Problem Set #4 Solutions

1. (a) i. Using integration by parts with $u = \ln|x|$, $dv = x$, and so $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$, we have

$$\int x \ln|x|\, dx = \frac{x^2 \ln|x|}{2} - \int \frac{x^2}{2x}\, dx$$
$$= \frac{x^2 \ln|x|}{2} - \frac{x^2}{4} + C$$

ii. Using integration by parts with $u = \arctan x$, $dv = dx$, and so $du = \frac{1}{1+x^2}$ and $v = x$, we have

$$\int \arctan x\, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx.$$

Now using substitution with $u = 1 + x^2$, we have the above is equal to

$$x \arctan x - \int \frac{1}{2u}\, du = x \arctan x - \frac{\ln|u|}{2} + C$$
$$= x \arctan x - \frac{\ln(1+x^2)}{2} + C.$$

(b) Using Integration by parts with $u = \cos^{n-1} x$ and $dv = \cos x\, dx$, we get that

$$\int \cos^n x\, dx = \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{n-2} x\, dx$$
$$= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x\, dx.$$

Adding $\int \cos^{n-2} x\, dx$ to both sides, we get

$$n \int \cos^n x\, dx = \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx.$$

Dividing by n on both sides gives the desired result.

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

(a) $\displaystyle\int_0^\infty \frac{x}{x^3+1} dx$

Solution $\frac{x}{x^3+1} \leq \frac{x}{x^3} \leq \frac{1}{x^2}$ for $x \geq 0$. By p-test we know that $\int_1^\infty \frac{1}{x^2} dx$ is convergent. Thus $\int_1^\infty \frac{x}{x^3+1} dx$ is convergent. Since $f(x) = \frac{x}{x^3+1}$ is continuous on the interval $[0,1]$, $\int_0^1 \frac{x}{x^3+1} dx$ is finite. Thus the integral converges.

(b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4 - x}} dx$

Solution $\frac{x+1}{\sqrt{x^4-x}} \geq \frac{x}{\sqrt{x^4-x}} \geq \frac{x}{\sqrt{x^4}} \geq \frac{1}{x}$ for $x \geq 1$. By p-test we know that $\int_1^\infty \frac{1}{x} dx$ is divergent. Thus this integral is divergent by Comparison Theorem.

(c) $\displaystyle\int_0^\infty \frac{\arctan x}{2 + e^x} dx$

Solution We cannot compare this integral with $\frac{1}{2+e^x}$ becasue $-\frac{\pi}{2} \leq \arctan x \leq \frac{\pi}{2}$. Instead we can consider $\frac{c}{2+e^x}$ for any $c \geq \frac{\pi}{2}$. Suppose $c = \pi$, then $\int_0^\infty \frac{\arctan x}{2+e^x} dx \leq \int_0^\infty \frac{\pi}{2+e^x} dx = \pi \int_0^\infty \frac{1}{2+e^x} dx$. We know that $\int_0^\infty \frac{1}{2+e^x} dx$ is convergent since $\frac{1}{2+e^x} \leq \frac{1}{e^x}$ whose integral from 0 to infinity is convergent. Thus $\pi \int_0^\infty \frac{1}{2+e^x} dx$ is also convergent. Therefore the integral is convergent by Comparison Theorem.

(d) $\displaystyle\int_0^{\pi} \frac{\sin^2(x)}{\sqrt{x}} dx$

Solution Since $0 \le \sin^2(x) \le 1$, $\displaystyle\int_0^{\pi} \frac{\sin^2(x)}{\sqrt{x}} dx \le \int_0^{\pi} \frac{1}{\sqrt{x}} dx$. $\displaystyle\int_0^{\pi} \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} \int_t^{\pi} \frac{1}{\sqrt{x}} dx = \lim_{t\to 0^+} 2\sqrt{x}\Big|_t^{\pi} = \lim_{t\to 0^+} (2\sqrt{\pi} - 2\sqrt{t}) = 2\sqrt{\pi}$. Thus $\displaystyle\int_0^{\pi} \frac{1}{\sqrt{x}} dx$ is convergent, and the integral is convergent by Comparison Theorem.

