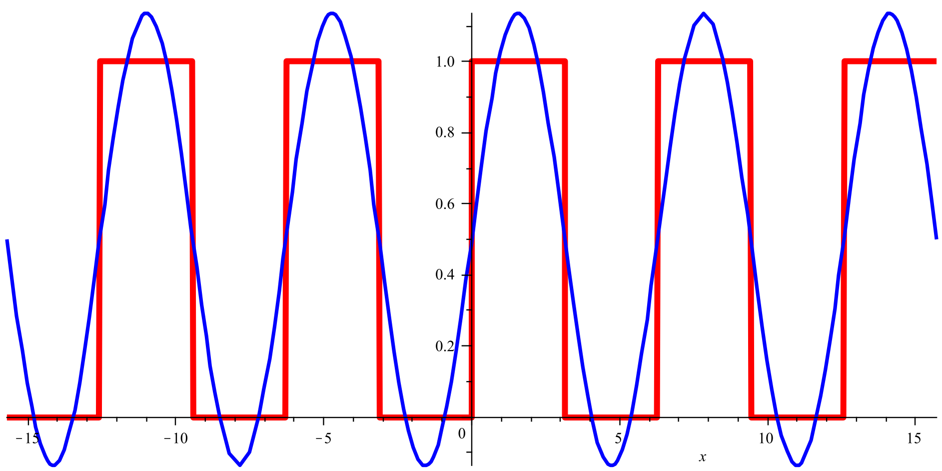


So $F_0 = \frac{1}{2}$.

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(x)dx = \frac{1}{\pi}\sin(x)\big|_0^{\pi} = \frac{1}{\pi}\left(\sin(\pi) - \sin(0)\right) = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)dx = \frac{1}{\pi}\int_{0}^{\pi}\sin(x)dx = \frac{1}{\pi}\left(-\cos(x)\right)\big|_0^{\pi} = \frac{1}{\pi}\left(-\cos(\pi) + \cos(0)\right) = \frac{2}{\pi}$$

$$F_1 = \frac{1}{2} + \frac{2}{\pi}\sin(x)$$



$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2x)dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(2x)dx = \frac{1}{\pi}\frac{1}{2}\sin(2x)\big|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{2}\sin(2\pi) - \sin(0)\right) = 0$$

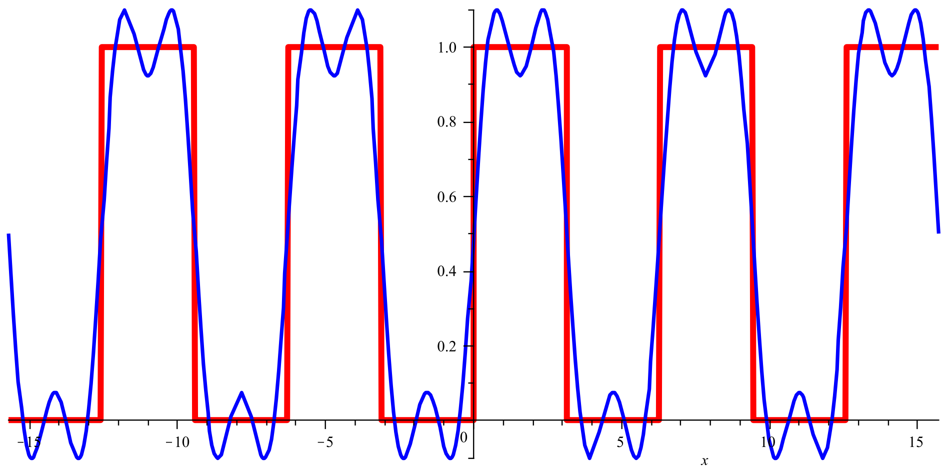
$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2x)dx = \frac{1}{\pi}\int_{0}^{\pi}\sin(2x)dx = \frac{1}{\pi}\left(-\frac{1}{2}\cos(2x)\right)\big|_0^{\pi} = \frac{1}{2\pi}\left(-\cos(2\pi) + \cos(0)\right) = 0$$

$$F_2 = \frac{1}{2} + \frac{2}{\pi}\sin(x) = F_1$$

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(3x)dx = \frac{1}{\pi}\int_{0}^{\pi}\cos(3x)dx = \frac{1}{\pi}\frac{1}{3}\sin(3x)\big|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{3}\sin(3\pi) - \sin(0)\right) = 0$$

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(3x)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(3x)dx = \frac{1}{\pi}\left(-\frac{1}{3}\cos(3x)\right)\Big|_0^{\pi} = \frac{1}{3\pi}\left(-\cos(3\pi)+\cos(0)\right) = \frac{2}{3\pi}$$

$$F_3 = \frac{1}{2} + \frac{2}{\pi}\sin(x) + \frac{2}{3\pi}\sin(3x)$$



$$a_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(4x)dx = \frac{1}{\pi}\int_{0}^{\pi} \cos(4x)dx = \frac{1}{\pi}\frac{1}{4}\sin(4x)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{4}\sin(4\pi) - \sin(0)\right) = 0$$

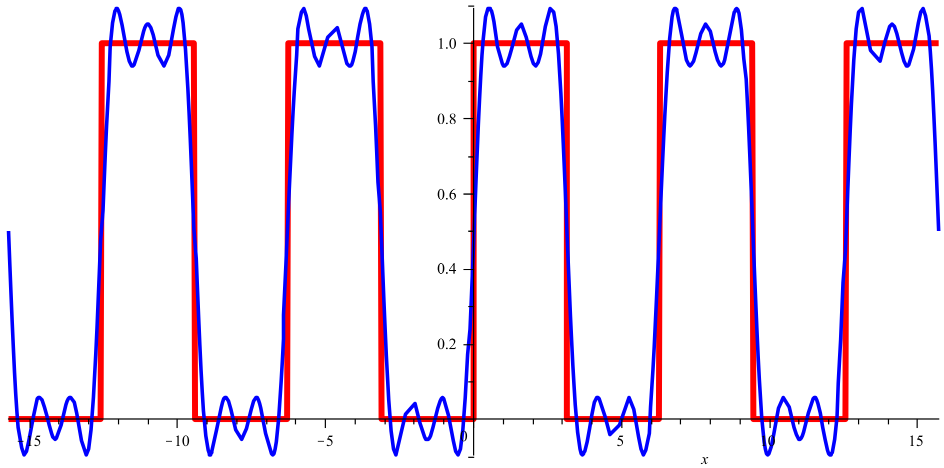
$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(4x)dx = \frac{1}{\pi}\left(-\frac{1}{4}\cos(4x)\right)\Big|_0^{\pi} = \frac{1}{4\pi}\left(-\cos(4\pi)+\cos(0)\right) = 0$$

