# Problem Set #3 Solutions

1. Suppose function $f$ passes through the following points:

| x | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| f(x) | 2 | -1 | 7 | 2 | 5 | 8 | 5 |

   (a) Approximate $\int_0^{12} f(x)dx$ using the **smallest** Riemann Sum with 3 rectangles of equal width.

   *Solution*

$$\begin{aligned}\int_0^{12} f(x)dx &\approx (-1)\cdot\frac{12}{3} + (2)\cdot\frac{12}{3} + (5)\cdot\frac{12}{3}\\ &= -4+8+20\\ &= 24\end{aligned}$$

   (b) Approximate $\int_0^8 (f(x))^2dx$ using the **Trapezoid Rule** with 2 rectangles of equal width.

   *Solution*

$$\begin{aligned}\int_0^{8} (f(x))^2dx &\approx \frac{2^2+7^2}{2}\cdot\frac{8}{2} + \frac{7^2+5^2}{2}\cdot\frac{8}{2}\\ &= 106+148\\ &= 254\end{aligned}$$

   (c) If by using a LHS, we could approximate $\int_1^5 f(x)dx$ by $\sum_{k=0}^{6} f(1+\frac{4k}{7})\cdot\frac{4}{7}$. If we instead want to approximate $\int_6^{10} f(x)dx$ with the same number of rectangles, how should we adjust the Riemann Sum?

   *Solution* $\sum_{k=0}^{6} f\big(6+\frac{4k}{7}\big)\cdot\frac{4}{7}$

2. Evaluate the following integrals. (First write down the definite integral and then use FTC or area to compute the integral.)

   (a) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(4-3\left(\frac{2i}{n}\right)^2+6\left(\frac{2i}{n}\right)^5\right)\cdot\frac{2}{n}$

   *Solution*

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(4-3\left(\frac{2i}{n}\right)^2+6\left(\frac{2i}{n}\right)^5\right)\cdot\frac{2}{n} &= \int_0^2 (4-3x^2+6x^5)dx\\ &= 4x-x^3+x^6\Big|_0^2\\ &= 8-8+64\\ &= 64\end{aligned}$$

   (b) $\lim_{n\to\infty}\sum_{i=0}^{n-1}\sqrt{4-\left(\frac{4i}{n}-2\right)^2}\cdot\frac{4}{n}$

   *Solution*

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1}\sqrt{4-\left(\frac{4i}{n}-2\right)^2}\cdot\frac{4}{n} &= \int_{-2}^{2}\sqrt{4-x^2}dx \quad \text{(see remark below for more explanation of this step)}\\ &= \text{area of semi-circle center at } (0,0) \text{ with radius 2.}\\ &= \frac{1}{2}\cdot\pi 2^2\\ &= 2\pi\end{aligned}$$

*Remark* Can also think of the integral first as $\int_0^4 \sqrt{4-(x-2)^2}dx$. Let $u = x-2, du = dx$, and $u$ goes from $-2$ to 2. Then it becomes $\int_{-2}^2 \sqrt{4-u^2}du$.

(c) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{-1}{n\cdot(1+\left(\frac{k}{n}\right)^2)}$

*Solution*

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \frac{-1}{n\cdot(1+\left(\frac{k}{n}\right)^2)} &= \int_0^1 \frac{-1}{1+x^2}dx \\ &= \cot^{-1}(x)\Big|_0^1 \\ &= \cot^{-1}(1)\end{aligned}$$

3. (a) *Solution* By FTC II, we have that

$$\begin{aligned}\frac{d}{dx}\int_0^x e^{-t^2}\,dt &= e^{-t^2}\Big|_{t=x} \\ &= e^{-x^2}\end{aligned}$$

(b) *Solution* Let $h(x) = \int_1^x \ln(t+t^2)\,dt$. By FTC II, we note that $h'(x) = \ln(x+x^2)$. The question, however, is asking us to find $\frac{d}{dx}h(x^2)$. Using Chain Rule,

$$\begin{aligned}\frac{d}{dx}h(x^2) &= 2xh'(x^2) \\ &= 2x\ln(x^2+x^4).\end{aligned}$$

(c) *Solution* To simplify notation, write $f(t) = \frac{\sin t + 1}{\cos^3 t}$, and we want to find

$$\frac{d}{dx}\int_0^t f(t)\,dt - \int_0^x f'(t)\,dt.$$

By FTC II, the first term is simply $f(x)$. By FTC I, on the other hand, the second term is $f(x) - f(0)$. Therefore, their difference is $f(0) = 1$.

