Problem Set #4 Solutions

1. (a) i. Using integration by parts with $u = \ln|x|$, $dv = x$, and so $du = \frac{1}{x} dx$ and $v = \frac{x^2}{2}$, we have

$$\int x \ln|x| \, dx = \frac{x^2 \ln|x|}{2} - \int \frac{x^2}{2x} \, dx$$

$$= \frac{x^2 \ln|x|}{2} - \frac{x^2}{4} + C$$

ii. Using integration by parts with $u = \arctan x$, $dv = dx$, and so $du = \frac{1}{1+x^2} \, dx$ and $v = x$, we have

$$\int \arctan x \, dx = x \arctan x - \int \frac{1}{2} \, du = x \arctan x - \frac{\ln|u|}{2} + C$$

$$= x \arctan x - \frac{\ln(1+x^2)}{2} + C.$$  

(b) Using Integration by parts with $u = \cos^{n-1}x$ and $dv = \cos x \, dx$, we get that

$$\int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int (1-\cos^2 x) \cos^{n-2} x \, dx.$$  

Adding $\int \cos^{n-2} x \, dx$ to both sides, we get

$$n \int \cos^n x \, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx.$$  

Dividing by $n$ on both sides gives the desired result.

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

(a) $\int_0^\infty \frac{x}{x^3 + 1} \, dx$

Solution $\frac{x}{x^3 + 1} \leq \frac{x}{x^3} \leq \frac{1}{x^2}$ for $x \geq 0$. By $p$-test we know that $\int_1^\infty \frac{1}{x^4} \, dx$ is convergent. Thus $\int_1^\infty \frac{x}{x^3 + 1} \, dx$ is convergent. Since $f(x) = \frac{x}{x^3 + 1}$ is continuous on the interval $[0,1]$, $\int_0^1 \frac{x}{x^3 + 1} \, dx$ is finite. Thus the integral converges.

(b) $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}} \, dx$

Solution $\frac{x+1}{\sqrt{x^4-x}} \geq \frac{x}{\sqrt{x^4}} \geq \frac{1}{x}$ for $x \geq 1$. By $p$-test we know that $\int_1^\infty \frac{1}{x} \, dx$ is divergent. Thus this integral is divergent by Comparison Theorem.

(c) $\int_0^\infty \frac{\arctan x}{2 + e^x} \, dx$

Solution We cannot compare this integral with $\frac{1}{2 + e^x}$ because $-\frac{\pi}{2} \leq \arctan x \leq \frac{\pi}{2}$. Instead we can consider $\frac{c}{2 + e^x}$ for any $c \geq \frac{\pi}{2}$. Suppose $c = \pi$, then $\int_0^\infty \arctan x \, dx \leq \int_0^\infty \pi \, dx = \pi \int_0^\infty \frac{1}{2 + e^x} \, dx$.

We know that $\int_0^\infty \frac{1}{2 + e^x} \, dx$ is convergent since $\frac{1}{2 + e^x} \leq \frac{1}{e^x}$ whose integral from 0 to infinity is convergent. Thus $\pi \int_0^\infty \frac{1}{2 + e^x} \, dx$ is also convergent. Therefore the integral is convergent by Comparison Theorem.
(d) \[ \int_0^\pi \frac{\sin^2(x)}{\sqrt{x}} \, dx \]

**Solution** Since \(0 \leq \sin^2(x) \leq 1\), \(\int_0^\pi \frac{\sin^2(x)}{\sqrt{x}} \, dx \leq \int_0^\pi \frac{1}{\sqrt{x}} \, dx\). \(\int_0^\pi \frac{1}{\sqrt{x}} \, dx = \lim_{t \to 0^+} \int_t^\pi \frac{1}{\sqrt{x}} \, dx = \lim_{t \to 0^+} 2\sqrt{x}\bigg|_t^\pi = \lim (2\sqrt{\pi} - 2\sqrt{t}) = 2\sqrt{\pi}\). Thus \(\int_0^\pi \frac{1}{\sqrt{x}} \, dx\) is convergent, and the integral is convergent by Comparison Theorem.