$$F_4 = \frac{1}{2} + \frac{2}{\pi}\sin(x) = F_3$$

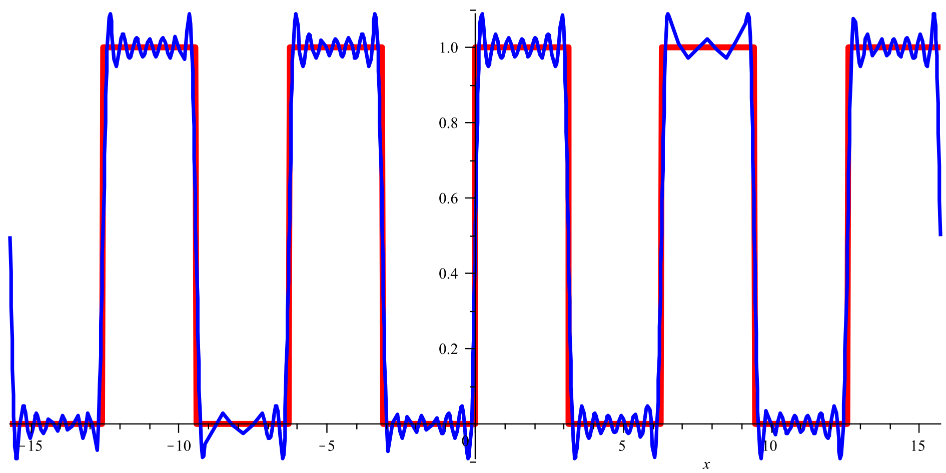
$$a_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(5x)dx = \frac{1}{\pi}\int_{0}^{\pi} \cos(5x)dx = \frac{1}{\pi}\frac{1}{5}\sin(5x)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{5}\sin(5\pi) - \sin(0)\right) = 0$$

$$b_5 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(5x)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(5x)dx = \frac{1}{\pi}\left(-\frac{1}{5}\cos(5x)\right)\Big|_0^{\pi} = \frac{1}{5\pi}\left(-\cos(5\pi)+\cos(0)\right) = \frac{2}{5\pi}$$

$$F_5 = \frac{1}{2} + \frac{2}{\pi}\sin(x) + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x)$$



Here's F_{13} as well:



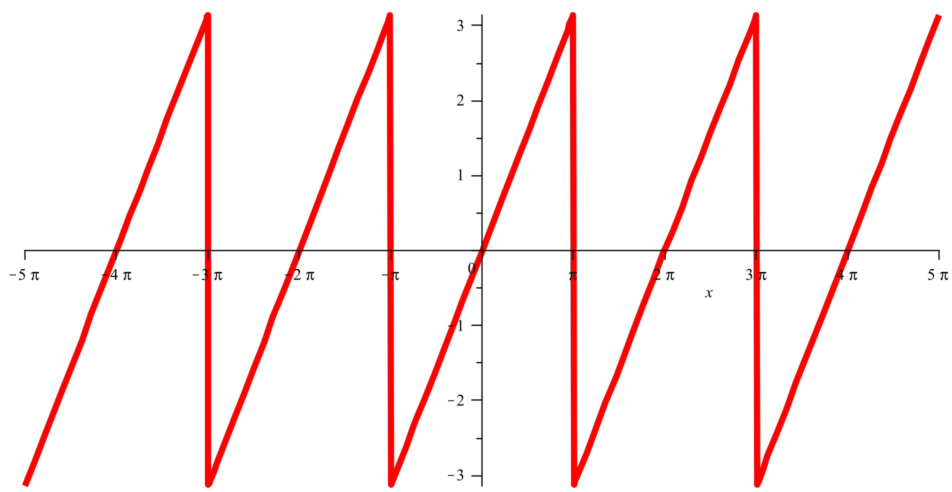
Before doing the next example we notice some simplifications. We are integrating over a symmetric interval. An odd function is symmetric with respect to the origin, so if we integrate over a symmetric interval we will always get 0. Conversely, an even function is symmetric with respect to the y-axis, so the area to the left of the y-axis is equal to the area to the right of the y-axis.

The functions $\sin(kx)$ are odd. The functions $\cos(kx)$ are even. If $f(x)$ is an even function then the $f(x)\sin(kx)$ are odd, so the $b_k = 0$. If $f(x)$ is odd then the $f(x)\cos(kx)$ are odd, so the $a_k = 0$. Another way to remember these simplifications is that an even function should be made up of even functions, so its Fourier Series consists entirely of cos terms. An odd function should be made up of odd functions, so its Fourier Series consists entirely of sin terms. This was true for Taylor Series as well. Recall that the Taylor Series for $\sin(x)$ contained only odd powers of x, while the Taylor Series for $\cos(x)$ contained only even powers.

Also, while we often will require a special argument for $k = 0$, usually we can compute a_k and b_k for all $k > 0$ simultaneously, as in the following examples. This will save us considerable work.

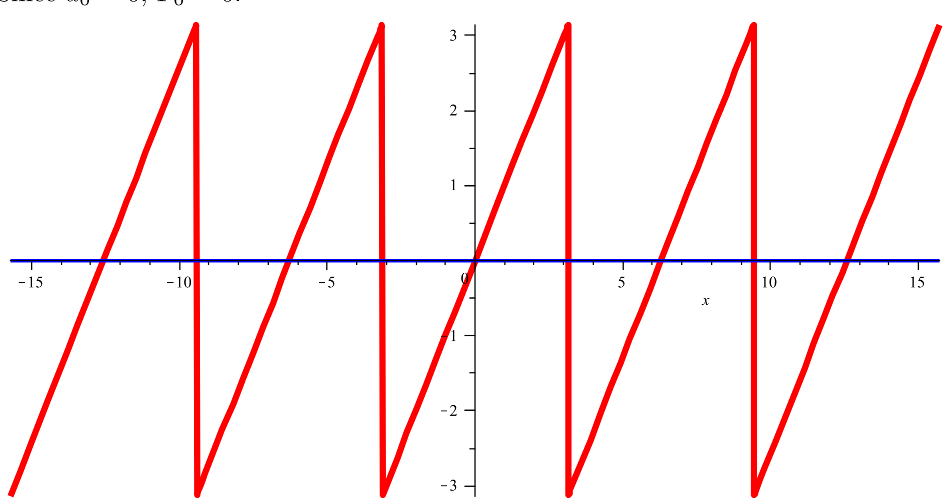
Example 3.2. Find some Fourier Polynomials for the 2π-periodic sawtooth wave defined by:

