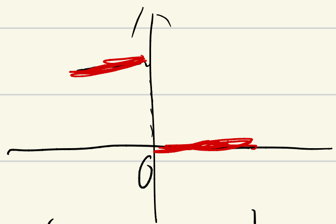
11.6

Fourier Series.

First go through the last Ex (Use the Taylor Polynomials to get the derivative at the center).

Sometimes, the Taylor series is not enough for us.

Ex1: 

discontinuous function

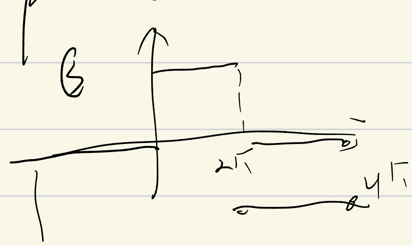
Ex2: $f(x) = e^{-\frac{1}{x^2}}$ $x \neq 0$ $\Longrightarrow$ In the problemset, we see

$f(x) = 0$ $x = 0$ $\underline{T(x) = 0}$!!! (derivatives at $x=0$)

not good.

So, sometimes, we want to use trig functions (sin, cos) to get a new series. The advantage of doing that we will see later.

First we should look at the periodic function. (Intuition: Maybe that the trig functions all have period)

Ex: ① $\sin(x)$, ② $\cos(x)$, ③ 

$\underline{f(x) = f(x+2\pi)}$ periodic function

Notice: the function are $\frac{2\pi}{n}$ periodic are also 2π periodic (所以)

$$f(x) = f(x+\frac{2\pi}{n}) = f(x+\frac{4\pi}{n}) \cdots = f(x+\frac{2n\pi}{n}) = f(x+2\pi).$$

For periodic function, we can consider only the periodical interval.

Now we can define the Fourier Series, for 2π-periodic function (or $[-\pi,\pi]$)

Defn: $F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \cos(nx)$ for $f(x)$

$f(x)$.

Notice, it's 2π-periodic

where: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

decompose periodic function to some trig series.

That make sense: at least for trig function:

check that $f(x) = \sin(4x)$ / $\cos(5x)$

$F(x) = f(x)$

Because: we know $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi\,\delta_{mn}$ $\Rightarrow$ $m=n$, $\delta_{mn}=1$; $m\neq n$, $\delta_{mn}=0$

$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\,\delta_{mn}$

$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$

$= 0$

$= 0$.

Ex: Cst fntn = Cnst, $\sin(nx)$, $\cos(nx)$ = itself.

Ex. 11:

$$f(x) = \begin{cases} -\frac{\pi}{2} & -\pi \le x < 0 \\ \frac{\pi}{2} & 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 1 \cdot -\frac{\pi}{2} + \int_0^{\pi} 1 \cdot \frac{\pi}{2} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{2}\cos(x)\,dx + \int_0^{\pi} \frac{\pi}{2}\cos(x)\,dx\right] \Rightarrow 0$$

cs odd fntn

Here actually we use different fntns.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin(nx)\,dx + \int_0^{\pi} \sin(nx)\,dx\right]$$

when n is even $\Rightarrow$ 0.

n is odd $\Rightarrow$ $a_n = \frac{1}{n}$

$$\therefore F(x) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin((2n-1)x)$$ (only cnt for the odd terms)

How cnt: $cf(x) \Rightarrow cF(x)$; $f(x)+1 \Rightarrow F(x)+1$.

linearity of Fourier series by linearity of integral and const mult.

Ex 2:

$$f(x) = \begin{cases} -x & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin(nx)\,dx = 0 \qquad \text{odd}$$

($f(x)$: even, $\sin(nx)$: odd)

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx\right)$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(mx)\,dx = \frac{2}{\pi}\int_0^{x} x\cdot d\left(\frac{\sin(mx)}{m}\right)$$

$$= \frac{2}{\pi m}\, x\cdot\sin(mx)\Big|_0^{\pi} - \frac{2}{\pi m}\int_0^{\pi}\sin(mx)\,dx = \frac{2}{\pi m^2}\cos(mx)\Big|_0^{\pi}$$

$m = \text{odd} \Rightarrow -\frac{4}{\pi m^2}$

$m = \text{even} \Rightarrow 0$.

$\therefore$ finally: $F(x) = \sum_{m=1}^{\infty} \frac{-4}{\pi(2m-1)^2}\cos((2m-1)x) + \frac{\pi}{2}$

(only odd term)

Notice: ① Fourier series use the global information (the integral)

Instead, Taylor series use local information ($f^{n}(x)$, derivative)

② Also, when $f(x)$ is odd ⇒ $f(x)\cdot\cos(nx)$ is odd ⇒ $a_n = 0$.

when $f(x)$ is even ⇒ $f(x)\cdot\sin(nx)$ is odd ⇒ $a_n = 0$.