Math 122L Weekly Graded Homework

Homework 1 (90 pts)

1. (10 pts) Suppose that function $f(x)$ is defined as

$$f(x) = \begin{cases} ax + b, \ x < 1, \\ x^2, \ x \geq 1. \end{cases}$$

   Find a, b such that $f(x)$ is differentiable at $x = 1$.

2. (10 pts) Find $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3}$.

3. (10 pts) Find $\lim\limits_{x \to 0} \dfrac{1 - \cos(x^2)}{x^3 \sin x}$.

4. (10 pts) Suppose that $f(x)$ is n times continuously differentiable, i.e. the nth order derivative of f is continuous. If $f(0) = f'(0) = f''(0) = ... = f^{(n)}(0) = 0$, prove that $\lim\limits_{x \to 0} \dfrac{f(x)}{x^n} = 0$. Here $f^{(n)}(x)$ is the nth order derivative.

5. (20 pts) Suppose that $f(x) = x - (a + b \cos x) \sin x$. **Use conclusions in problem 4** to solve the following problems.

   (a) Express $f'(x), f''(x), f'''(x)$ with a, b.

   (b) If $a + b = 1$, prove that $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 0$.

   (c) Find a, b such that $\lim\limits_{x \to 0} \dfrac{f(x)}{x^3} = 0$.

   (d) For the a, b in question (c), find the largest integer n such that $\lim\limits_{x \to 0} \dfrac{f(x)}{x^n} = 0$ but $\lim\limits_{x \to 0} \dfrac{f(x)}{x^{n+1}} \neq 0$. For this n, find $\lim\limits_{x \to 0} \dfrac{f(x)}{x^{n+1}}$.

   Here are some trigonal equality you might want to employ: $\sin 2x = 2 \sin x \cos x, \cos 2x = \cos^2 x - \sin^2 x$.

6. (10 pts) Suppose that $f(x)$ is twice continuously differentiable and $f'(1) \neq 0, f(x) \neq f(1)$ for any $x \neq 1$. Also, we have $f'(1) = 1, f''(1) = 2$. Find $\lim\limits_{x \to 1} \dfrac{1}{f(x) - f(1)} - \dfrac{1}{f'(1)(x-1)}$.

7. (10 pts) Zibu is doing Math 122L homework on L'Hospital's Rule, which asks him to find $\lim_{x\to+\infty} \dfrac{x+\sin x}{x-\sin x}$. Zibu made the following statement:

   Because $|\sin x| \le 1$, so both $x+\sin x$ and $x-\sin x$ approaches infinity as $x \to +\infty$, so this is an indeterminant type of ∞/∞. By L'Hospital's Rule, we have

$$\lim_{x\to+\infty} \frac{x+\sin x}{x-\sin x} = \lim_{x\to+\infty} \frac{1+\cos x}{1-\cos x}.$$

   However, the limit on the right hand side does not exist since $1-\cos x$ can be 0 for infinitely many times as x approaches, so the limit does not exists.

   Is Zibu's answer (the limit does not exist) and application of L'Hospital's Rule right? If not, explain where he is wrong and find the right limit.

8. (10 pts) Suppose that there is a point $M(x,y)$ moving on the curve $x^2+xy+y^2=1$. At point $A\left(\dfrac{\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right)$, its speed on x direction $v_x = 1$ ft/s. Now we are going to compute its speed on y direction v_y at this point using 2 different methods.

   (a) Suppose that the y coordinate of M is a function of the x coordinate of M, i.e. y can be written as $y=y(x)$. Find $y'(x)$ at A by taking derivative with respect to x on both sides of the equation of the curve.

   (b) Use Chain Rule to compute v_y.

   (c) Find v_y Directly by taking derivative with respect to t on both sides of the equation.