$$f(x) = x \quad \text{for } -\pi < x < \pi$$



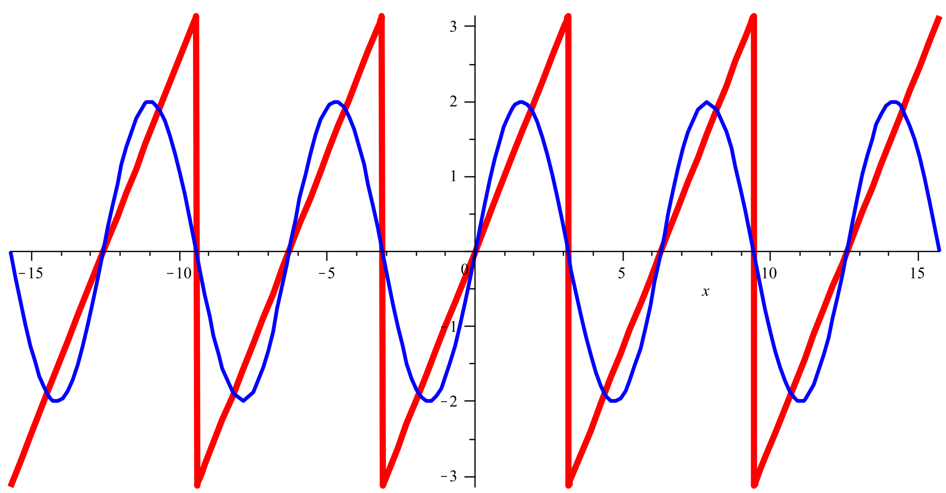
On the interval $[-\pi, \pi]$ this function is odd, so the $a_k = 0$ and we need only compute the b_k. If $f(x)$ is odd then $f(x)\sin(kx)$ is even, so we may compute integrals on $[0, \pi]$ and double the result.

Since $a_0 = 0$, $F_0 = 0$.

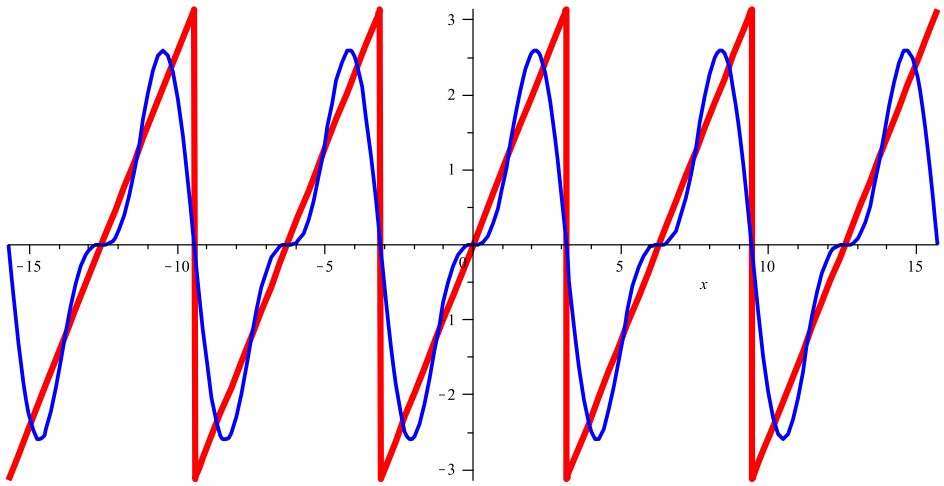


$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(kx)dx \\
&= \frac{2}{k\pi}\int_{0}^{\pi} kx\sin(kx)dx \\
& \text{substitute } w = kx, \text{ so } dx = \frac{1}{k}dw \\
&= \frac{2}{k^2\pi}\int_{0}^{k\pi} w\sin(w)dw \\
&= \frac{2}{k^2\pi}\int_{0}^{k\pi} w\sin(w)dw \\
&= \frac{2}{k^2\pi}\left(\sin(w) - w\cos(w)\right)\Big|_{0}^{k\pi} \\
&= \frac{2}{k^2\pi}\left(-k\pi(-1)^k\right) \\
&= \frac{2}{k}\cdot(-1)^{k+1}
\end{aligned}
$$

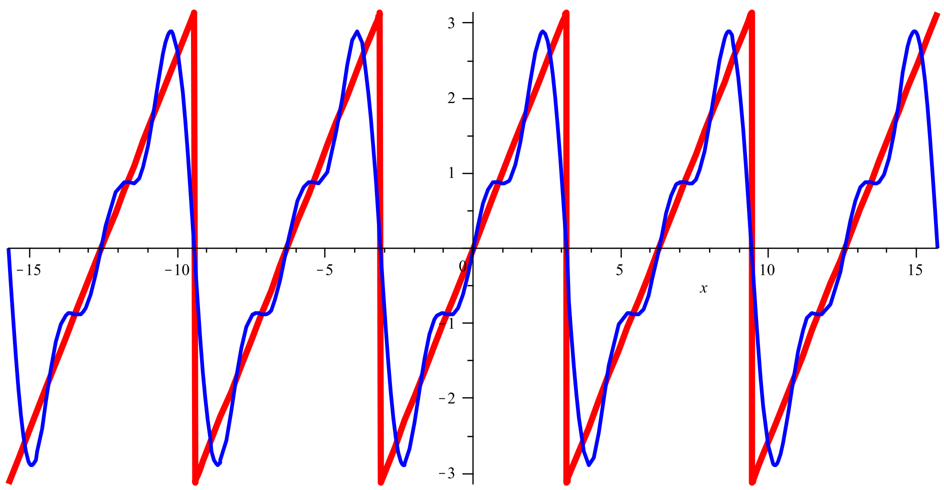
So $b_1 = 2$ and $F_1 = b_1\sin(x) = 2\sin(x)$.



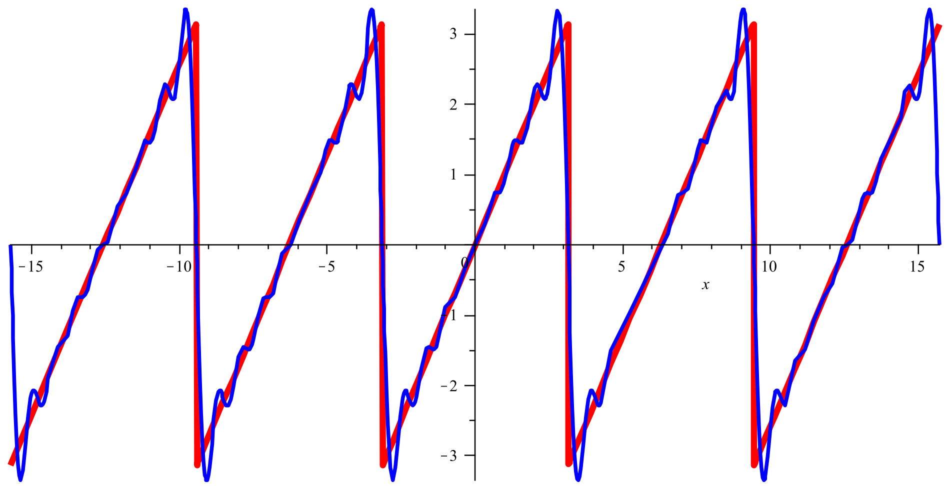
$b_2 = -1$, so $F_2 = 2\sin(x) - \sin(2x)$.



