10.16. Alternating Series and Absolute Convergence

An alternating series is a series whose terms alternate between being positive and negative. The nth term of an alternating series is of the form $a_n = (-1)^{n-1} b_n$ where $b_n = |a_n|$ ( )
(So b_n is a positive number)

The Alternating Series Test: If the alternating series

$$\sum (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 \cdots \qquad (b_n > 0)$$

if $b_n \xrightarrow[b_n \geq b_{n+1}]{\text{monotone decreasing}} 0$, then we know that the series is convergent.

[Intuition: Person jump up and down, with smaller step, finally you'll stop...

Ex: alternating harmonic series: Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converge?

Yes. $\left|\frac{-1}{n}\right| \to 0$.

Ex: $\sum \frac{(-1)^n 3n}{4n-1}$ convergent? divergent.

No. $\Rightarrow$ test for divergence.

Ex $\sum (-1)^{n+1} \frac{n^2}{n^3+1}$?

Yes $\lim b_n \to 0$ $\underline{b_n \geq b_{n+1}}$

Alternating series estimation Theorem: If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $b_{n+1} \leq b_n$

(ii) $\lim b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$

Ex: Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$ correct to three decimal place.

$b_{n+1} \leq \frac{1}{1000} \Longrightarrow n = 6$

Absolute Convergence:

Given any series $\sum a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = \lim_{n \to \infty} \sum_{k=1}^{n} |a_k|$$

whose terms are the absolute value of the terms of the original series

A series $\sum a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n| < \infty$

Note: if $\sum a_n$ is positive (or a_n), then $|a_n| = a_n$, then

convergence $\Longleftrightarrow$ absolute convergence.

Theorem: If a series $\sum a_n$ is absolute convergent, then it's also convergent.

Ex: Are the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converge?

absolute converge?

(1) Yes $\Rightarrow$ alternating series

(2) Yes $\Rightarrow$ integral test

Ex: (Converge but not A.C.)

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

converges? Yes A.C $\Rightarrow$ No $\Rightarrow$ integral test for $\frac{1}{n}$

Ex: Is the series $\sum \frac{\cos n}{n^2}$ convergent? (Yes,

Actually, Absolute converge $\Rightarrow$ $\sum \left|\frac{\cos n}{n^2}\right| < \sum \frac{1}{n^2}$

Comparison Theorem.

More Example:

$$\sum \frac{(-1)^n \sqrt{n}}{n+4}$$

$$\sum \frac{(-1)^n n}{n+5}$$