Problem Set #2 Solutions

1. (a) Since the interval between measurements is 10 s, the right hand sum is

$$10(4 + 10 + 16 + 21 + 28) = 790.$$

So our estimate is that the plane takes 790 m to take off.

(b) Although our estimate is longer than the available runway, since the speed is increasing, the right hand sum is an overestimate, so we cannot be sure that the actual distance travelled by the plane will be longer than 700 m, so we cannot conclude whether the runway is long enough.

(c) Using trapezoid rule gives an estimate of

$$10\left(\frac{0+4}{2} + \frac{4+10}{2} + \frac{10+16}{2} + \frac{16+21}{2} + \frac{21+28}{2}\right) = 650$$

(d) Although our estimate is shorter than the available runway, we don't have any information about the concavity of $v(t)$, so we don't know whether our estimate is an under- or overestimate. Therefore, we cannot conclude whether the runway is long enough.

2. $\displaystyle\int_{-1}^{3}(3-2x)dx$

(a) Evaluate this integral as a limit of Riemann sums. (Your answer should be a number.)

Solution With n subintervals we have

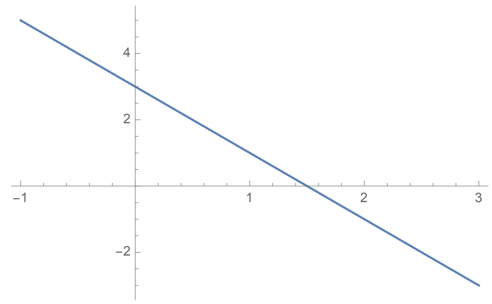
$$\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}$$

Thus $x_0 = -1, x_1 = -1 + 4/n, x_2 = -1 + 8/n, x_3 = -1 + 12/n$, and, in general, $x_i = -1 + 4i/n$. Hence, using right endpoints,we have

$$\begin{aligned}
\int_{-1}^{3}(3-2x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\Big(-1+\frac{4i}{n}\Big)\frac{4}{n} \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\Big[3-2\Big(-1+\frac{4i}{n}\Big)\Big] \\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\Big(5-\frac{8i}{n}\Big) \\
&= \lim_{n\to\infty}\Big(\frac{4}{n}\sum_{i=1}^{n}5-\frac{32}{n^2}\sum_{i=1}^{n}i\Big) \\
&= \lim_{n\to\infty}\Big[\frac{4}{n}\cdot 5n-\frac{32}{n^2}\cdot\frac{n(n+1)}{2}\Big] \\
&= \lim_{n\to\infty}\Big[20-\frac{16n(n+1)}{n^2}\Big] \\
&= 20-16 \\
&= 4
\end{aligned}$$

(b) Evaluate the integral by interpreting it in terms of areas. Plot the function and indicate the corresponding area(s).

Solution The areas that are enclosed by this function are two triangles one above the x-axis and one below the x-aix in the figure below.



The vertices of the triangle above are $(-1,0),(-1,5),(\frac{3}{2},0)$, and the vertices of the triangle below are $(\frac{3}{2}),(3,0),(3,-3)$.

$$\begin{aligned}\int_{-1}^{3}(3-2x)dx &= \text{Areas enclosed by the function counting sign} &(1)\\ &= 5\cdot\frac{5}{2}\cdot\frac{1}{2}+\Big(-3\cdot\frac{3}{2}\cdot\frac{1}{2}\Big) &(2)\\ &= \frac{25}{4}-\frac{9}{4} &(3)\\ &= 4 &(4)\end{aligned}$$

3. (a) We choose a=0 as the starting point, and $\Delta x = 1/n$ be the width of each rectangle. Then $x_i = a + i\cdot\Delta x = i/n$, and $b = x_n = 1$. For S_n, we have

$$\begin{aligned}S_n &= \sum_{i=1}^{n}\frac{n}{i^2+n^2}\\ &= \sum_{i=1}^{n}\frac{1}{n}\cdot\frac{n^2}{i^2+n^2}\\ &= \sum_{i=1}^{n}\Delta x\cdot\frac{1}{\left(\frac{i}{n}\right)^2+1}\\ &= \sum_{i=1}^{n}\Delta x\cdot\frac{1}{x_i^2+1}.\end{aligned}$$

So our integrand $f(x)$ should be $f(x) = \dfrac{1}{1+x^2}$, and $S_n = \sum_{i=1}^{n}\Delta x\cdot f(x_i) = \text{RHS}(n)$. remark: Student can get full credit as long as they point out $a, b, f, \Delta x$ and transform S_n properly into a RHS, they don't have to write exactly the same as the solution.

(b) By definition of integral, we have

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\text{RHS}(n) = \int_0^1\frac{1}{1+x^2}\mathrm{d}x.$$

Because $1/(1+x^2)$ is continuous on $[0,1]$, and $\arctan(x)$ is an antiderivative of it, so by Fundamental Theorem of Calculus, we have

$$\int_0^1\frac{1}{1+x^2}\mathrm{d}x = \arctan(x)|_0^1 = \frac{\pi}{4}.$$

Thus limit of S_n as n approaches infinity is $\dfrac{\pi}{4}$.

(c) Let's compute S_1, S_2, S_3.

$$S_1 = \sum_{i=1}^{1} \frac{1}{i^2+1} = \frac{1}{2} = 0.5.$$

$$S_2 = \sum_{i=1}^{2} \frac{2}{i^2+4} = \frac{2}{5} + \frac{2}{8} = \frac{13}{20} = 0.65.$$

$$S_3 = \sum_{i=1}^{3} \frac{3}{i^2+9} = \frac{3}{10} + \frac{3}{13} + \frac{3}{18} = \frac{7}{15} + \frac{3}{13} = \frac{136}{195} \approx 0.697.$$

From (b), we know that $S_\infty = \pi/4 > 0.75$, so $S_1 < S_\infty, S_2 < S_\infty, S_3 < S_\infty$.