$b_3 = \frac{2}{3}$, so $F_3 = 2\sin(x) - \sin(2x) + \frac{2}{3}sin(3x)$.



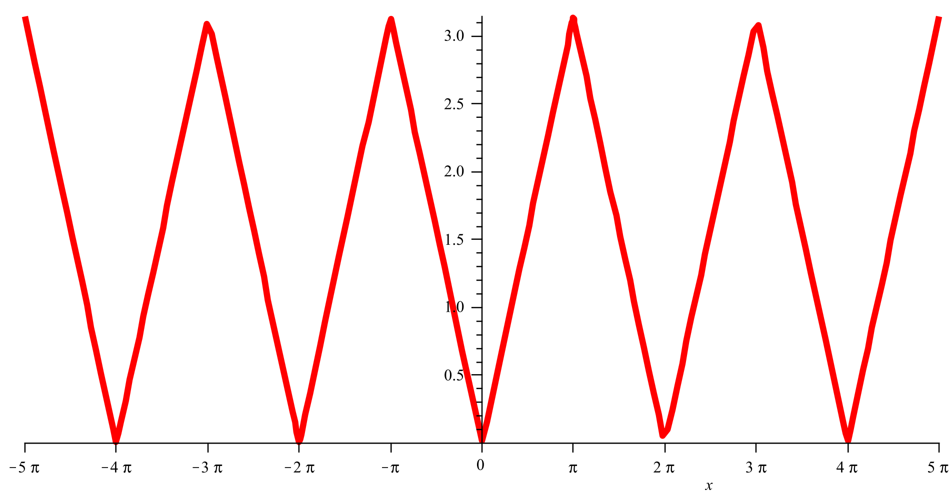
Here is F_8:



Example 3.3. Here's a triangular wave with period 2π:

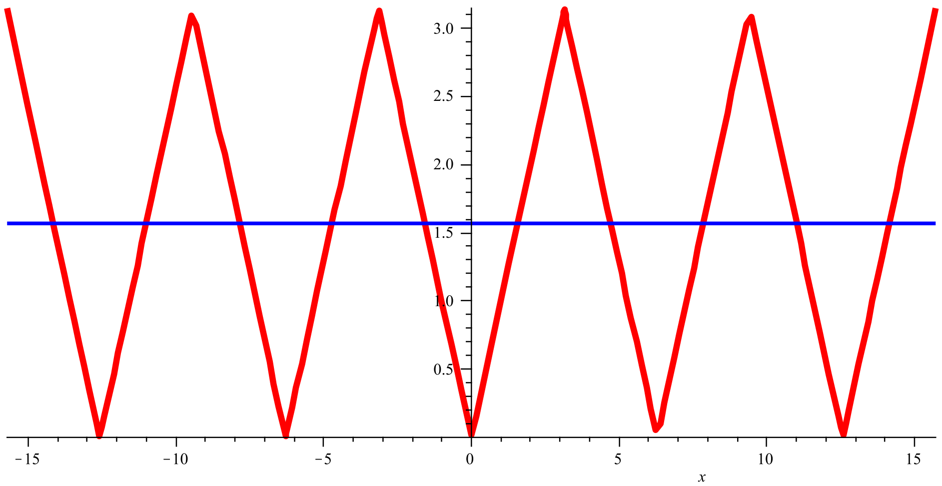
$$f(x) = |x| \qquad \text{for } -\pi \leq x \leq \pi$$

This wave is symmetric about the y-axis, so it is an even function, and $b_k = 0$ for all k. The Fourier Series will contain only cos terms. We simplify by integrating on the interval $[0, \pi]$ and doubling the result. On this interval $|x| = x$.



$$a_0 = \frac{2}{2\pi}\int_0^{\pi} f(x)\cos(0x)dx = \frac{2}{2\pi}\int_0^{\pi} x dx = \frac{2x^2}{4\pi}\Big|_0^{\pi} = \frac{\pi}{2}$$

$$F_0 = \frac{\pi}{2}$$

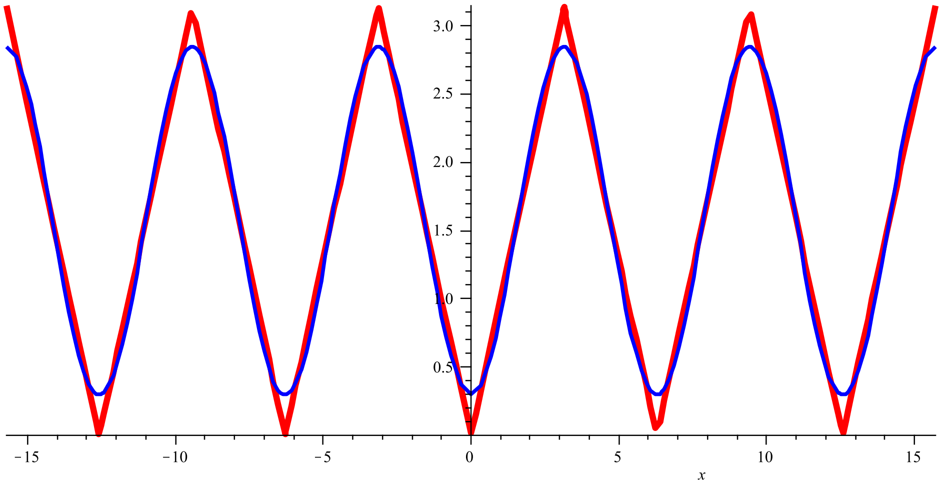


For $k > 0$

$$
\begin{aligned}
a_k &= \frac{2}{\pi}\int_0^\pi f(x)\cos(kx)dx \\
&= \frac{2}{\pi}\int_0^\pi x\cos(kx)dx \\
&= \frac{2}{k\pi}\int_0^\pi kx\cos(kx)dx \\
&\text{substitute } w = kx, \text{ so } dx = \frac{1}{k}dw \\
&= \frac{2}{k^2\pi}\int_0^{k\pi} w\cos(w)dw \\
&= \frac{2}{k^2\pi}\left(\cos(w) + w\sin(w)\right)\Big|_0^{k\pi} \\
&= \frac{2}{k^2\pi}\left((-1)^k - 1\right) \\
&= \begin{cases} -\frac{4}{k^2\pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}
\end{aligned}
$$

