HW 2:

5(b))

$$\lim \sum \frac{2k^2}{n^3}$$

$$x_k = \frac{2k}{n}$$

$$= \lim \sum_{k=1}^{n} \left(\frac{k}{2}\cdot\frac{2}{n}\right)^2 \frac{2}{n}$$

$$= \lim \sum \frac{x_k^2}{4}\,\Delta x$$

$$= \int_0^2 \frac{x^2}{4}\,dx$$

$$= \frac{2}{3}$$

Week 4. 9.16

Integration by Parts.

Last time, we use chain rule to get antiderivatives, now, we want to use product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integrate two side, we get

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x) + C.$$

$\Longrightarrow$ as $g'(x)dx = dg(x)$, $df(x) = f'(x)dx$.

$$\int f(x)dg(x) + g(x)df(x) = f(x)g(x).$$

Reorganize it, we get:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)dx + C.$$

$u = f(x),\ v = g(x).$

or $$\int u\,dv = uv - \int v\,du$$

General idea: $\boxed{du \text{ is simpler than } u}$

Ex1: $\int x\sin(x)\,dx$ $\qquad u = x,\ v = -\cos x$

$= -x\cdot\cos(x) + \int \cos x\,dx$

$= \sin(x) - x\cdot\cos x + C$

Ex2: $\int \ln(x)\,dx$

$= x\ln(x) - \int x\cdot\frac{1}{x}dx$

$= x\ln(x) - x + C$

Ex3: $\int t^2e^t\,dt$ $\qquad \int 2t\,de^t$ $\qquad \shortparallel$

$= \int t^2 de^t = t^2e^t - \int 2te^t dt$

$= t^2e^t - 2te^t + \int 2e^t dt$

$= t^2e^t - 2te^t + 2e^t + C.$

Ex 4: $\int e^x \sin(x)\,dx$ $\qquad -\left(e^x\cos(x) + \int \sin(x) e^x\,dx\right)$

$$= \int \sin(x)\,de^x = \sin(x)e^x - \int \cos(x)e^x\,dx =$$

$$= (\sin(x) - \cos(x))e^x - \int \sin(x)e^x\,dx$$

$$\Longrightarrow 2\int e^x \sin(x)\,dx = (\sin(x) - \cos(x))e^x + C$$

$$\Longrightarrow \int e^x \sin(x)\,dx = \frac{\sin(x) - \cos(x)}{2}e^x + C'$$

For definite integrals:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Ex: $\int_0^1 \tan^{-1}(x)\,dx$

$$= x\tan^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2}$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Ex 5: Prove the reduction formula:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$u = \sin^{n-1}(x)$, $v = -\cos(x)$

$$\int u\,dv$$

$$= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\,dx$$

$$= -\sin^{n-1}(x)\cos(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}x\,dx$$

$$= -\sin^{n-1}(x)\cos(x) - (n-1)\int \sin^n(x)\,dx + (n-1)\int \sin^{n-2}(x)\,dx$$

$$\Longrightarrow n\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)$$

$$\Longrightarrow \int \sin^n(x)\,dx = \frac{-\cos(x)\cdot\sin^{n-1}(x)}{n} + \frac{n-1}{n}\cdot\int \sin^{n-2}(x)\,dx$$

Ex 5: $\int x\cdot\sin(x)\cos(x)\,dx = \int x\cdot\sin(x)\,d\sin(x)$

$$= x\cdot\sin^2 x - \int \sin(x)\,d(x\sin(x))$$

$$= x\sin^2(x) - \int \sin^2(x)\,dx - \int \sin(x)\cos(x)\,x\,dx$$

$$\Longrightarrow 2\int \sin(x)\cdot\cos(x)\,dx = x\cdot\sin^2(x) - \int \sin^2(x)\,dx$$

$$\int \sin(x)\cos(x)\,dx = \frac{1}{2}x\sin^2(x) - \frac{1}{4}x + \frac{1}{8}\sin(2x) + C$$

Ex 6: $\int e^{6x}\sin(e^{3x})\,dx$

$e^{3x} = u, \Longrightarrow dv = e^{3x}\sin(e^{3x}) \Longrightarrow v = -\frac{1}{3}\cos(e^{3x})$

$$\int e^{6x}\sin(e^{3x})\,dx = -\frac{1}{3}e^{3x}\cos(e^{3x}) + \int e^{3x}\cdot\cos(e^{3x})\,dx$$

$$= -\frac{1}{3}e^{3x}\cos(e^{3x}) + \frac{1}{3}\sin(e^{3x}) + C$$

Ex7: $\int_0^{\frac{1}{2}} \cos^{-1}(x)\,dx$.

recall: $(\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \int_0^{\frac{1}{2}} \cos^{-1}(x)\,dx = x\cos^{-1}(x)\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{6} - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{d(1-x^2)}{\sqrt{1-x^2}}$$

$$= \frac{\pi}{6} - \sqrt{1-x^2}\Big|_0^{\frac{1}{2}}$$

$$= \frac{\pi}{6} + \left(1 - \frac{\sqrt{3}}{2}\right)$$

Ex8: (求) $f(0) = g(0) = 0$, f'', g'' 連續, 求

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)$$

Ans: $\int_0^a f(x)g''(x)\,dx = \int_0^a f(x)\,dg'(x)$

$$= f(x)g'(x)\Big|_0^a - \int_0^a f'(x)g'(x)\,dx \quad (分部積分)$$

$$= f(a)g'(a) - \int_0^a f'(x)\,dg(x) \quad (分部積分)$$

$$= f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx.$$