10.1:

A sequence is a list of numbers written in definite order.

a_n is also the nth term of the sequence.

Ex: Find formula for a_n: $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \left(1-\frac{1}{n}\right)\right)$

Ex: Find a formula for the general term:

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots \right. \quad \frac{n+2(-1)^n}{5^n}$$

Convergent / divergent: If $\lim a_n = c$, we say the sequence convergent.

otherwise, it's divergent.

limit's property:

$\lim(a_n \pm b_n) = \lim a_n \pm \lim b_n$

$\lim\limits_{n\to\infty} c\,a_n = c \lim\limits_{n\to\infty} a_n$

$\lim(a_n b_n) = (\lim a_n)(\lim b_n)$

$\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ if $\lim b_n \neq 0$

$\lim a_n^p = (\lim a_n)^p$ if $p>0$, $a_n > 0$

The squeeze (sandwich Theorem) If $a_n \le c_n \le b_n$, for $n \ge n_0$, and $\lim a_n = \lim b_n = L$,

then $\lim_{n\to\infty} c_n = L$.

Absolute value Thm: If $\lim |a_n| = 0$, then $\lim a_n = 0$

Ex: $\left\{\frac{(-1)^n}{n}\right\} \Rightarrow$ convergent.

Thm: If $a_n \to L$, and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Ex: Does the sequence $\left\{\frac{n!}{n^n}\right\}$ converge?

$$\Rightarrow \left|\frac{n!}{n^n}\right| \le \frac{1}{n}$$

Ex: For what values of r is $\{r^n\}$ convergent?

Thm: Converge of a geometric sequence if and only if $-1 < r \le 1$, divergent for any other r.

Increasing: $a_{n+1} > a_n$, decreasing: $a_{n+1} < a_n$, for all n.

Monotonic: either decreasing or increasing.

Ie: $|c_n| \le M$. $c_n \ge M$ for each n, $c_n \le M$ for all n

Monotone sequence thm: Every bdd / monotone seq $\Rightarrow$ convgt

A series is the sum of the terms: $\sum_{n=1}^{\infty} c_n$.

The n^{th} partial sum is: $\sum_{k=1}^{n} a_k$

If the sequence (s_n) of part sum convgt and $\lim s_n = s$ exist which number, $\Rightarrow$ $\sum c_n$ is called convgt. and we write

$$\sum c_n = s.$$

s is the the limit of the series. If (s_n) is divgt $\Rightarrow$ series divgt.

Ex: telescope series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) \Rightarrow \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1-\frac{1}{k+1} \rightarrow \underset{\text{convgt}}{1}$$

$$\sum_{k=1}^{\infty}(\ln(k+1)-\ln(k)) \Rightarrow \sum_{k=1}^{n}\ln(k+1)-\ln k = \ln(n+1) \rightarrow \infty$$

Ex: $\sum_{k=1}^{\infty}\frac{k}{k+1}$ $\Rightarrow$ Hard to write the partial sum.

$\sum\left(1-\frac{1}{k+1}\right)$

How to tell whether it's divergent or convergent?

Talk about it later (Tomorrow

18.02

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

(Intuition: As $\sum_{n=1}^{n} a_n$ exist, the latter terms should be small.

Important: The inverse is not true: $\lim a_n = 0$, $\sum a_n$ might be divergent?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ (A very tricky approach in Textbook)

If you're interested in it, read it. Or, we'll use integral test to prove it generally.

The test for divergent: $\lim a_n \neq 0$ or does not have a limit:

contrapositive: if convergent $\Longrightarrow$ contradiction $\Longrightarrow$ divergent.

Ex: $\sum_{n=1}^{\infty} \frac{n}{n+1}$, $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

A very important example:

geometric series:

$$a_n = a_{n-1} \cdot r \quad \text{or} \quad \boxed{a_n = a_1 \cdot r^{n-1}}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a r^{n-1}$$

partial sum: $S_n = \sum_{k=1}^{n} a_k$

How to find it?

when $r = 1 \implies S_n = n$.

when $r \neq 1$:

$$S_n = a + ar + ar^2 \cdots ar^{n-1}$$

$$rS_n = ar + \cdots \quad ar^n$$

$$(1-r) S_n = a - ar^n$$

$$\implies S_n = \frac{a(1-r^n)}{1-r}$$

by definition when $|r| < 1 \implies \lim S_n$ exist $\implies S_n$ conv.

Also: $\sum_{k=m}^{n-1} r^k = r^m \sum_{k=m}^{n-1} r^{k-m}$ let $(k-m)=s$

$= r^m \sum_{k-m=0}^{n-1-m} r^{(k-m)} = r^m \frac{1-r^{n-m}}{1-r}$ (of finite value)

$b_1 = a_m$

$s_k = c_{m+k}$

$\Rightarrow \sum_{k=m}^{\infty} r^k = r^m \sum r^k = \frac{r^m}{1-r}$ when $|r|<1$

Hence the probability of flip coin / toll a die:

$P(X=n) = (\frac{1}{2})^n \Rightarrow$ finite

$\sum_{n=1}^{\infty} P(X=n) = \sum (\frac{1}{2})^n = \frac{1}{2} \cdot \frac{1-\frac{1}{2}^n}{1-\frac{1}{2}} = 1 \Rightarrow$ is a prob

$P(X=n) = \frac{1}{6}(\frac{5}{6})^{n-1} \Rightarrow$ geometric series:

$a_1 = \frac{1}{6}$, $c_2 = \frac{1}{6} \cdot \frac{5}{6}$

$\therefore P(X=n) = \sum_{n=0}^{\infty} \frac{1}{6}(\frac{5}{6})^n = \lim_{n\to\infty} \frac{1}{6} \cdot \frac{1-(\frac{5}{6})^n}{1-\frac{5}{6}} = \frac{1}{6} \cdot 6 \cdot (1-0)$

$= 1 \Rightarrow$ geometric series well prob sp.

Sometimes, the geometric example is not

Ex: $\sum 2^{2n} 5^{1-n}$?

$\sum 2^{2n} 5^{1-n} = 5 \cdot 4^n \cdot 5^{-n} = \sum 5 \cdot (\frac{4}{5})^n$

converge / diverge ?

Ex 2: $5 - \frac{5}{3} + \frac{5}{9} - \frac{5}{27} + \cdots$