Math 122L Weekly Graded Homework

Homework 6 (100 pts)

To get full credit for a question, clear explanation is necessary. Only submission in paper is accepted.

1. (15 pts) Suppose that we flip a die until we get either a 5 or 6. Let X represent the time of flipping when we stop.

   (a) What is the range of X?

   (b) Find the probability of $\mathbb{P}(X=1), \mathbb{P}(X=2), \mathbb{P}(X=3)$.

   (c) Find $\mathbb{E}X$.

2. (20 pts) Determine whether the following sequence converges. Find the limit if it converges and explain the reason if it diverges.

   (a) $a_n = \frac{1}{n^3}, n = 1, 2, ...;$

   (b) $a_n = \frac{\ln n}{n^2}, n = 1, 2, ...;$

   (c) $a_n = \frac{\cos n}{\sqrt{n}}, n = 1, 2, ...;$

   (d) $a_n = \sqrt[n]{2}, n = 1, 2, ...$

3. (15 pts) Determine whether the series converges or diverges.

   (a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2};$

   (b) $\sum_{n=1}^{\infty} \frac{1}{|\cos n|\sqrt{n}};$

   (c) $\sum_{n=1}^{\infty} e^{-n} n^{2019};$

4. (10 pts) Recall that L'Hospital's Rule helps compute indeterminant form of $0/0$ or ∞/∞. The counterpart of L'Hospital's Rule for sequences is called Stolz theorem:

Theorem 1. *(Stolz, 0/0 type) Suppose that* $\{a_n\}, \{b_n\}$ *converges to 0 and decreasing, and limit*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$$

exists. Then we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

Theorem 2. *(Stolz, */∞ type) Suppose that* $\{b_n\}$ *is increasing and approaches* $+\infty$ *as* $n \to \infty$*, and limit*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$$

exists. Then we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

Now use Stolz theorem to compute the following limits:

(a) $\lim_{n\to\infty} \frac{1! + 2! + ... + n!}{n!}$;

(b) $\lim_{n\to\infty} \frac{n}{2^n}$.

5. (25 pts) Suppose that sequence a_n satisfies $a_1 = \frac{1}{2}$, $a_{n+1} = a_n - a_n^2$.

   (a) Compute a_2, a_3, a_4.

   (b) Prove that $0 < a_n < 1$ for any $n = 1, 2, ...$ (Hint: assume $0 < a_n < 1$, is it ture that $0 < a_{n+1} < 1$?)

   (c) Prove that $\lim_{n\to\infty} a_n$ exists. (Hint: use monotonic sequence theorem)

   (d) Find $\lim_{n\to\infty} a_n$.

   (e) Find $\lim_{n\to\infty} na_n$. (Hint: use Stolz theorem)

6. (15 pts) Determine whether the following improper integrals converges.

   (a) $\int_0^1 \frac{\ln x}{\sqrt{x}} \mathrm{d}x$;

   (b) $\int_0^{\pi/4} \frac{1}{\sqrt{\sin x}} \mathrm{d}x$; (Hint: what is $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$?)

   (c) $\int_1^{\infty} e^{-x^2} x^2 \mathrm{d}x$.