Ex 2: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

10.09.

Recall the definition of sequences, series, and convergence of them:

$$\{a_n\}, \quad \sum_{n=1}^{\infty} a_n, \quad S_n = \sum_{k=1}^{n} a_k$$

$\left(\underline{\overline{|a_n| \to 0 \Rightarrow a_n \to 0}} \right)$

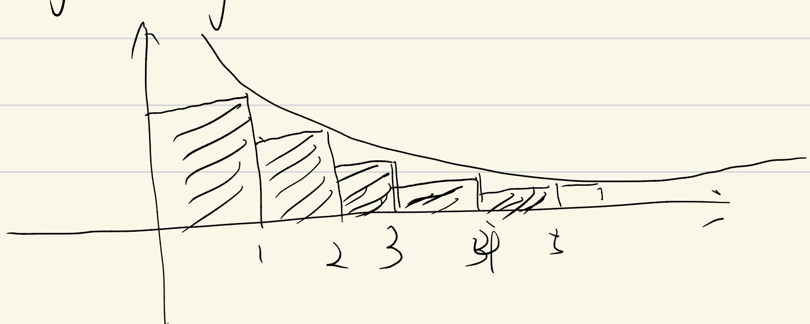
Motivation: we know that, the test of divergence is insufficient to check the convergence / divergence.

Test Ex: harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} \cdots$

Also, $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots \quad \left(= \frac{\pi^2}{6} \right)$

which is quite amazing.

Integral test:
Consider function $f(x) = \frac{1}{x^2}$



the sum $\Rightarrow$ the areas of these rectangles

Quest: rectangles you drew give an underestimate or overestimate of the area under the curve? $\Rightarrow$ underestimate

$\therefore$ from the second rectangle

$$\Rightarrow \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2} \le \int_1^{\infty} \frac{1}{x^2}\,dx = 1.$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \le 1+1 = 2.$$

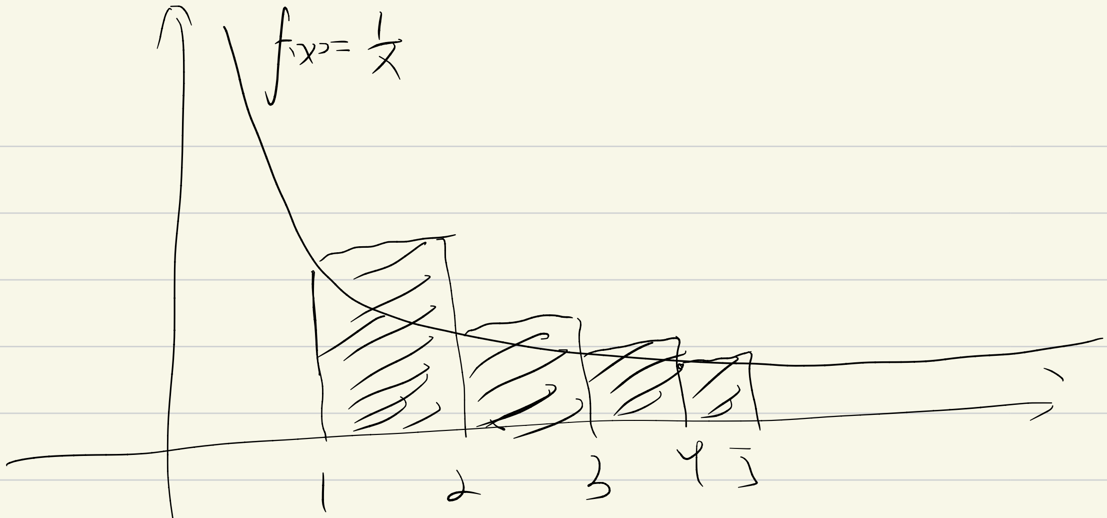
of course. $a_n \ge 0 \Rightarrow 0 \le \sum a_n \le 2$

we can see the partial sum is increasing and bdd.

by the Theorem we introduce before $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 2.$

Another example: $\sum \frac{1}{n}$

Consider the function: $f(x) = \frac{1}{x}$, Draw a picture on the graph of the function f represent the sum of the series: $\sum \frac{1}{n}$



Look $\sum_1^\infty c_n = \sum_1^\infty \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$ ——

The rectangle we drew overestimate? underestimate $\Longrightarrow$ overestimate.

$\therefore \quad \sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^\infty \frac{1}{x}\,dx \to \infty$ divergent

$\therefore \quad \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$, $\Longrightarrow$ divergent.

