Probability

Important principle of probability:

(a) $0 \leq \text{P(any event)} \leq 1$

(b) $\displaystyle\sum_{\textit{all outcomes}} \text{P(outcomes)} = 1$

1. A *sample space* is the set of all possible outcomes from some experiment. What is the sample space of the roll of a single six-sided die? What is the sample space of a roll of two dice (hint: there are 36 different outcomes)?

2. What is the probability of rolling a 7 on a single roll of two dice?
What is the probability of rolling an 11 on a single roll of two dice?

3. What is the probability of rolling a 7 on the first roll of the dice and then rolling an 11 on the second roll of the dice?

4. Let $\mathbb{P}$ be shorthand for probability. If event A is rolling ≤ 3 on the first roll of one die and event B is rolling ≥ 4 on the second roll of one die, what is the $\mathbb{P}(A \text{ and } B)$? Does $\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B)$?

Definition: A and B are *independent* if and only if $\mathbb{P}(A \text{and} B) = \mathbb{P}(A)\mathbb{P}(B)$.

5. Let C=the event of rolling between 5 and 8 (inclusively) on a single roll of two dice. Let D=the event of rolling between 7 and 10 (inclusively) on a single roll of two dice.
   (a) What is $\mathbb{P}(C)$?

   What is $\mathbb{P}(D)$?

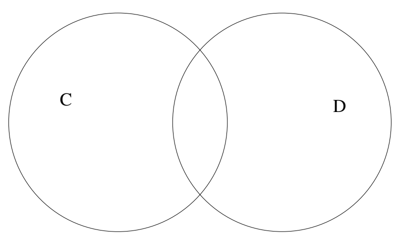
   What is $\mathbb{P}(C \text{ or } D)$?

   What is $\mathbb{P}(C \text{ and } D)$?

   (b) Does $\mathbb{P}(C \text{ or } D) = \mathbb{P}(C) + \mathbb{P}(D)$?

   (c) Are C and D independent events

6. Write a formula that gives $\mathbb{P}(C \text{ or } D)$ as some combination of $\mathbb{P}(C)$, $\mathbb{P}(D)$ and $\mathbb{P}(C \text{ and } D)$. (The picture below may help you.) This formula is called the Addition Rule.



7. A bowl contains 16 chips, of which 6 are red, 7 are white, and 3 are blue. If three chips are taken at random and with replacement.
   (a) Find the probability that each of the chips are red.

   (b) Find the probability that none of the chips are red.

   (c) Find the probability that at least one chip is red.

Suppose S is the sample space, the set of all possible outcomes of an experiment. A random variable, X, is a function that assigns a real number to each element of the sample space, S. In symbolic terms, we can say $X : S \to \mathbb{R}$.

Example:

$$\begin{aligned}\text{Let } S &= \text{possible outcomes of flipping a coin 3 times}\\ &= \{\text{HHH,HHT,HTH,THH,HTT,THT,TTH,TTT}\}\\ \text{Let } X &= \text{the number of heads}\end{aligned}$$

1. What is the domain of X?

   What is the range of X?

2. Let S= set of outcomes of rolling two dice. Consider the random variable X=sum of the two dice and whose domain is S. How many elements are in the domain of X? What is the range of X? Consider the random variable, Y=product of the two dice and whose domain is S. How many elements are in the domain of Y? What is the range of Y

3. Consider again the bowl which contains 16 chips, of which 6 are red, 7 are white, and 3 are blue. Let R=sample space of outcomes of the experiment of choosing three chips at random and with replacement and let Z=the number of red chips chosen.

   Is Z a random variable?

   Find the following probabilities:

   $\mathbb{P}(Z = 0) =$_____

   $\mathbb{P}(Z = 1) =$_____

   $\mathbb{P}(Z = 2) =$_____

   $\mathbb{P}(Z = 3) =$_____

   What is the sum of these probabilities?

   What is $\mathbb{P}(Z = \frac{1}{2})$?

