# Power Series

10.29.

Recall the Ratio Test. It's important tool to tell whether the infinite sum is convergent.

(Have seen some example from Midterm 2)

A power series is a series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where $x$ is a variable and $c_n$'s are constants called the coefficients.

If we chose some value of $x$ plug in, the power series then become a series of constants that we can test for converge or divergence.

Note: power series may converge for some values of $x$ and diverge for other values.

Ex 1 $\quad f(x) = \sum_{x=0}^{\infty} (-1)^n x^n \qquad -1 \le x < 1$, converge.

Ex 2 $\quad f(x) = \sum_{n=1}^{\infty} (x-1)^n$

Notice that sometimes we can have $a=0$, then we have the form $f(x) = \sum_{n=0}^{\infty} c_n x^n$

Ex3: For which value of $x$ is the series ( Textbook P553 )

$$\sum n!\,x^n$$

① Ratio test Usage, For $\lim |x|>0$

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{c_{n+1}}{c_n}\right| = \lim (n+1)|x| = \infty$$

② $x=0$, converge

∴ it's only converge at $x=0$ !

( Textbook P553 )

Ex4: $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

Let $c_n = (x-3)^n/n$. then

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{1}{1+\frac{1}{n}}(x-3)\right| \to |x-3| \quad \text{as } n \to \infty.$$

∴ $|x-3|<1$, converge

$|x-3|>1$, diverge

$|x-3|=1 \Rightarrow x=2$ or $4$

① $2 \Rightarrow \sum \frac{(-1)^n}{n}$ converge

② $4 \Rightarrow \sum \frac{1}{n}$ diverge

Not power series: $\sum \frac{\sin(nx)}{n!}$

Also, note if we have $\sum_{n=0}^{\infty} x^n$, the geometric series, but power series, but very special.

Ex 6: Find the domain of the function:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(-1)^{n+1} x^{2n+2}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \text{ for all } x.$$

$\therefore$ converge for all $x$!

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities.

1. The series converges only when $x = a$.
2. The series converges for all $x$.
3. There is a positive number $R$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number $R$ in the third case in the theorem above is called the radius of convergence of the power series.

In case 1) above, we say that $R=0$; in case 2, we say that $R=\infty$.

The interval of convergence of a power series is the interval that consists of all values of $x$ for which the series converges.

In case 1: only a single pt a.

In case 2: $(-\infty, \infty)$.

In case 3:

$(a-R, a+R)$, $(a+R, a+R]$, $[a-R, a+R)$, $[a-R, a+R]$.

Example: Find the radius of convergence and interval of convergence of the series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = 3\sqrt{\frac{1+\frac{1}{n}}{1+(\frac{2}{n})}}\,|x| \to 3|x|$$

$\therefore |x| < \frac{1}{3}$, conv.

① $x = \frac{1}{3}$, $= \sum \frac{(-1)^n}{\sqrt{n+1}}$ conv.

② $x = -\frac{1}{3}$ $= \sum \frac{1^n}{\sqrt{n+1}}$ diverge

$\therefore$ interval $[-\frac{1}{3}, \frac{1}{3}]$

Ex: $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Let $a_n = n(x+2)^n/3^{n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \quad \text{as } n\to\infty.$$

$\therefore$ $|x+2|<3 \Rightarrow -5<x<1$.

At $x=-5$, $\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum (-1)^n n$.

diverges.

$x=1$, $\sum_{n=0}^{\infty} \frac{n3^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$. diverge

$\therefore$ interval of convergence is

$(-5, 1)$

Think about: $\sum c_n x^n$ converge at $x=2$, diverge at $x=10$.

What can we tell from these conditions?