$$= 2\underbrace{\int_{-1}^{1} f(t)\,dt}_{=0} + 2\overbrace{\int_{1}^{3} f(t)\,dt}^{=14} \quad (\text{as } f(t) \text{ is odd})$$

$$= 14$$

Average value of f on the $[a,b]$.

$$f_{average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

09.11. Partial Fraction Decomposition and Trig Substitution:

(More tricky!)

①

$$\int \left[\frac{2}{(x-1)} + \left(\frac{3}{x+2}\right)\right] dx \quad (\checkmark) \quad (\text{We need a common denominator and then simplify it})$$

$$= \int \frac{5x+1}{(x-1)(x+2)}\,dx \Longrightarrow \text{How to undo it?}$$

How to do reverse $\frac{(5x+1)}{(x-1)(x+2)} \Longrightarrow \frac{2}{(x-1)} + \frac{3}{(x+2)}$?

Let's do $\frac{4}{x^2-1}$.

① First, factor the denominator into irreducible terms

$$(x^2-1) = \underline{(x-1)}\,\underline{(x+1)}$$

② Next, write the fraction as a sum of parts with unknown numerators

$$\frac{4}{x^2-1} = \frac{a}{x-1} + \frac{b}{x+1}$$

(where there're only order of 0. (常数)?)

$$4 = a(x+1) + b(x-1)$$

$$0x + 4 = (a+b)x + a-b$$

③ Finally, combine the sum into one fraction and then solve for the unknown coefficient.

$$\frac{4}{(x^2-1)} = \frac{a(x+1)}{(x-1)(x+1)} + \frac{b(x-1)}{(x-1)(x+1)}$$

$$\Longrightarrow \begin{cases} a+b=0 \\ a-b=4 \end{cases} \Longrightarrow \begin{matrix} a=2 \\ b=-2 \end{matrix}$$

In general: we have two polynomials $P(x)$, $Q(x)$

$\frac{P(x)}{Q(x)}$, where $Q(x)$'s degree is higher than that of $P(x)$.

we can decompose it in the following:

Example: compute the integral:

$$\int_0^1 \frac{2x+3}{x^3+2x^2+x}\,dx = ?$$

$$(x^3+2x^2+x) = x(x^2+2x+1) = x(x+1)^2 \Rightarrow$$

$$\therefore \frac{2x+3}{x^3+2x^2+x} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2}, \text{ solve } a, b, c.$$

$$= \frac{a}{x} + \frac{b'x+c'}{(x+1)^2}$$

How about degree $P(x) \geq Q(x)$?

Use the division of polynomials:

$$\int \frac{4x^3-3x+5}{x^2-2x}\,dx \overset{?}{=} \int \underline{4x + \frac{5x+5}{x^2-2x}}\,dx$$

$$\begin{array}{r|l} & 4x+0 \\ \hline x^2-2x & 4x^3-3x+5 \\ & 4x^3-8x \\ \hline & 5x+5 \end{array}$$

Ex: $\frac{3x^2+2x+4}{x+3}\,dx$?

How abt: $\underline{\int \frac{1}{2+x^2}\,dx}$?

u-substitution:

$$\int \frac{dx}{1+x^2} = \arctan x$$

$$y = \frac{x}{\sqrt{2}} \Longrightarrow \int \frac{1}{\sqrt{2}}\left(\frac{1}{2+2y^2}\,dy\right) = \frac{1}{2a}\arctan(y)$$

$$= \frac{1}{2a}\arctan\left(\frac{x}{a}\right)$$

(2) Recall these useful equality:

$$\sin^2(x) = 1-\cos^2(x).$$

$$\sec^2(x) = 1+\tan^2(x)$$

$$\tan^2(x) = \sec^2(x)-1.$$

General guideline

1) when meet $\sqrt{a^2-x^2} \Longrightarrow x = a\cdot\sin\theta$

2) when meet $\sqrt{a^2+x^2}$, try $x = a\cdot\tan\theta$

3) when $\sqrt{x^2-a^2}$, try $x = a\cdot\sec(\theta)$

Ex 1: $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

$(\tan\theta)' = \frac{1}{\cos^2\theta}\,d\theta$

$= (1+x^2)\,d\theta$

$x = \tan\theta \Longrightarrow dx = \frac{d\theta}{\cos^2\theta}$

$$LHS = \int \frac{|\cos\theta|}{\sin^2\theta\,\cos^2\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{d\sin\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C$$

$dx = 3\cos\theta\, d\theta$

Ex2: $\int \frac{x^2}{\sqrt{9-x^2}}\, dx =$

$x = 3\sin\theta \Longrightarrow \int \frac{9\sin^2\theta \cdot 3\cos\theta}{3\cos\theta}\, d\theta$

$= \int 9\sin^2\theta\, d\theta$

$= \int \frac{9}{2}(\sin^2\theta - \cos^2\theta) + \frac{9}{2}(\sin^2\theta + \cos^2\theta)\, d\theta$

$= \int -\frac{9}{2}\cos 2\theta\, d\theta + \int \frac{9}{2}\, d\theta$

$= -\frac{9}{4}\sin 2\theta \quad \frac{9}{2}\theta + C$

[$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\Longrightarrow \sin^2\theta$

$1 - 2\sin^2\theta = \cos 2\theta$

$\Longrightarrow \boxed{\frac{1-\cos 2\theta}{2} = \sin^2\theta}$]

$2\cos^2\theta - 1 = \cos 2\theta$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$\frac{a^2}{4}\sin\left(2\arctan\frac{x}{a}\right) +$

$x = a\cdot\sin\theta$

$\int \sqrt{a^2 - x^2}\, dx = \int a\cdot\cos\theta\, a\cos\theta\, d\theta$

$= \int a^2\cos^2\theta\, d\theta$

$= a^2\left(\frac{\cos 2\theta + 1}{2}\right) d\theta$

$= \frac{a^2}{4}\sin 2\theta + \frac{a^2\theta}{2} + C$

arctan

$=$