4. If Z is a random variable and $x \in \mathbb{R}$, then $p(x) = \mathbb{P}(Z = x)$ is called a mass density or a probability mass density or a probability density function or a pdf. Note that $\sum_{all\ x} p(x) = 1$.

   The range of $p(x)$ is always a subset of what interval?

   Note that in number 3 above you described a pdf. What is the domain of that pdf?

   What is the range of that pdf?

5. Consider the experiment: roll a single die until a 6 appears. Let X=number of flips until a 6 appears.

   What are the possible values of X (note that the sample space is not finite)?

   What is $p(1)$?

   $p(2)$?

   $p(3)$?

   $p(4)$?

   $p(n)$?

   What is $\sum_{k=1}^{\infty} p(k)$?

6. You own one share of stock for three years and each year the value of the stock either rises by one dollar (with probability=$\frac{2}{3}$) or declines by one dollar (with probability=$\frac{1}{3}$); it never stays the same. Suppose the yearly changes are independent of each other. What is the sample space? What is the probability of each element in the sample space? Let X be the profit after three years. What are the possible values of X? What is the pdf of X?

7. If a fair coin is tossed three times and X=the number of heads minus the number of tails, write down the pdf for X.

8. Circle the correct words in the following statement: The (domain/range) of the mass density function f is the (domain/range) of the random variable X

Homework

9. Consider the experiment of flipping a biased coin (which comes up heads with probability $\frac{3}{4}$) twice. Let X be the number of heads minus the number of tails. What are the possible values of X? Find the pdf of X.

10. Consider the experiment of flipping a fair coin three times. Let X be the number of heads minus the number of tails. What are the possible values of X? Find the pdf of X.

11. An encyclopedia salesman visits three customers each day and with each he has a probability of $\frac{1}{4}$ of making a sale (assume also that these events are independent). For each sale he earns a commission of $100 and if he makes three sales in one day, he earns a $50 bonus from his company. Let X be his daily earnings. What are the possible values of X? What is the pdf?

12. A die is painted so that three sides are red, two sides are blue, and one side is green.
    (a) What is the probability that the die will come up blue?

    What is the probability that the die will not come up red?

    What is the probability that the face showing is either red or blue?

    (b) Suppose now that the die is rolled twice. Denote the nine possible outcomes as RR,RB, etc. Find the probability of each element of the sample space.

    What is the probability that at least one roll is red?

    What is the probability that neither roll is blue?

    What is the probability that the two rolls will have different colors?

    (c) Use the addition rule to find the following:

    the probability that either both rolls are red or both rolls are blue.

    the probability that either both rolls are red or exactly one roll is blue.

    the probability that either at least one roll is red or exactly one roll is blue.

    the probability that either at least one roll is red or at least one roll is blue.

13. If we roll a fair die, what is the probability that after 6 rolls we:

    (a) do not get a 6?

    (b) get a 6 on the first roll, but not after?

    (c) get exactly one 6?

14. Suppose that a fair die is rolled twice. Let A be the event that the first roll is ≤ 4. Let B be the event that the second roll is ≤ 4. Let C be the event that the sum of the rolls is ≤ 4.

    (a) Prove that A and B are independent.

    (b) Prove that A and C are not independent. Why does this make sense?

15. In a version of the game Two Finger Morra, each of two players show one or two fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he or she wins an amount (in dollars) equal to the sum of the fingers shown by the two players. If none or both of the players guess correctly, then no one wins anything. If X=amount of winnings for player one, describe the probability density function, assuming all sixteen possible events are equally likely.

	Player 1 shows 1, guesses 1	Player 1 shows 1, guesses 2	Player 1 shows 2, guesses 1	Player 1 shows 2, guesses 2
Player 2 shows 1, guesses 1				
Player 2 shows 1, guesses 2				
Player 2 shows 2, guesses 1				
Player 2 shows 2, guesses 2				

Probability: Expected Value

1. Suppose a class has the following twenty quiz grades: 2 3 3 3 4 4 4 4 4 4 5 5 5 5 5 6 6 6 6. Describe two ways to compute the average grade.

2. Find the average of the items with numerical values $x_1, x_2, \ldots, x_n$ that occur with frequency $p_1, p_2, \ldots, p_n$ respectively.

