Problem Set #6

Submit your answers on a separate sheet on the day assigned by your instructor. You must explain all answers. You may (and, in fact, are encouraged) to work with your peers on problem sets, but your final answers must be your own. Please write an acknowledgement for anyone who provides you help.

1. Use Integral Test to determine whether the following series are convergent or divergent.
   (a) $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$
   (b) $\sum_{k=1}^{\infty} k^2 e^{-k^3}$
   (c) $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$
2. (a) Use Comparison Test to determine whether the series is convergent or divergent.
      i. $\sum_{n=2}^{\infty} \frac{n^3}{n^4+1}$
      ii. $\sum_{n=0}^{\infty} \frac{1+\cos n}{10^n}$
      iii. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

   (b) Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$.
      i. Use the Comparison Test to show that this series converges.
      ii. Note that this series satisfies the conditions of the Integral Test. Thus, we can use the associated error bounds to say that if we approximate $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$ by its 10th partial sum, the resulting error is bounded above by $\int_{10}^{\infty} \frac{1}{x^3+1}\, dx$. The value of this integral is difficult to find but we know that it is bounded above by $\int_{10}^{\infty} \frac{1}{x^3}\, dx$. Use this to find an upper bound of the error.

   (c) Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1+a_n)$ is convergent.
3. Determine whether the following series is convergent or divergent. If convergent, state the type of convergence (conditional convergence or absolute convergence). Explain your answer completely.
   (a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$
   (b) $\sum_{k=1}^{\infty} \frac{\cos(k)\sin(k)}{k^2}$
   (c) $\sum_{k=1}^{\infty} \frac{\pi + 3\sin k}{3\sqrt{k}}$
   (d) $\sum_{k=1}^{\infty} \frac{(-1)^k \ln k}{k}$
4. Use Ratio test to determine whether the series is convergent or divergent.
   (a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^5}{4^n}$

(b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ where k is a integer and $k \geq 2$

5. Consider the series $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$. We will approximate the value of this series using partial sums $s_n = \sum_{k=1}^{n} \frac{1}{1+k^2}$.

(a) Use Integral Test to show the series converges.

(b) Compute s_{15}. (You may want to use a computer or a calculator for this step)

(c) Find an upper bound on the error for your approximation in part b).

(d) Find a lower bound on the error for your approximation in part b).

(e) Use parts c) and d) to write the value of the series $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$, accurate to two decimal places.