Theorem: (The Integral test):

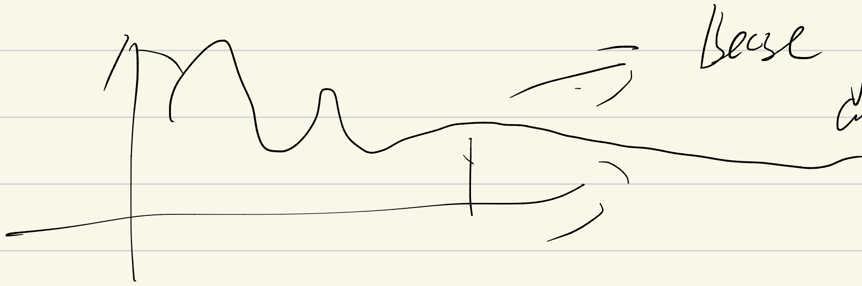
Suppose f is continuous, positive, decreasing function on $[1, \infty)$, and let $c_n = f(n)$, then the series $\sum_{n=1}^{\infty} c_n$ is convergent if and only if $\int_1^\infty f(x)\,dx$ is convergent. In other words:

1. If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_1^\infty c_n$ is convergent.
2. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} c_n$ is divergent.

Note: When we use the Integral test, it's not necessary that the index the index of the integral:

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \int_4^{\infty} \frac{1}{(x-3)}\,dx$$

Note: It's not necessary that f be always decreasing. It's only necessary that f is eventually decreasing: say, starting after x reaches some number N.



Because first N terms, finite, can be ignored

Ex: $\sum_{n=1}^{\infty} \frac{\ln n}{n} \Rightarrow \int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \ln(n)\Big|_1^t = \infty$

Ex: convergence of p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} \Leftrightarrow \int_1^{\infty} \frac{1}{x^p}\,dx$

Thm: $\sum \frac{1}{n^p}$ converges if and only if $p>1$

Estimating the sum of series:

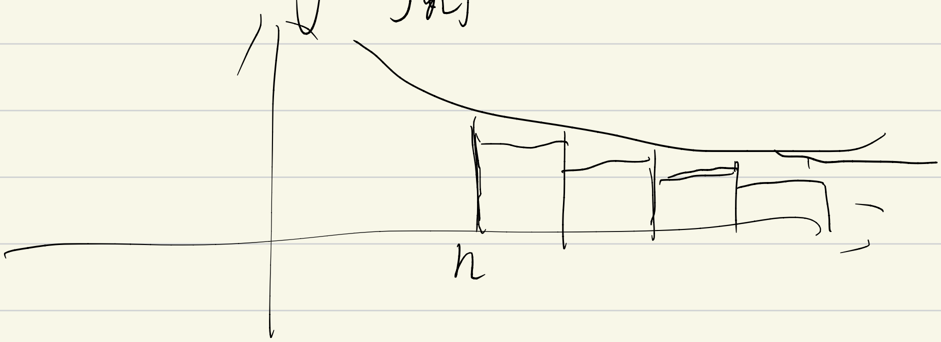
We want to find the approximation of the sum s, as $\lim s_n = s$.

We can use the partial sum, but how good is the approximation?

We can estimate the remainder

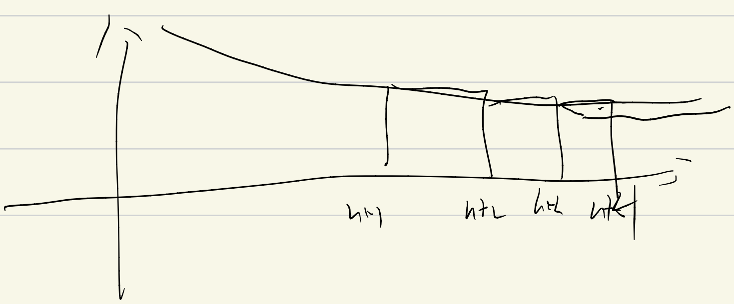
$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

Draw a graph of a continuous, positive, decreasing function that shows R_n is an underestimate of $\int_n^\infty f(x)\,dx$: $R_n = a_{n+1} + a_{n+2}$



$$\int_n^\infty f(x)\,dx \geq \sum_{i=n+1}^{\infty} a_i$$

Draw picture the graph of f that shows R_n is an overestimate of $\int_{n+1}^\infty f(x)$



Thm:

Remainder Estimate for the Integral test: Suppose $f(k) = a_k$, where f is continuous, positive, decreasing function for $x \geq a$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

If we add s_n to each side of the inequalities in the above remainder estimate, we get:

Approximation of the sum of series s:

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx$$

Ex: Use $n=10$ to estimate the sum of series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$, give bounds of error

Very important for computation.

How large should n be in order for the error in approximating $\sum \frac{1}{n^5}$ to be within [illegible]?

$$\boxed{R_n \leq \int_{n}^{\infty} f(x)\,dx} \Rightarrow n$$

Remk: The integral test can't tell us the exact value:

eg: $\sum_{k=1}^{\infty} e^{-k}$, but sometimes we can use other formulas

(geometric formula).