(d) Claim: $S_n < S_\infty, n = 1, 2, ...$

i. Notice that $f(x)$ is decreasing on $[0, 1]$, so RHS always underestimates the integral, so

$$S_n < \int_0^1 \frac{1}{1+x^2} \mathrm{d}x = S_\infty.$$

(A picture of RHS is also accepted.)

ii. Just repeat the proof of FTC. For any given n, we have

$$\int_0^1 \frac{1}{1+x^2} \mathrm{d}x = \arctan(1) - \arctan\left(\frac{n-1}{n}\right) + \arctan\left(\frac{n-1}{n}\right) - \arctan\left(\frac{n-2}{n}\right) + ...$$
$$+ \arctan\left(\frac{1}{n}\right) - \arctan(0)$$
$$= \sum_{i=1}^{n} \arctan\left(\frac{i}{n}\right) - \arctan\left(\frac{i-1}{n}\right).$$

By MVT, there exists $x_i^* \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$ s.t.

$$\arctan\left(\frac{i}{n}\right) - \arctan\left(\frac{i-1}{n}\right) = \frac{1}{n} \cdot (\arctan)'(x_i^*) = \frac{1}{n} \cdot \frac{1}{1+(x_i^*)^2}.$$

Because $f(x) = 1/(1+x^2)$ is decreasing, so $1/(1+(x_i^*)^2) > 1/(1+x_i^2)$. So

$$\int_0^1 \frac{1}{1+x^2} \mathrm{d}x = \sum_{i=1}^{n} \arctan\left(\frac{i}{n}\right) - \arctan\left(\frac{i-1}{n}\right)$$
$$= \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(x_i^*)^2}$$
$$> \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(x_i)^2} = S_n.$$

This is exactly $S_\infty > S_n, n = 1, 2, ...$

4. The key is to find that f is either increasing/decreasing/easy to find maximal in each interval

(a) we can find that there are two parts of sum terms:from 1 to n and from n+1 to 2n, where on each part the maximum is either Left hand value and Right hand value.$S_{2n}^{max} = \Sigma_{i=0}^{n} m_i * \Delta x_i + \Sigma_{i=n+1}^{2n} m_i \Delta x_i = \Sigma_{i=1}^{n} (\frac{n-i+1}{n})^2 \frac{1}{n} + \Sigma_{i=n+1}^{2n} (1 - \frac{i-n}{n})^2 \frac{1}{n} = 2\Sigma_{i=1}^{n} (\frac{i}{n})^2 \frac{1}{n}$(Or just by symmetry)

(b) for $i < n+1$ and $i > n+1$,the maximum is either Left hand value and Right hand value. When $i = n+1$,we know it's both right hand value and left hand value $m_{n+1} = (\frac{1}{2n+1})^2$. So $S_{2n+1}^{max} = \Sigma_{i=1}^{n} (1 - \frac{2i-2}{2n+1})^2 \frac{2}{2n+1} + \Sigma_{i=n+2}^{2n+1} (\frac{2i-2n-1}{i-n})^2 \frac{2}{2n+1} + \frac{2}{(2n+1)^3} = 2\Sigma_{i=1}^{n} (\frac{2i+1}{2n+1})^2 \frac{2}{2n+1} + \frac{2}{(2n+1)^3}$

(c) $lim_{n\to\infty} S_{2n} = \int_{-1}^{1} x^2 dx = 2/3$ as we can divide the sum it two parts: from 1 to n and from n+1 to 2n. On each part, the upper Darboux sum is either RHS and LHS. So the limit converge to integral. In the case of S_{2n+1}^{max}, all are the same except for the n+1 th term. But notice that it will go to zero as $n \to infinity$, so we know $S_n^{max} \to 2/3$

5. (a)

$$\begin{aligned}
\lim_{x\to 0} \frac{5^x - 2^x}{x^5 - x} &= \left(\frac{0}{0}\right) \qquad \text{(Indeterminate form)} \\
&= \lim_{x\to 0} \frac{\ln(5)5^x - \ln(2)2^x}{5x^4 - 1} \qquad \text{(By L'HR)} \\
&= \ln(2) - \ln(5) \\
&= \ln\left(\frac{2}{5}\right)
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to\infty} x^{\frac{1}{x}} &= \lim_{x\to\infty} e^{\ln(x^{\frac{1}{x}})} \\
&= e^{\lim_{x\to\infty} \frac{\ln(x)}{x}}
\end{aligned}$$

Evaluating the limit of the exponent,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)}{x} &= \left(\frac{\infty}{\infty}\right) \qquad \text{(Indeterminate form)} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \qquad \text{(By L'HR)} \\
&= 0
\end{aligned}$$

Therefore, the original limit evaluates to $e^0 = 1$.

(c)

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x^2)}{\sin^2 x} &= \frac{0}{0} \qquad \text{(Indeterminate form)} \\
&= \lim_{x\to 0} \frac{x\cos(x^2)}{\sin x \cos x} \qquad \text{(By L'HR)} \\
&= \left(\frac{0}{0}\right) \qquad \text{(Indeterminate form)} \\
&= \lim_{x\to 0} \frac{\cos(x^2) - 2x^2\sin(x^2)}{\cos^2 x - \sin^2 x} \qquad \text{(By L'HR)} \\
&= 1
\end{aligned}$$