4. Use u-substitution to find the following integrals.

(a) $\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

*Solution* Let $u = \sqrt{x}, du = \frac{1}{2}\frac{1}{\sqrt{x}}dx$. $x$ goes from 4 to 9, then $u$ goes from 2 to 3.

$$\begin{aligned}\int_4^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx &= \int_2^3 2e^u du \\ &= 2e^u\Big|_2^3 \\ &= 2e^3 - 2e^2\end{aligned}$$

(b) $\int_0^1 \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx$

*Solution* Let $x = \sin(u), u = \sin^{-1}(x), dx = \cos(u)du$. $x$ goes from 0 to 1, then $u$ goes from 0 to $\frac{\pi}{2}$.

$$\begin{aligned}\int_0^1 \frac{\sin^{-1}(x)}{\sqrt{1-x^2}}dx &= \int_0^{\frac{\pi}{2}} \frac{u}{\sqrt{1-\sin^2(u)}} \cdot \cos(u)du \\ &= \int_0^{\frac{\pi}{2}} u \cdot du \\ &= \frac{1}{2}u^2\Big|_0^{\frac{\pi}{2}} \\ &= \frac{\pi^2}{8}\end{aligned}$$

(c) If $f$ is continuous and even, and $\int_{-9}^{9} f(x)dx = 8$, find $\int_0^3 xf(x^2)dx$.

*Solution* Let $u = x^2, du = 2xdx$. $x$ goes from 0 to 3, then $u$ goes from 0 to 9.

$$\begin{aligned}\int_0^3 xf(x^2)dx &= \int_0^9 \frac{1}{2}f(u)du \\ &= \frac{1}{2}\int_0^9 f(x)dx \\ &= \frac{1}{2}\cdot\Big(\frac{1}{2}\int_{-9}^9 f(x)dx\Big) \\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot 8 \\ &= 2\end{aligned}$$

5. (a) We need to decompose the integrand into partial fraction. Suppose $A, B$ satisfy

$$\frac{13x+7}{(3x-1)(2x+5)} = \frac{A}{3x-1} + \frac{B}{2x+5},$$

then we have

$$\begin{aligned}\frac{13x+7}{(3x-1)(2x+5)} &= \frac{A}{3x-1} + \frac{B}{2x+5} \\ &= \frac{A(2x+5)+B(3x-1)}{(3x-1)(2x+5)} \\ &= \frac{(2A+3B)x+5A-B}{(3x-1)(2x+5)}.\end{aligned}$$

Thus $2A+3B = 13, 5A-B = 7$. So we have $A = 2, B = 3$. Thus

$$\begin{aligned}\int \frac{13x+7}{(3x-1)(2x+5)}\mathrm{d}x &= \int \frac{2}{3x-1} + \frac{3}{2x+5}\mathrm{d}x \\ &= \frac{2}{3}\ln|3x-1| + \frac{3}{2}\ln|2x+5| + C.\end{aligned}$$

(b) trig substitution works for this integral, but an easier way is using simple u-substitution and linearity: Let $u = x^2$, then

$$\begin{aligned}\int \frac{2x+1}{\sqrt{1-x^2}}\mathrm{d}x &= \int \frac{2x}{\sqrt{1-x^2}}\mathrm{d}x + \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\ &= \int \frac{1}{\sqrt{1-u}}\mathrm{d}u + \arcsin x \\ &= -2\sqrt{1-u} + \arcsin x + C \\ &= -2\sqrt{1-x^2} + \arcsin x + C.\end{aligned}$$

(c) Let $x = \tan\theta$, then $dx = \sec^2\theta \mathrm{d}\theta$. We have

$$
\begin{aligned}
\int \frac{1}{(1+x^2)^2}\mathrm{d}x &= \int \frac{1}{(\sec^2\theta)^2} \cdot \sec^2\theta \mathrm{d}x \\
&= \int \cos^2\theta \mathrm{d}\theta \\
&= \int \frac{1+\cos 2\theta}{2}\mathrm{d}\theta \\
&= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C
\end{aligned}
$$

Remember that $x = \tan\theta$, so $\theta = \mathrm{acrtan}x$ and

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\tan\theta\cos^2\theta = \frac{2x}{1+x^2}.$$

So

$$\int \frac{1}{(1+x^2)^2}\mathrm{d}x = \frac{\mathrm{arctan}x}{2} + \frac{x}{2(1+x^2)} + C.$$

6. Define the dirichlet function(see wiki or google it, which is a quite important example in math) on [0,1]:$f(x) = 0$ when $x$ is irrational number,while $f(x) = 1$ when $x$ is rational number.

   The key is to notice that the minimum of f in each interval is 0, why the midpoint,right/left hand point all have f=1, as they are rational($i/2n$)

   (a) RHS(3)=LHS(3)=MPS(3)=1
   (b) RHS(n)=LHS(n)= MPS(n)=1
   (c) Min(3)=0
   (d) No, as $\lim_{n\to\infty}$Min(n)=0, while $\lim_{n\to\infty}$RHS(n)=1(Just a reminder that it's not enough to check only the four sums we've learned!)