3. If $p_1, p_2, \ldots, p_n$ represent the probabilities associated, respectively, with the outcomes $x_1, x_2, \ldots, x_n$ of a random variable, then we call $\sum_{i=1}^{n} p_i x_i$ the <u>expectation</u>, <u>expected value</u>, <u>mean</u>, or <u>average value</u>. More generally, if $p(x)$ is the probability density function for a random variable X, we call $\sum_{all\ x} xp(x)$ the expected value of X and use the notation $\mathbb{E}(X)$.

4. Suppose the following table gives the percentage of passengers (including the driver) in cars passing through a certain intersection:

1	2	3	4	5	6	7
.52	.27	.11	.05	.02	.02	.01

Find P($2 \leq x \leq 5$). Find the expected value of the distribution.

5. Consider once again the bowl which contains 16 chips, of which 6 are red, 7 are white, and 3 are blue and consider the sample space of outcomes of the experiment of choosing three chips at random and with replacement. Let Z=the number of red chips chosen. Determine the following probabilities:

   $\mathbb{P}(Z = 0)$=________

   $\mathbb{P}(Z = 1)$=________

   $\mathbb{P}(Z = 2)$=________

   $\mathbb{P}(Z = 4)$=________

   Find the expected value of Z.

6. Suppose you run a (very small) insurance agency and suppose that:
   (i) each of your three clients has a 20% probability of getting into an accident during the year
   (ii) for each accident this year, you must pay \$4,000
   (iii) you pay your friend, Rae the Reinsurer, a flat fee of \$200 and in exchange, Rae promises to reimburse you \$3,000 in case all three of your clients have accidents this year.

   (a) How much should you charge each client so that your expected net profit is \$500? (First, decide what the random variable is and its possible outcomes. Then after writing down the pdf for that random variable, it should be easy to compute the expected value).

   (b) What is Rae's expected profit from you?

Homework

7. Find $\mathbb{E}(X)$ of the random variable X in problems 9-11 on page 87 in this coursepack.

8. Consider the following game. We choose a ball at random from an urn containing 7 red, 3 green and 2 amber balls. We win \$2 if we choose a green ball, we lose \$1 if we choose a red ball, and we don't win or lose any money if we choose an amber ball. Let X represent our winnings. What values could X possibly assume? What is the probability density function? Find $\mathbb{E}(X)$.

9. Consider the following game. We choose two balls at random (with replacement) from an urn containing 6 red balls and 4 green balls. You win \$2.00 each time you pick a green ball and lose \$1.50 each time you pick a red ball. Let X represent your winnings.

   (a) What values could X possibly assume? What is the pdf? What is the expected value of X?

   (b) Continue to assume that you win \$2.00 each time you pick a green ball. In order to make it a fair game (i.e. with expected value zero) how much should you lose when you pick a red ball?

10. Suppose we roll two six sided dice. Let X represent the absolute value of the difference of the values that come up. What values could X possibly assume? What is the probability density function? What is the expected value of X? The following may be helpful:

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

11. Consider the following game played at a casino. A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, for $i = 1, 2, 3$, then the player wins i dollars. On the other hand, if the number bet by the player does not appear on any of the dice, then the player loses \$1. What is the player's expected winnings?

12. Consider again the die from example 12 on page 92 of this coursepack and assume that a casino offers a game with the following rules:
(i) you pay \$10 to play
(ii) if both rolls are red, the casino pays you \$10 (for a net gain of \$0)
(iii) if both rolls are blue, the casino pays you \$25 (for a net gain of \$15)
(iv) if both rolls are green the casino pays you \$100 (for a net gain of \$90)
What is the player's expected winnings

13. Suppose that we have a biased coin which comes up tails $\frac{5}{8}$ of the time. Consider the experiment of flipping the coin until a heads comes up and let X be the number of flips required. What values can X take? What is the pdf of X? Find an infinite sum for $\mathbb{E}(X)$ and use a spreadsheet to estimate $\mathbb{E}(X)$.