Review

1. Suppose the power series $\sum_{k=0}^{\infty} c_k(x+1)^k$ converges at $x = 2$ and diverges at $x = -8$.

   (a) Determine for which values of x this power series *must* converge, or state that we do not have enough information to do so.

   (b) Determine for which values of x this power series *must* diverge, or state that we do not have enough information to do so.

   (c) Find the following limits, or state that there is not enough information to do so.

      i. $\lim_{k\to\infty} c_k$

      ii. $\lim_{k\to\infty} (-3)^k c_k$

      iii. $\lim_{k\to\infty} 6^k c_k$

2. Suppose T_n is the nth degree Taylor polynomial of $f(x)$ and F_n is the nth partial sum of the Fourier series for $f(x)$.

   (a) When can we say $\lim_{n\to\infty} (f(x) - T_n(x)) = 0$?

   (b) When can we say $\lim_{n\to\infty} (f(x) - F_n(x)) = 0$?

3. Use the fact that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ to express the following power series as functions (assuming x is in the interval of convergence).

   (a) $\sum_{k=1}^{\infty} kx^{k-1}$

   (b) $\sum_{k=2}^{\infty} k(k-1)x^{k-2}$

   (c) $\sum_{k=1}^{\infty} kx^k$

   (d) $\sum_{k=1}^{\infty} x^k$

4. Find the sum of the following series:

   (a) $\ln(4) + \frac{(\ln(4))^2}{2!} + \frac{(\ln(4))^3}{3!} + \frac{(\ln(4))^4}{4!} + \cdots$

   (b) $2 - \frac{2^3}{3} + \frac{2^5}{5} - \frac{2^7}{7} + \cdots$

   (c) $1 - 0.5 + \frac{(0.5)^2}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^4}{4} - \cdots$

5. Find the following limits:

(a) $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1}(2k+1)!}$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \sin\left(\pi + \frac{k\pi}{n}\right)\frac{\pi}{n}$

(c) $\lim_{n\to\infty} \sum_{k=0}^{n} \left(\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+2}\right)\right)$

6. Let $f(x) = xe^{-x}$.

   (a) Find the second degree Taylor polynomial for $f(x)$, centered at $x = 4$.

   (b) Find $f^{(20)}(0)$

7. Consider the differential equations

   - $\frac{dy}{dx} = (y+2)(x^2+y^2)$
   - $\frac{dy}{dx} = \frac{1}{2}(y^2-1)$

   (a) Find the equilibrium solutions of each differential equation.

   (b) Solve the differential equation that is separable.