3. Evaluate the following improper integrals.

(a) \(\int_0^{\infty} x e^{-x^2} \, dx\)

**Solution** Notice that \((e^{-x^2})' = -2x \cdot e^{-x^2}\), so by definition of improper integrals and FTC, we have

\[
\int_0^{\infty} x e^{-x^2} \, dx = \lim_{t \to +\infty} \int_0^t x e^{-x^2} \, dx
\]

\[
= \lim_{t \to +\infty} -e^{-x^2}\bigg|_0^t
\]

\[
= \lim_{t \to +\infty} 1 - e^{-t^2}
\]

\[
= \frac{1}{2}.
\]

(b) \(\int_0^{1/\sqrt{e}} \frac{1}{x(\ln x)^2} \, dx\)

**Solution** the integrand is not continuous at 0, so we can’t use FTC directly. By definition, we have

\[
\int_0^{1/\sqrt{e}} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to 0^+} \int_t^{1/\sqrt{e}} \frac{1}{x(\ln x)^2} \, dx
\]

\[
= \lim_{t \to 0^+} \int_t^{1/\sqrt{e}} \frac{1}{(\ln x)^2} \, d(\ln x)
\]

\[
= \lim_{t \to 0^+} \int_{\ln t}^{-1/2} \frac{1}{u^2} \, du
\]

\[
= \lim_{t \to 0^+} -\frac{1}{u} \bigg|_{\ln t}^{-1/2}
\]

\[
= \lim_{t \to 0^+} 2 + \frac{1}{\ln t}
\]

\[
= 2.
\]

The last equality holds due to \(\lim_{t \to 0^+} \ln t = -\infty\).

(c) \(\int_1^{\infty} \frac{\ln x}{x^3} \, dx\)

**Solution** Notice that \((1/x^2)' = -2/x^3\), by definition and integration by parts, we have

\[
\int_1^{\infty} \frac{\ln x}{x^3} \, dx = \lim_{t \to +\infty} \int_1^t \frac{\ln x}{x^3} \, dx
\]

\[
= \lim_{t \to +\infty} -\ln x \bigg|_1^t \frac{1}{2x^2} + \int_1^t \frac{1}{2x^2} \, d(\ln x)
\]

\[
= \lim_{t \to +\infty} -\ln t \bigg|_1^{1/t} - \frac{1}{4x^2} \bigg|_1^t
\]

\[
= \lim_{t \to +\infty} -\frac{\ln t}{2t^2} + \frac{1}{4} - \frac{1}{4t^2}.
\]
Remember that \( e^{2x} \) dominates \( x \) as \( x \) approaches infinity, so \( \lim_{t \to +\infty} -\frac{\ln t}{2t^2} = 0 \). So the limit above should be \( 1/4 \).

(d) \( \int_0^\infty \frac{1}{\sqrt{x}(1 + x)} \, dx \).

**Solution** This integral is improper for two reasons: The interval \([0, \infty)\) is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

\[
\int_0^\infty \frac{1}{\sqrt{x}(1 + x)} \, dx = \int_0^1 \frac{1}{\sqrt{x}(1 + x)} \, dx + \int_1^\infty \frac{1}{\sqrt{x}(1 + x)} \, dx.
\]

Let \( u = \sqrt{x}, u^2 = x, 2udu = dx \).

\[
\lim_{t \to 0+} \int_0^1 \frac{1}{\sqrt{x}(1 + x)} \, dx = \lim_{t \to 0+} \int_{\sqrt{t}}^1 \frac{2}{1 + u^2} \, du = \lim_{t \to 0+} 2 \arctan(u) \bigg|_{1}^{\sqrt{t}} = \lim_{t \to 0+} 2(\arctan(1) - \arctan(\sqrt{t})) = \frac{\pi}{2}.
\]

Similarly, \( \lim_{s \to \infty} \int_1^s \frac{1}{\sqrt{x}(1 + x)} \, dx = \lim_{s \to \infty} \int_{1}^{s} \frac{2}{1 + u^2} \, du = \lim_{s \to \infty} 2 \arctan(u) \bigg|_{1}^{s} = \lim_{s \to \infty} 2(\arctan(s) - \arctan(1)) = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}.
\]

Therefore \( \int_0^\infty \frac{1}{\sqrt{x}(1 + x)} \, dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi \).

