11.04.

Find New Taylor Series from the old.

Recall the Taylor series you see in the lab:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

When $a=0$ Maclaurin Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty, +\infty)$$

$$\ln(1+x) = \sum (-1)^n \frac{x^n}{n} \qquad (-1, 1)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad (-\infty, \infty)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad (-\infty, \infty)$$

Ex:

$$f(x) = \sum_{n=1}^{\infty} c_n (x-a)^n + c_0$$

$$f(x^2) = ? \qquad \text{or} \qquad f(g(x)) = ?$$

$$\Rightarrow \sin(x^2) = ? \qquad \cos(x^2) = ?$$

$$\Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

Ex: $\frac{1}{1+x^2} = ?$

$$\frac{1}{1+x} = \sum_{n=1}^{\infty} (-1)^n x^n + 1$$

$$\frac{1}{1+x^2} = \sum_{n=1}^{\infty} (-1)^n x^{2n} + 1$$

How to do $x^k f(x) = ?$ $\Longrightarrow$ $x^k f(x) = \sum_{n=0}^{\infty} c_n x^{n+k} + c_0 x^k$

Ex: $\frac{x^3}{x+2} = ?$

$$\frac{1}{x+2} = \frac{1}{2} \cdot \frac{1}{\frac{x}{2}+1} = \frac{1}{2}\left(\sum_{n=1}^{\infty}\left(-\frac{x}{2}\right)^n\right)$$

$$\therefore \frac{x^3}{x+2} = \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n \cdot x^{n+3}$$

How to do differentiation and integration of Power Series?

Theorem: if the power series $\sum c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum c_n(x-a)^n$$

is differentiable and therefore continuous on the interval $(a-R, a+R)$ and

1. $f'(x) = c_1 + 2c_2(x-a) \cdots = \sum n c_n (x-a)^{n-1}$ (differentiable condition)

2. $\int f(x)\,dx = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{(n+1)}$ (integration condition)

The radius of the power series in Equation (1) and (2) are both R

$$\int c_n(x-a)^n dx = \frac{c_n(x-a)^{n+1}}{n+1}$$

$$(c_n(x-a)^n)' = c_n n \cdot (x-a)^{n-1}$$

Example: $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$\int \ln(1+x)\, dx = ?$

$$\Longrightarrow \quad = \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)n} \quad > \quad n=1.$$

Ex: $\int f(x) = \arctan(x)$? $\Longrightarrow$ hard to write the derivative directly

let $f'(x) = \frac{1}{1+x^2} = \sum_{n=1}^{\infty} (-1)^n x^{2n} + 1$

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + x + \underset{0}{\underset{\shortparallel}{C}} \quad \text{as} \quad \arctan(0) = 0.$$

Ex: $x^2 \cos(x) = ?$

$$\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + 1$$

$$x^2 \cos x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n)!} + x^2.$$