Problem Set #4

1. (a) Integrate the following.
   i. $\int x \ln|x|\, dx$
   ii. $\int \arctan x\, dx$

   (b) Use Integration by Parts to prove the following recursion formula:
   $$\int \cos^n x\, dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent.
   (a) $\int_0^\infty \frac{x}{x^3+1}dx$
   (b) $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}dx$
   (c) $\int_0^\infty \frac{\arctan x}{2+e^x}dx$
   (d) $\int_0^\pi \frac{\sin^2(x)}{\sqrt{x}}dx$

3. Evaluate the following improper integrals.
   (a) $\int_0^\infty xe^{-x^2}\mathrm{d}x$
   (b) $\int_0^{1/\sqrt{e}} \frac{1}{x(\ln x)^2}\mathrm{d}x$
   (c) $\int_1^\infty \frac{\ln x}{x^3}\mathrm{d}x$
   (d) $\int_0^\infty \frac{1}{\sqrt{x}(1+x)}dx.$

4. Evaluate the following indefinite/definite integrals.
   (a) $\int \sin^3(x)\cos^2(x)dx$
   (b) $\int \arcsin(x)dx$
   (c) $\int_1^2 \frac{2x^3+7x^2+8x+6}{2x^2+7x+5}dx$
   (d) $\int_0^\infty \frac{1}{4+x^2}dx$

5. Suppose that $f(x)$ is a continuous function on $\mathbb{R}$. Prove the claim if true and give a counterexample if false:
   (a) If $\lim_{x\to+\infty} f(x) = 0$, then $\int_0^\infty f(t)\mathrm{d}t$ converges.
   (b) If $\int_0^\infty f(t)\mathrm{d}t$ converges, then $\lim_{x\to+\infty} f(x) = 0$.
   (c) If $\lim_{x\to+\infty} f(x) = a$ and $\int_0^\infty f(t)\mathrm{d}t$ converges, then $a = 0$.