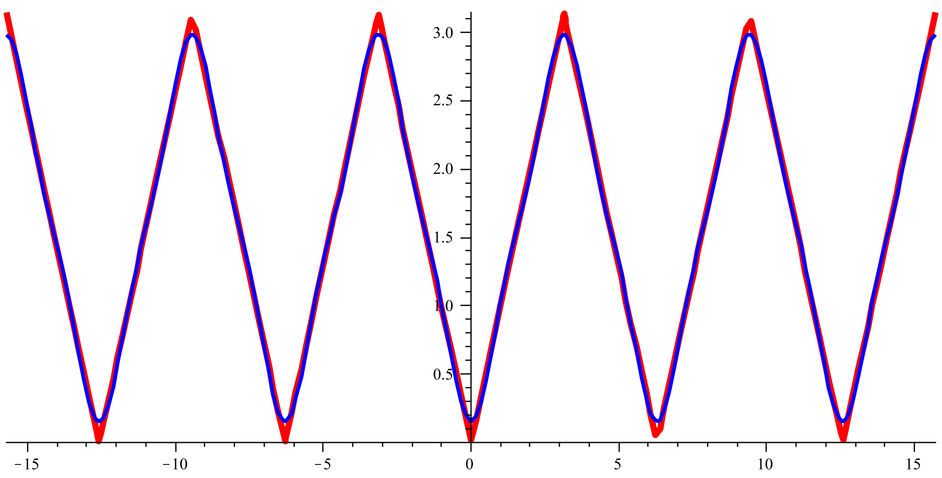
So $a_1 = -\frac{4}{\pi}$ and

$$F_1 = \frac{\pi}{2} - \frac{4}{\pi}\cos(x)$$



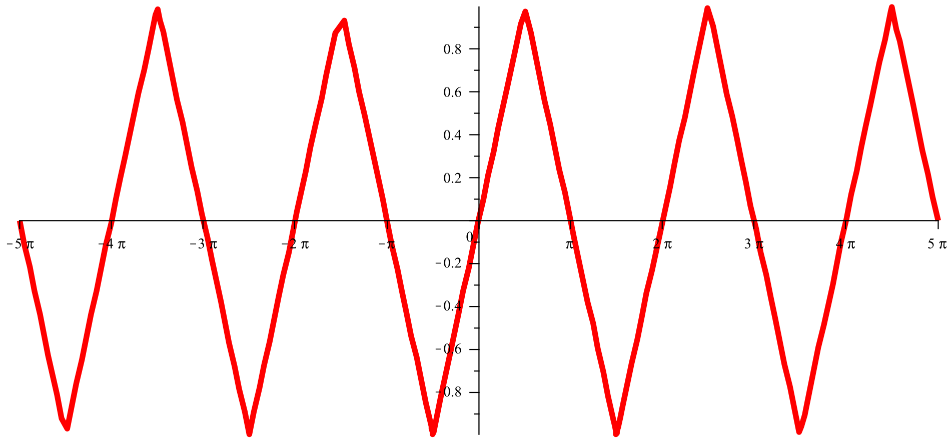
$a_2 = 0$ so $F_2 = F_1$.
$a_3 = -\frac{4}{9\pi}$ so

$$F_3 = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x)$$



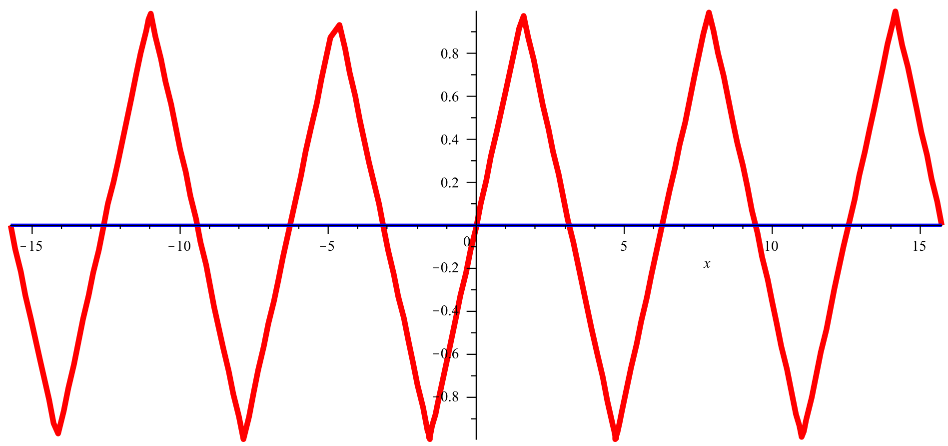
Example 3.4. Compute the Fourier Polynomials for the 2π-periodic triangular wave given by:

$$f(x) = \begin{cases} -\frac{2}{\pi}x - 2 & \text{for } -\pi \le x \le -\frac{\pi}{2} \\ \frac{2}{\pi}x & \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ -\frac{2}{\pi}x + 2 & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases}$$



Notice that $f(x)$ is an odd function, so the a_k are all zero, we need only compute the b_k. $f(x)\sin(kx)$ is even, so we compute the integral on the interval $[0, \pi]$ and double the result.

So $F_0 = a_0 = 0$.



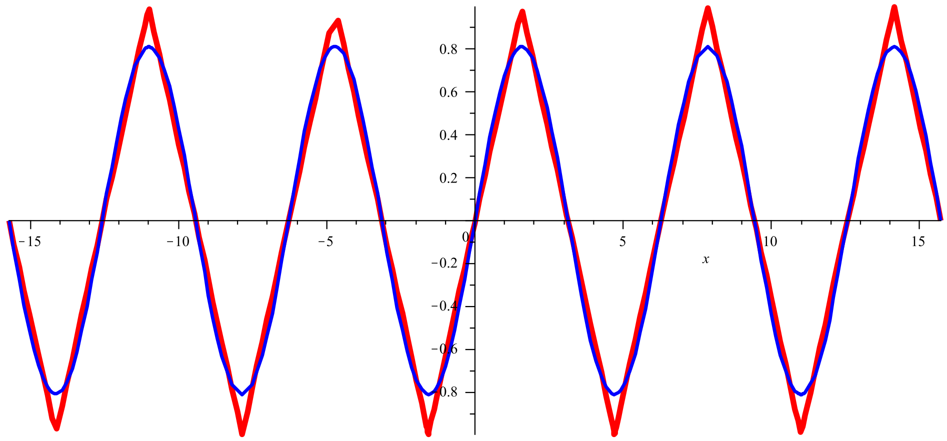
All other $a_k = 0$.

