Problem Set #5 Solutions

1. (a) We win \$2 when exactly one coin comes up heads. There are eight possible outcomes overall, three of which (HTT, THT, TTH) have one head. Therefore the probability that $X = 2$ is $\frac{3}{8}$.

   (b) Similarly to part a), we see that $P(X=1) = \frac{1}{8}, P(X=2) = \frac{3}{8}, P(X=4) = \frac{3}{8}, P(X=8) = \frac{1}{8}$. Therefore,

$$E(X) = 1p(1) + 2p(2) + 4p(4) + 8p(8) = \frac{1}{8} + \frac{2\cdot 3}{8} + \frac{4\cdot 3}{8} + \frac{8}{8} = \frac{27}{8}.$$

   (c) If a coin has probability p of coming up heads, then the probability of coming up tails is $(1-p)$. Then the probability of flipping three tails is $(1-p)^3$, the probability of flipping one head and two tails is $3p(1-p)^2$. (Note that the factor of 3 is to account for the fact that each of the three coins coming up heads is counted separately). The probability of flipping two heads is $3p^2(1-p)$. The probability of flipping three heads is p^3.

   Therefore the expected value is

$$\begin{aligned} E(X) &= 1P(X=1) + 2P(X=2) + 4P(X=4) + 8P(X=8) \\ &= (1-p)^3 + 2\cdot 3p(1-p)^2 + 4\cdot 3p^2(1-p) + 8p^3 \\ &= ((1-p)+2p)^3 \\ &= (1+p)^3 \end{aligned}$$

   (Note: you can also expand the terms and then factor to find the above).

   Since the expected value is $\frac{64}{27}$, we set $(1+p)^3 = \frac{64}{27}$, and thus $(1+p) = \frac{4}{3}$. Therefore $p = \frac{1}{3}$.

2. (a) Determine whether the sequence converges or diverges. If it converges, find the limit.

   i. $a_n = \dfrac{(-1)^{n-1}n}{n^2+1}$

   Solution $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \dfrac{n}{n^2+1} = 0$. Thus $\lim_{n\to\infty} a_n = 0$

   ii. $a_n = \left(1 + \dfrac{2}{n}\right)^n$

   Solution $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\dfrac{2}{n}\right)^n = e^2$ by setting $y = \left(1+\frac{2}{x}\right)^x$ and compute the limit of $\ln y$.

   (b) Determine whether the series converges or diverges. If it converges, find the limit.

   i. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$

   Solution $\lim_{n\to\infty} a_n = 0$. Inclusive. $S_n = \displaystyle\sum_{i=2}^{n} \frac{1}{i(i+2)} = \sum_{i=2}^{n} \frac{1}{2}\left(\frac{1}{i} - \frac{1}{i+2}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2}\right)$. Thus $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

   ii. $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

   Solution $\lim_{n\to\infty} a_n = 0$. Inclusive. $S_n = \displaystyle\sum_{i=1}^{n} \ln\frac{i}{i+1} = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \cdots + \ln\frac{n}{n+1} = \ln\frac{1}{n+1}$.

   Thus $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\frac{1}{n+1} = -\infty$. Therefore the series diverges.

   (c) If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = \frac{n-1}{n+1}$$

   find a_n and $\sum_{n=1}^{\infty} a_n$.

Solution $a_n = S_n - S_{n-1} = \dfrac{n-1}{n+1} - \dfrac{n-2}{n} = \dfrac{2}{n(n+1)}$.

$\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 1.$

(d) If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = S_{n-1} + \cos(S_{n-1}) \qquad S_1 = 1$$

Suppose this series converges to a finite number, L where $0 < L < 4$. Find $\displaystyle\lim_{n\to\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$.

Solution $\displaystyle\lim_{n\to\infty} a_n = 0$ since the series is convergent.
Since the series converges to L, we have $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$. Taking the limit on both sides of the equation above, we get $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} \cos(S_{n-1})$ which gives us $L = L + \cos(L)$. Thus $L = \dfrac{\pi}{2} = \displaystyle\sum_{n=1}^{\infty} a_n$.

3. (a) Notice that this is a geometric series with $a = -\dfrac{1}{3}$ and $r = -\dfrac{1}{3}$, so the series is convergent, and the sum is $\dfrac{a}{1-r} = -\dfrac{1}{3} \cdot \dfrac{1}{1 - \dfrac{-1}{3}} = -\dfrac{1}{4}$.

(b) Consider function $f(x) = \left(1 + \dfrac{2}{x}\right)^x$, we have

$$\ln f(x) = x \ln\left(1 + \frac{2}{x}\right)$$

and by L'Hospital's Rule, we have

$$\begin{aligned}\lim_{x\to+\infty} \ln f(x) &= \lim_{x\to+\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x\to+\infty} \frac{\frac{x}{x+2}\cdot\frac{-2}{x^2}}{-\frac{1}{x^2}} \\ &= 2.\end{aligned}$$

Thus the limit of $a_k = (1 + 2/k)^k$ should be e^2, and by test for divergence, the series is divergent.

(c) Notice that $\displaystyle\lim_{k\to+\infty} \frac{k}{\sqrt{k^2+4}} = \lim_{k\to+\infty} \frac{1}{\sqrt{1+4/k^2}} = 1$, so by test for divergence, the series is divergent.

(d) Because $2 > 1$, so for any $k > 0$, $2^{1/k} > 1$, and it can not have a limit 0. So by test for divergence, the series is divergent.
In fact, the sequence does have a limit of 1. Suppose that $2^{1/k} = 1 + a_k$, we only need to prove that a_k has a limit 0. We have

$$\begin{aligned}2 = (1 + a_k)^k &= 1 + ka_k + \frac{k(k-1)}{2} a_k^2 + ... \\ &> 1 + ka_k.\end{aligned}$$

Thus $a_k < 1/k$, so it has a limit 0 by squeeze theorem.

4. (a) We stop only we get 1 or 3. So we rolls two times means until the last time, we don't get 1 or 3. Since each roll is independent we know that

$$P(X = 2) = \frac{2}{6} * \frac{4}{6} = \frac{2}{9}$$

(b) We don't get 1 and 3 until the kth roll, so we know that

$$P(X = k) = \frac{2}{6} * (\frac{4}{6})^{k-1}$$

(c) There are two way to compute the probability:

(1)use the geometric series $P(X \geq 3) = \sum_{k=3}^{\infty} P(X = k) = \sum_{k=3}^{\infty} \frac{2}{6} * (\frac{4}{6})^{k-1} = \frac{4}{9}$

(2)Notice that $P(X \geq 3) = 1 - P(X = 1) - P(X = 2) = 1 - \frac{1}{3} - \frac{2}{9} = \frac{4}{9}$

5. (a) We may use telescope series like $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$

(b) We may use two convergent geometic series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$ and $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2^k}$, $\sum_{k=1}^{\infty} \frac{a_k}{b_k} = \sum_{k=1}^{\infty} 1$ is divergent.