4. (a) \( \int \sin^3(x) \cos^2(x) \, dx \)

**Solution**

\[
\int \sin^3(x) \cos^2(x) \, dx = \int -\sin^2(x) \cos^2(x) \, d(\cos(x))
\]

\[
= \int (\cos^2(x) - 1)(\cos^2(x)) \, d(\cos(x))
\]

\[
= \frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + C
\]

(b) \( \int \arcsin(x) \, dx \)

**Solution**

\[
\int \arcsin(x) \, dx = \arcsin(x) x - \int x \, d(\arcsin(x))
\]

\[
= \arcsin(x) x - \int \frac{x \, dx}{\sqrt{1 - x^2}}
\]

\[
= \arcsin(x) x - \int \frac{d(1 - x^2)}{2\sqrt{1 - x^2}}
\]

\[
= \arcsin(x) + \sqrt{1 - x^2} + C
\]

(c) \( \int_1^2 \frac{2x^3 + 7x^2 + 8x + 6}{2x^2 + 7x + 5} \, dx \)
Solution

\[
\int_1^2 \frac{2x^3 + 7x^2 + 8x + 6}{2x^2 + 7x + 5} \, dx = \int_1^2 \frac{3x + 6}{2x^2 + 7x + 5} \, dx + x \text{ polynomials division}
\]

\[
= \int_1^2 \left( \frac{A}{2x + 5} + \frac{B}{x + 1} + x \right) \, dx \quad \text{by solving linear equation}
\]

\[
= \int_1^2 \left( \frac{1}{2x + 5} + \frac{1}{x + 1} + x \right) \, dx
\]

\[
= \left( \frac{1}{2} \ln(2x + 5) + \ln(x + 1) + \frac{1}{2}x^2 \right) \bigg|_1^2
\]

\[
= \frac{1}{2} \ln(9) + \ln(3) + \frac{3}{2}
\]

(d) \( \int_0^\infty \frac{1}{4 + x^2} \, dx \)

Solution Let \( x = 2u, \, dx = 2 \, du \), then

\[
\int_0^\infty \frac{1}{4 + x^2} \, dx = \int_0^\infty \frac{2 \, du}{4 + 4u^2}
\]

\[
= \lim_{t \to \infty} \frac{1}{2} (\arctan(t) - \arctan(0))
\]

\[
= \frac{\pi}{4}
\]

5. (a) If \( \lim_{x \to +\infty} f(x) = 0 \), then \( \int_0^\infty f(t) \, dt \) converges.

Solution False. Consider \( f(x) = 1/x \) when \( x > 1 \) and \( f(x) = 1 \) when \( 0 \leq x \leq 1 \). Because \( \int_1^\infty \frac{1}{x} \, dx \) diverges, so the improper integral also diverges.

(b) If \( \int_0^\infty f(t) \, dt \) converges, then \( \lim_{x \to +\infty} f(x) = 0 \).

Solution False. Consider \( f(x) \) defined as \( f(x) = 1 \) if and only if \( x \in [n, n + 1/2^n] \), where \( n = 2, 3, \ldots \), and \( f(x) = 0 \) anywhere else. Then

\[
\int_0^\infty f(x) \, dx = \sum_{n=2}^\infty \frac{1}{2^n} = \frac{1}{2}
\]

However, \( f(x) \) does not have a limit at infinity since it will attain 1 for infinitely many times.

(c) If \( \lim_{x \to +\infty} f(x) = a \) and \( \int_0^\infty f(t) \, dt \) converges, then \( a = 0 \).

Solution True. By definition, the integral converges if and only if the following limit exists: \( \lim_{t \to +\infty} \int_0^t f(x) \, dx \). If \( a \neq 0 \) (without loss of generality we assume it is positive), then there exists \( X \) s.t. if \( x > X \), then we have \( f(x) > a/2 \). Thus \( \int_0^{t_1} f(x) \, dx - \int_0^{t_2} f(x) \, dx > (t_1 - t_2) a/2 \) holds for any \( t_1, t_2 \) where \( t_1 > t_2 > X \). This contradicts to the convergence of \( \int_0^t f(x) \, dx \).