$$
\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)dx \\
&= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{2}{\pi}x\sin(kx)dx + \int_{\frac{\pi}{2}}^{\pi}\left(-\frac{2}{\pi}x+2\right)\sin(kx)dx\right) \\
&= \frac{2}{\pi}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(kx)dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} x\sin(kx)dx + 2\int_{\frac{\pi}{2}}^{\pi}\sin(kx)dx\right) \\
&\text{substitute } w = kx\text{, so } dx = \frac{1}{k}dw \\
&= \frac{2}{\pi}\left(\frac{2}{k^2\pi}\int_0^{\frac{k\pi}{2}} w\sin(w)dw - \frac{2}{k^2\pi}\int_{\frac{k\pi}{2}}^{k\pi} w\sin(w)dw + \frac{2}{k}\int_{\frac{k\pi}{2}}^{k\pi}\sin(w)dw\right) \\
&= \frac{2}{\pi}\left(\frac{2}{k^2\pi}\left(\sin(w)-w\cos(w)\right)\Big|_0^{\frac{k\pi}{2}} - \frac{2}{k^2\pi}\left(\sin(w)-w\cos(w)\right)\Big|_{\frac{k\pi}{2}}^{k\pi} - \frac{2}{k}\cos(w)\Big|_{\frac{k\pi}{2}}^{k\pi}\right) \\
&\text{the result of this computation will depend on whether } k \text{ is odd or even} \\
&\text{Assuming } k \text{ is odd} \\
&= \frac{2}{\pi}\left(\frac{2}{k^2\pi}\left((-1)^{\frac{k-1}{2}}\right) - \frac{2}{k^2\pi}\left(k\pi-(-1)^{\frac{k-1}{2}}\right) + \frac{2}{k}\right) \\
&= \frac{4}{k^2\pi^2}\left((-1)^{\frac{k-1}{2}} - k\pi + (-1)^{\frac{k-1}{2}} + k\pi\right) \\
&= (-1)^{\frac{k-1}{2}}\frac{8}{k^2\pi^2} \\
&\text{On the other hand, if } k \text{ is even} \\
&= \frac{2}{\pi}\left(\frac{2}{k^2\pi}\left(-\frac{k\pi}{2}(-1)^{\frac{k}{2}}\right) - \frac{2}{k^2\pi}\left(-k\pi+\frac{k\pi}{2}(-1)^{\frac{k}{2}}\right) - \frac{2}{k}\left(1-(-1)^{\frac{k}{2}}\right)\right) \\
&= \frac{4}{k^2\pi^2}\left(\left(-\frac{k\pi}{2}(-1)^{\frac{k}{2}}\right) - \left(-k\pi+\frac{k\pi}{2}(-1)^{\frac{k}{2}}\right) - k\pi\left(1-(-1)^{\frac{k}{2}}\right)\right) \\
&= 0
\end{aligned}
$$

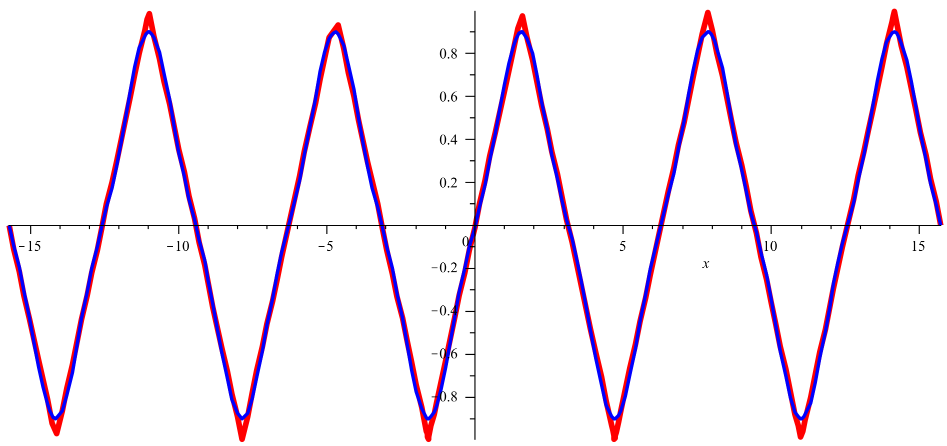
So we have found:

$$
b_k = \begin{cases} (-1)^{\frac{k-1}{2}}\frac{8}{k^2\pi^2} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}
$$

$$
F_1 = b_1\sin(x) = \frac{8}{\pi^2}\sin(x)
$$



$$F_3 = \frac{8}{\pi^2}\sin(x) - \frac{8}{9\pi^2}\sin(3x)$$



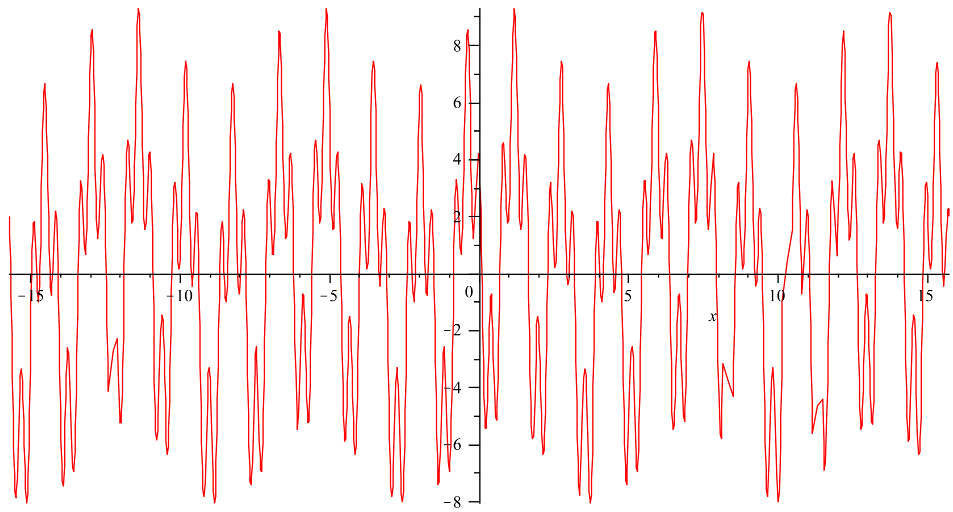
4. Harmonics and Energy

The *k-th harmonic* of a function $f(x)$ is the function $a_k \cos(kx) + b_k \sin(kx)$ from the Fourier Series of $f(x)$.