3. Evaluate the following improper integrals.

(a) $\displaystyle\int_0^{\infty} xe^{-x^2} \mathrm{d}x$

Solution Notice that $(e^{-x^2})' = -2x \cdot e^{-x^2}$, so by definition of improper integrals and FTC, we have

$$\begin{aligned}
\int_0^{\infty} xe^{-x^2} \mathrm{d}x &= \lim_{t\to+\infty} \int_0^{t} xe^{-x^2} \mathrm{d}x \\
&= \lim_{t\to+\infty} -\frac{e^{-x^2}}{2}\Big|_0^t \\
&= \lim_{t\to+\infty} \frac{1}{2} - e^{-t^2} \\
&= \frac{1}{2}.
\end{aligned}$$

(b) $\displaystyle\int_0^{1/\sqrt{e}} \frac{1}{x(\ln x)^2} \mathrm{d}x$

Solution the integrand is not continuous at 0, so we can't use FTC directly. By definition, we have

$$\begin{aligned}
\int_0^{1/\sqrt{e}} \frac{1}{x(\ln x)^2} \mathrm{d}x &= \lim_{t\to 0^+} \int_t^{1/\sqrt{e}} \frac{1}{x(\ln x)^2} \mathrm{d}x \\
&= \lim_{t\to 0^+} \int_t^{1/\sqrt{e}} \frac{1}{(\ln x)^2} \mathrm{d}(\ln x) \\
&= \lim_{t\to 0^+} \int_{\ln t}^{-1/2} \frac{1}{u^2} \mathrm{d}u \\
&= \lim_{t\to 0^+} -\frac{1}{u}\Big|_{\ln t}^{-1/2} \\
&= \lim_{t\to 0^+} 2 + \frac{1}{\ln t} \\
&= 2.
\end{aligned}$$

The last equality holds due to $\displaystyle\lim_{t\to 0^+} \ln t = -\infty$.

(c) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3} \mathrm{d}x$

Solution Notice that $(1/x^2)' = -2/x^3$, by definition and integration by parts, we have

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^3} \mathrm{d}x &= \lim_{t\to+\infty} \int_1^{t} \frac{\ln x}{x^3} \mathrm{d}x \\
&= \lim_{t\to+\infty} -\frac{\ln x}{2x^2}\Big|_1^t + \int_1^t \frac{1}{2x^2} \mathrm{d}(\ln x) \\
&= \lim_{t\to+\infty} -\frac{\ln t}{2t^2} + \int_1^t \frac{1}{2x^3} \mathrm{d}x \\
&= \lim_{t\to+\infty} -\frac{\ln t}{2t^2} - \frac{1}{4x^2}\Big|_1^t \\
&= \lim_{t\to+\infty} -\frac{\ln t}{2t^2} + \frac{1}{4} - \frac{1}{4t^2}.
\end{aligned}$$

Remember that e^{2x} dominates x as x approaches infinity, so $\lim_{t\to+\infty} -\frac{\ln t}{2t^2} = 0$. So the limit above should be 1/4.

(d) $\int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx$.

Solution This integral is improper for two reasons: The interval $[0,\infty)$ is infinite and the integrand has an infinite discontinuity at 0. Evaluate it by expressing it as a sum of improper integrals of Type 2 and Type 1 as follows:

$$\begin{aligned}\int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx &= \int_0^1 \frac{1}{\sqrt{x}(1+x)} dx + \int_1^\infty \frac{1}{\sqrt{x}(1+x)} dx \\ &= \lim_{t\to 0+} \int_t^1 \frac{1}{\sqrt{x}(1+x)} dx + \lim_{s\to\infty} \int_1^s \frac{1}{\sqrt{x}(1+x)} dx\end{aligned}$$

Let $u = \sqrt{x}, u^2 = x, 2udu = dx$.
$\lim_{t\to 0+} \int_t^1 \frac{1}{\sqrt{x}(1+x)} dx = \lim_{t\to 0+} \int_{\sqrt{t}}^1 \frac{2}{1+u^2} du = \lim_{t\to 0+} 2\arctan(u)\Big|_{\sqrt{t}}^1 = \lim_{t\to 0+} 2(\arctan(1) - \arctan(\sqrt{t})) = \frac{\pi}{2}$.
Similarly, $\lim_{s\to\infty} \int_1^s \frac{1}{\sqrt{x}(1+x)} dx = \lim_{s\to\infty} \int_1^s \frac{2}{1+u^2} du = \lim_{s\to\infty} 2\arctan(u)\Big|_1^s = \lim_{s\to\infty} 2(\arctan(s) - \arctan(1)) = 2(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{\pi}{2}$.
Therefore $\int_0^\infty \frac{1}{\sqrt{x}(1+x)} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

