9.10

U-Substitution (technical part)

1.

Motivation: $\int = \int x\,dx$, $\int \sin x\,dx$, $\int \cos(x)\,dx$, $\int \frac{dx}{\sqrt{1-x^2}}$

we know some guys $F(x)$, $F'(x)=f(x)$

How about the others? $\int \sin x \cos x\,dx$?

Idea: Sometimes, we can try chain rule.

$$(f(g(x)))' = f'(g(x))\cdot g'(x)$$

$$\underline{f(g(x))} \overset{(+C)}{=} \underline{\int f'(g(x))\, g'(x)\,dx}$$

$$= \int f'(g(x))\,\underline{|d(g(x))|} \Longrightarrow \text{find } d\,g(x)$$

Ex 1: $\int x\cdot\sqrt{1+x^2}\,dx = ?$ [find antiderivative

(1) let $u = x^2$, then

the differential of u is $du = 2x\,dx$

例: $\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{1+u}\,du = \int (1+u)^{\frac{1}{2}}\,du = \left[\frac{2}{3}(1+u)^{\frac{3}{2}} + C\right]$

$= \frac{2}{3}(1+x^2)^{\frac{3}{2}}$ $\Longrightarrow$ Use chain rule to check it (Ex)

ILM: This method works whenever we have an integral of the form

$$\int f'(g(x))\,dg(x) = \int f'(g(x))\,g'(x)\,dx$$

It's just the inverse of chain rule

u-substitution rule for indefinite integrals: If $u = g(x)$ is differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \underline{F(g(x))} + C$$

General rule for knowing when to use u-substitution: Look for "inside function" $g(x)$ with $g'(x)$ is multiplied on the outside (of course, sometimes you can be tricky and do something else). It's more like art than science or math to find $g(x)$!!! [Be careful, $g(x)$, $f(x)$ should be continuous]

Ex1: $\int x^3 \cdot \cos(x^4+2)\,dx = ?$

$$= \int \frac{1}{4}\cos(x^4+2)\,dx^4$$

$$= \int \frac{1}{4}\cos(x^4+2)\,d(x^4+2)$$

$$= \frac{1}{4}\sin(x^4+2) + C$$

Ex2: (Two possible for u). Evaluate $\int \sqrt{2x+1}\,dx = ?$

Solution 1: $u = 2x+1 \Rightarrow du = 2dx$

$$\Rightarrow \int \sqrt{u} \cdot \frac{du}{2} \Rightarrow = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

Solution 2: $\sqrt{2x+1} = u \Rightarrow du = \dfrac{dx}{\sqrt{2x+1}}$

$$\Rightarrow \int \sqrt{2x+1}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$$

Ex3: $\int \sin^2(x)\,dx$ (Common trick: $d(\cos x) = -\sin(x)\,dx$; $(\cos x) = u$)

$$= \int -\cos x\,d(\cos x)$$

$$= \int -u \cdot du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$$

More Ex:

$$\int \frac{2x}{\sqrt{1+x^4}}\,dx$$

$$\int \frac{x}{1+x}\,dx$$

1.2

Definite Integral: (Combine u-substitution and FTC)

If g' is continuous on $[a,b]$ and f is continuous on the range of $u=g(x)$

Then: $\int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a))$

Ex 1: $\int_1^2 \frac{dx}{(3-5x)^2} = -\int_1^2 \frac{\frac{1}{5}\, d(3-5x)}{(3-5x)^2}$

$$= \frac{\frac{1}{5}}{(3-5x)}\Big|_1^2$$

$$= -\frac{1}{35} + \frac{1}{10} = \frac{1}{14}$$

Ex 2: $\int_1^e \frac{\ln(x)}{x}\, dx$

$$= \int_1^e \ln(x)\, d(\ln(x))$$

$$= \frac{1}{2}[\ln(x)]^2 \Big|_1^e$$

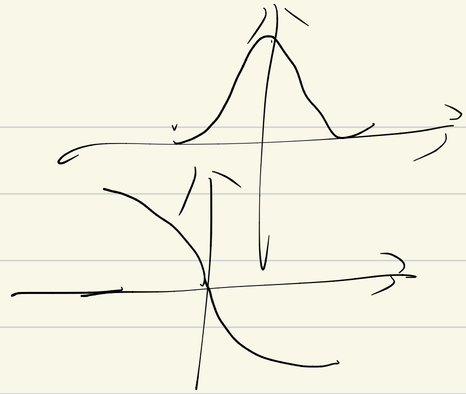
$$= \frac{1}{2}.$$

1.3: Symmetric functions:

We can use u-substitution for definite integrals to simplify the calculation of integrals of functions that have symmetry property.

$f(x) = f(-x) \Longleftrightarrow$ even function

$f(x) = -f(-x) \Longleftrightarrow$ odd



Even or odd

$\sin(x)$, $\cos(x)$, x^3, $x^5 + \sin(x)$, $\tan(x)$.

Integral of symmetric functions: Suppose f is continuous on $[-a, a]$, Then

1. f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

[Pf idea: use $(-x) = u$]
and $\int_{-a}^{0} f(x) + \int_{0}^{a} f(x) = \int_{-a}^{a} f(x)\,dx$

2. f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$

[Ex 1]
Evalue: $\int_{-2}^{2} (x^6+1)\,dx = ? = \frac{1}{7}x^7 + x \Big|_{-2}^{2}$

(2) $\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4}\,dx = ? = 0$ (odd function)

Ex 2: $\int_{1}^{3} f(x)\,dx = 7$, Find $\int_{1}^{3} 4f(2x-3)\,dx$

$\int_{1}^{3} 4f(2x-3)\,dx = \int_{1}^{3} 2f(2x-3)\,d(2x-3) = 2\int_{-1}^{3} f(t)\,dt$

$$= 2\underbrace{\overbrace{\int_{-1}^{1} f(t)\,dt}^{=0}} + 2\overbrace{\int_{1}^{3} f(t)\,dt}^{=14} \qquad (\text{as } f(t) \text{ is odd}).$$

$$= 14$$

Average value of f on the $[a,b]$.

$$f_{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

09.11. Partial Fraction Decomposition and Trig Substitution:

(More tricky!)

①

$$\int\left[\frac{2}{(x-1)} + \frac{3}{(x+2)}\right]dx \quad \checkmark)$$

(We need a common denominator and then simplify it)

$$= \int \frac{5x+1}{(x-1)(x+2)}\,dx \Longrightarrow$$

How to undo it?