Example 4.1. Consider the function:

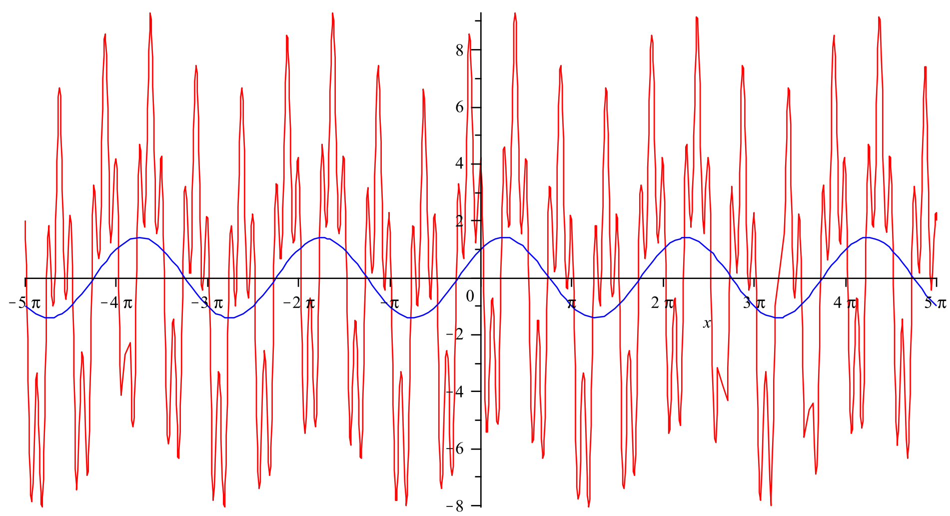
$$f(x) = \sin(x) + \cos(x) - 5\sin(4x) + 3\cos(16x)$$

Note that this is already a Fourier Series, so there is no calculation to do, just like a polynomial was its own Taylor Series.

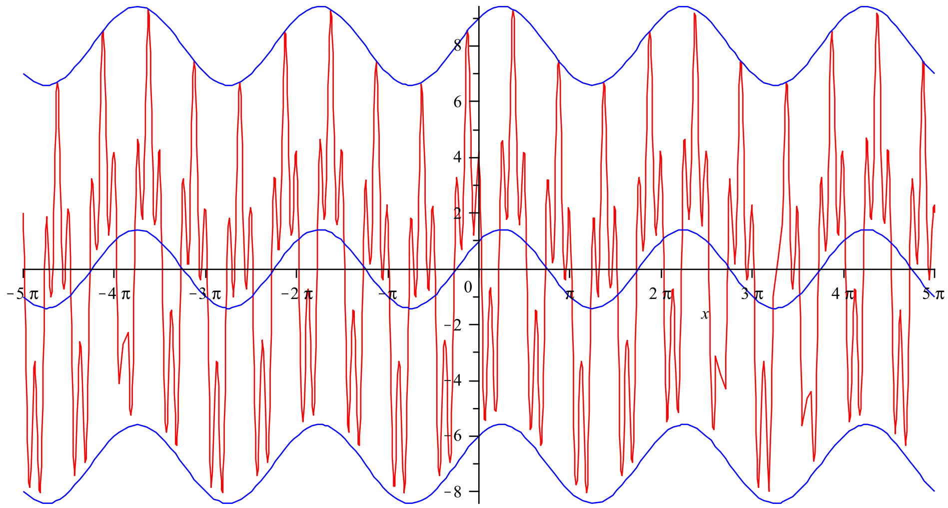


There are three harmonics at work here, and by plotting graphs of the harmonics together with graphs of the function we can see how each harmonic contributes to the overall picture.

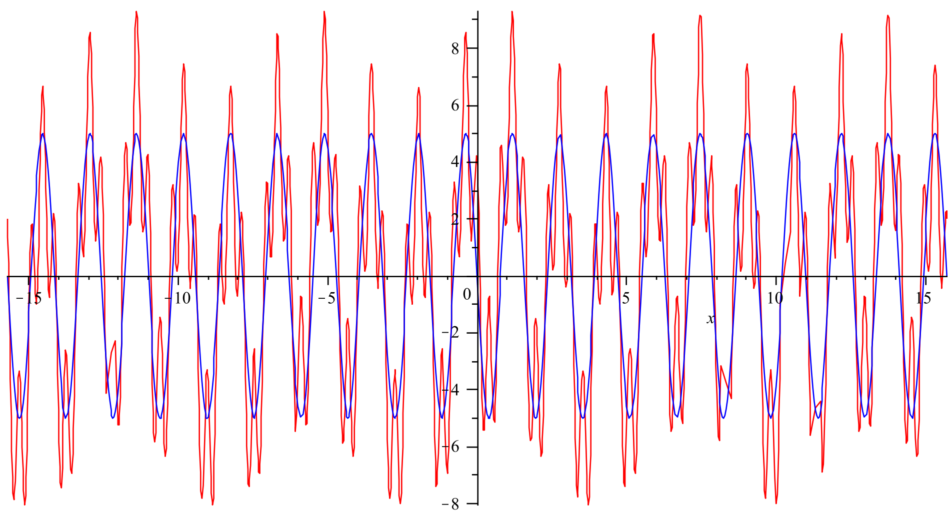
The first harmonic is $\sin(x) + \cos(x)$. Notice that this accounts for the lowest frequency shape of the graph.



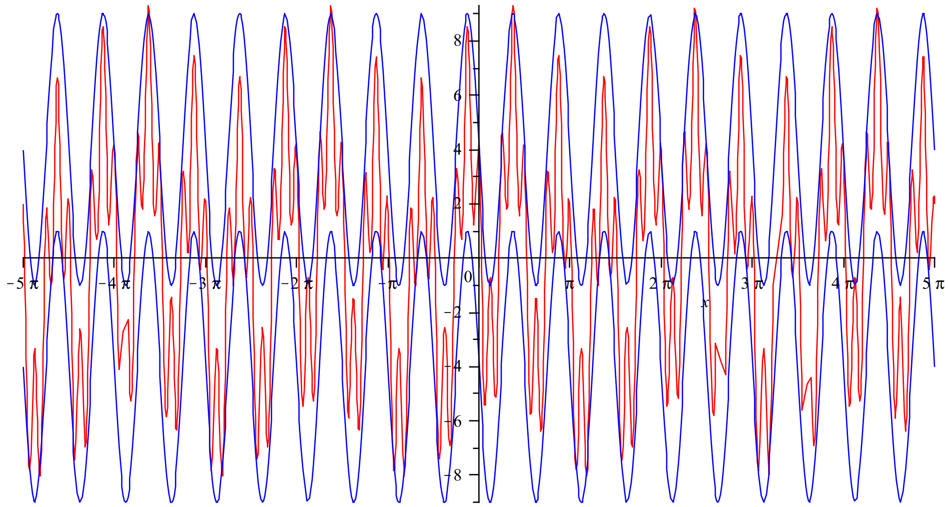
In fact, we make this relationship even more obvious by graphing two vertical translates of the first harmonic. Notice how the total function matches the contours of the first harmonic.