4. (a) $\int \sin^3(x)\cos^2(x) dx$

Solution

$$\begin{aligned}\int \sin^3(x)\cos^2(x) dx &= \int -\sin^2(x)\cos^2(x) d(\cos(x)) \\ &= \int (\cos^2(x) - 1)(\cos^2(x)) d(\cos(x)) \\ &= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C\end{aligned}$$

(b) $\int \arcsin(x) dx$

Solution

$$\begin{aligned}\int \arcsin(x) dx &= \arcsin(x)x - \int x d(\arcsin(x)) \\ &= \arcsin(x)x - \int \frac{xdx}{\sqrt{1-x^2}} \\ &= \arcsin(x)x - \int \frac{d(1-x^2)}{2\sqrt{1-x^2}} \\ &= \arcsin(x) + \sqrt{1-x^2} + C\end{aligned}$$

(c) $\int_1^2 \frac{2x^3 + 7x^2 + 8x + 6}{2x^2 + 7x + 5} dx$

Solution

$$\begin{aligned}
\int_1^2 \frac{2x^3+7x^2+8x+6}{2x^2+7x+5}dx &= \int_1^2 \frac{3x+6}{2x^2+7x+5} + xdx \qquad \textit{polynomials division}\\
&= \int_1^2 \left(\frac{A}{2x+5} + \frac{B}{x+1} + x\right) dx \qquad \textit{by solving linear equation}\\
&= \int_1^2 \left(\frac{1}{2x+5} + \frac{1}{x+1} + x\right) dx\\
&= (\frac{1}{2}\ln(2x+5) + \ln(x+1) + \frac{1}{2}x^2)\Big|_1^2\\
&= \frac{1}{2}\ln(\frac{9}{7}) + \ln(\frac{3}{2}) + \frac{3}{2}
\end{aligned}$$

(d) $\displaystyle\int_0^\infty \frac{1}{4+x^2}dx$

Solution Let x=2u,dx=2du,then

$$\begin{aligned}
\int_0^\infty \frac{1}{4+x^2}dx &= \int_0^\infty \frac{2du}{4+4u^2}\\
&= \lim_{t\to\infty} \frac{1}{2}(\arctan(t) - \arctan(0))\\
&= \frac{\pi}{4}
\end{aligned}$$

5. (a) If $\lim_{x\to+\infty} f(x) = 0$, then $\displaystyle\int_0^\infty f(t)\mathrm{d}t$ converges.

Solution False. consider $f(x) = 1/x$ when $x > 1$ and $f(x) = 1$ when $0 \leq x \leq 1$. Because $\displaystyle\int_1^\infty \frac{1}{x}\mathrm{d}x$ diverges, so the improper integral also diverges.

(b) If $\displaystyle\int_0^\infty f(t)\mathrm{d}t$ converges, then $\lim_{x\to+\infty} f(x) = 0$.

Solution False. Consider $f(x)$ defined as $f(x) = 1$ if and only if $x \in [n, n+1/2^n]$, where $n = 2, 3, ...$, and $f(x) = 0$ anywhere else. Then

$$\int_0^\infty f(x)\mathrm{d}x = \sum_{n=2}^\infty \frac{1}{2^n} = \frac{1}{2}.$$

However, $f(x)$ does not have a limit at infinity since it will attain 1 for infinitely many times.

(c) If $\lim_{x\to+\infty} f(x) = a$ and $\displaystyle\int_0^\infty f(t)\mathrm{d}t$ converges, then $a = 0$.

Solution True. By definition, the integral converges if and only if the following limit exists: $\lim_{t\to+\infty}\int_0^t f(x)\mathrm{d}x$. If $a \neq 0$(without loss of generality we assume it is positive), then there exists X s.t. if $x > X$, then we have $f(x) > a/2$. Thus $\int_0^{t_1} f(x)\mathrm{d}x - \int_0^{t_2} f(x)\mathrm{d}x > (t_1 - t_2)a/2$ holds for any t_1, t_2 where $t_1 > t_2 > X$. This contradicts to the convergence of $\int_0^t f(x)\mathrm{d}x$.