Problem Set #5

Submit your answers on a separate sheet on the day assigned by your instructor. You must explain all answers. You may (and, in fact, are encouraged) to work with your peers on problem sets, but your final answers must be your own. Please write an acknowledgement for anyone who provides you help.

1. Suppose you play a game where you flip three fair coins. You win $\$2^k$, where k denotes the number of heads that come up.
   (a) What is the probability that we win \$2?
   (b) How much money would you be willing to spend to play this game? That is, what are your expected winnings?
   (c) Suppose instead that the coins are not fair, but they each have the same probability p of coming up heads. Suppose moreover that the expected winnings are $\$\frac{64}{27}$. Find the value of p.

2. (a) Determine whether the sequence converges or diverges. If it converges, find the limit.
      i. $a_n = \dfrac{(-1)^{n-1}n}{n^2+1}$
      ii. $a_n = \left(1+\dfrac{2}{n}\right)^n$
   (b) Determine whether the series converges or diverges. If it converges, find the limit.
      i. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$
      ii. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$
   (c) If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = \frac{n-1}{n+1}$$

   find a_n and $\sum_{n=1}^{\infty} a_n$.
   (d) If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = S_{n-1} + \cos(S_{n-1}) \qquad S_1 = 1$$

   Suppose this series converges to a finite number, L where $0 < L < 4$. Find $\displaystyle\lim_{n\to\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$.

3. Determine whether the series converges or diverges. If it converges, find the limit.
   (a) $\displaystyle\sum_{k=1}^{\infty} \left(-\frac{1}{3}\right)^k$;
   (b) $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{2}{k}\right)^k$;
   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+4}}$;
   (d) $\displaystyle\sum_{k=1}^{\infty} 2^{1/k}$.

4. Consider we roll a fair die. We keep rolling it until we get 1 or 3. The times we roll before we stop is a random variable, X.
   (a) What's the probability of that we roll 2 times before stopping? Compute the $P(X=2)$.
   (b) What's the probability of that we roll k times before stopping? Compute the $P(X=k)$.

(c) What's the probability of that we roll more than 3 before stopping? Compute the $P(X \geq 3)$.

5. (a) Find two divergent series, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $\sum_{k=1}^{\infty}(a_k + b_k)$ is convergent. (Hint: you may consider telescope type series)

   (b) Find two convergent series, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $\sum_{k=1}^{\infty} \frac{a_k}{b_k}$ is divergent. (Hint: you may consider two geometric series.)