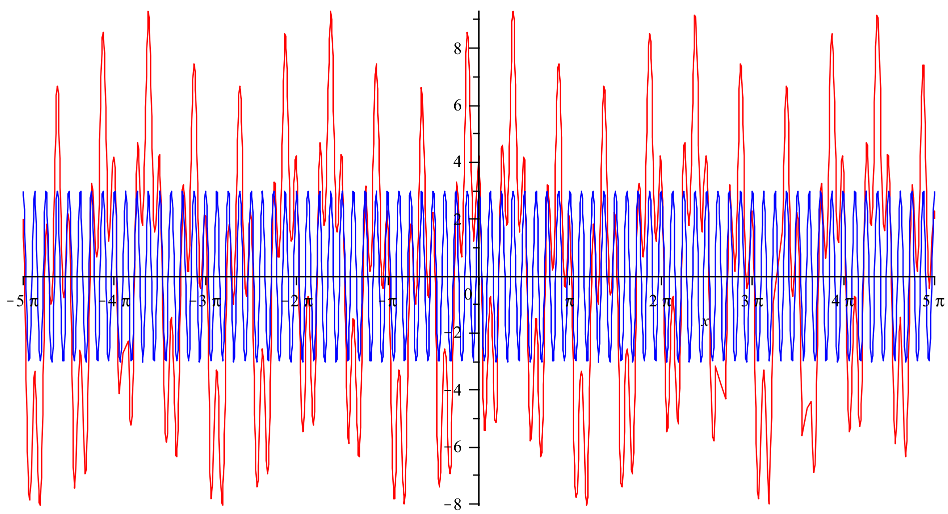
The fourth harmonic is $-5\sin(4x)$. This accounts for the intermediate frequency.



Again, we can make this relationship even more explicit by graphing two vertical translates of the fourth harmonic.



The only other non-zero harmonic is the 16th, $3\cos(16x)$. This accounts for the high frequency behavior of the graph.



The *energy* $E(f)$ of a 2π-periodic function $f(x)$ is defined to be:

$$E(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx$$

We can compute that the energy of the k-th harmonic is $a_k^2 + b_k^2$.

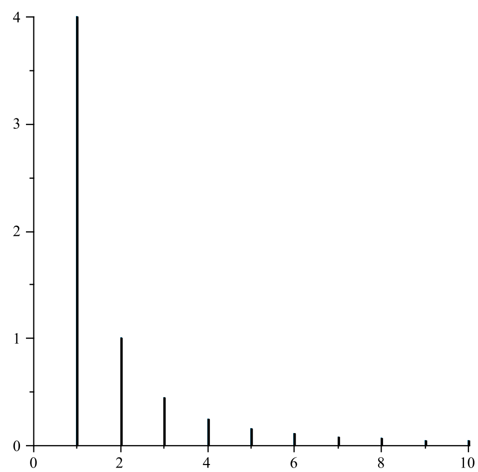
The Energy Theorem tells us that the energy of a 2π-periodic function is equal to the sum of the energies of the harmonics:

$$E(f) = a_0^2 + (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + \dots$$

If we graph the energies of the k-th harmonics vs. k we get the *Energy Spectrum* for $f(x)$.

The energy spectrum has application to sound. If the energy of a function is concentrated all in or around one value of k then the sound corresponding to that waveform would sound like a pure tone, like a generic, lifeless computer tone. If the energy is spread out over different harmonics the corresponding sound would seem richer and fuller. Different types of musical instruments have different characteristic energy spectra, and this accounts for the different sounds that the instruments make, even if they are all playing the same note. We will (hopefully) hear some examples in class.

Example 4.2. For the sawtooth wave example we calculated that $a_k = 0$ and $b_k = \frac{2}{k} \cdot (-1)^{k+1}$, so the k-th harmonic is $\frac{2}{k} \cdot (-1)^{k+1} sin(kx)$ and the energy of the k-th harmonic is $b_k^2 = \frac{4}{k^2}$. Then energy spectrum for this wave is:

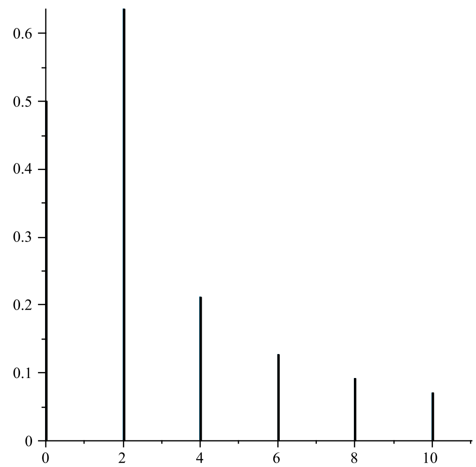


Example 4.3. For the square wave from the first example we calculated:

$$
\begin{aligned}
a_0 &= \frac{1}{2} \\
a_k &= 0 \quad \text{for } k > 0 \\
b_k &= 0 \quad \text{for } k \text{ even} \\
b_k &= \frac{2}{k\pi} \quad \text{for } k \text{ odd}
\end{aligned}
$$

So the 0-th harmonic is $\frac{1}{2}$ and all other even harmonics are 0. The odd harmonics are $\frac{2}{k\pi} \sin(kx)$.

The energy spectrum is:



5. Homework Problems

Exercise 1. *Show that the energy of the k-th harmonic* $a_k \cos(kx) + b_k \sin(kx)$ *is* $a_k^2 + b_k^2$.

Exercise 2. *For the function* $f(x) = 4\sin(2x) + 2\cos(8x)$ *sketch the function and its non-zero harmonics on the interval* $[-5\pi, 5\pi]$.

For the following 2π-periodic functions, sketch the wave on the interval $[-5\pi, 5\pi]$, compute the Fourier coefficients, sketch the third Fourier Polynomial (F_3), and sketch the energy spectrum up to $k = 3$.

Exercise 3.

$$f(x) = \begin{cases} 1 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ -1 & \text{for } -\pi < x < -\frac{\pi}{2} \text{ and } \frac{\pi}{2} < x < \pi \end{cases}$$

Exercise 4.

$$f(x) = x^2 \quad \text{for } -\pi < x < \pi$$

Exercise 5.

$$f(x) = \begin{cases} \frac{2}{\pi}x + 1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \frac{2}{\pi}x - 1 & \text{for } \frac{\pi}{2} < x < \pi \end{cases}$$