Section 5.10 in Textbook

Improper Integral

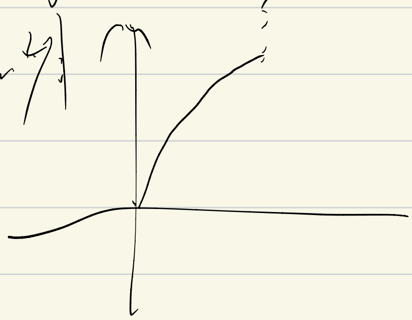
9.17/9.18.

Type of Discontinuity:

① Removable discontinuity

$\lim_{x\to a} f(x) = \lim_{x\to a} f(x) \neq f(a)$

② Jump discontinuity



$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

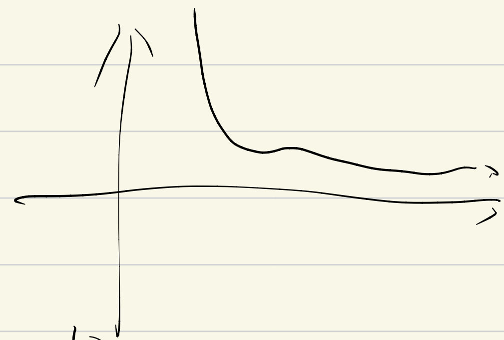
③ : infinite discontinuity: $\lim_{x\to a^-/a^+} f(x)$ DNE

Recall the condition of FTC: ① Continuous ② $[a,b]$ closed, bounded.

① If our interval is infinite, or ② f has discontinuity on $[a,b]$, we call the integral: improper integral.

Suppose we want to evaluate

$$\int_1^{\infty} \frac{1}{x^2} dx$$



let $t > 1$, Then $\int_1^t \frac{1}{x^2} dx = (1 - \frac{1}{t})$

we can take limit of this value

$$\lim \int_1^t \frac{dx}{x^2} = \lim (1 - \frac{1}{t}) = 1.$$

Therefore, we say that the area of the region is equal to 1, and we define

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t \to \infty} \int_1^t \frac{1}{x^2} dx$$

Definition of an improper integral of type 1:

(a) $\int_a^t f(x)$ exists for every number. then

$$\int_a^{\infty} f(x) dx = \lim_{t \to \infty} \int_a^t f(x) dx$$, provided this limit exists

(b) If $\int_t^b f(x) dx$ exist for every number $t \leq b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \to -\infty} \int_t^b f(x) dx$$, provided this limit exists.

$\int_{-\infty}^b f(x) dx$ and $\int_a^{+\infty} f(x) dx$ called converge if the corresponding limit exists (and $\neq \infty$), called diverge if the limit DNE.

(c) If both $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$$

In particular, a can be any finite number

Remark: $\int_{-\infty}^{+\infty} f(x)\,dx$ ~~does~~ doesn't equal to $\lim_{t\to+\infty}\int_{-t}^{t} f(x)\,dx$ exist.

Example $\boxed{f(x) = x}$

Ex: $\int_1^{\infty} \frac{dx}{x}$ is convergent or divergent?

Answer $\int_1^t \frac{dx}{x} = \ln(t) \to \infty$ as $t \to \infty$. divergent

Ex2: $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_0^{+\infty} \frac{1}{1+x^2} + \int_{-\infty}^{0} \frac{dx}{1+x^2}$ (0)

$$= \lim_{t\to\infty}(\arctan(t) - \arctan(0)) + \lim_{t\to-\infty}(\arctan(0) - \arctan(t))$$

$= \pi$ ✓, convergent.

Ex3: When $\int_1^{\infty} \frac{dx}{x^p}$ convergent? P's rule.

$p = 1 \Rightarrow$ $\ln(t) \to$ diverge

解 $\Rightarrow \lim_{x\to\infty} \frac{1}{(1-p)}(x^{1-p}-1)$

when $p > 1$, converge

$p < 1$, divergent.

Definition of an improper integral of type 2:

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx \quad \text{if the limit exist}$$

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx \quad \text{if the limit exist.}$$

The improper integral $\int_a^b f(x)\,dx$ is called convergent if the corresponding limit exist and is divergent if the limit DNE.

(c). If f has a discontinuity at c, where $a<c<b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(Else $\int_a^c f(x)dx + \lim_{t\to c}\left(\int_c^t f(x)dx + \int_t^b f(x)dx\right)$)

Example: Find $\int_2^5 \frac{1}{\sqrt{x-2}}dx$ (discontinuous at point 2)

$= \int_t^5 \left(\frac{1}{\sqrt{x-2}}\right)dx \quad 2\sqrt{x-2}\Big|_t^5 \to 2\sqrt{3}$ ⇒ converge.

Ex2. Evaluate: $\int_0^3 \frac{dx}{x-1}$

$\int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} \to$ diverge.

$\ln(x-1)\Big|_0^1 + \ln(x-1)\Big|_1^3$ ($\ln 0 = -\infty$)

9.18. (Ex 1)

Ex3: Evaluate $\int_0^1 \ln(x)dx$.

$\int_t^1 \ln(x)dx = x\ln(x) - x\Big|_t^1$

$= (-1) - (t\ln(t) - t)$

$\lim_{t\to 0^+}(\quad) = (-1)$ 1. ✓)

L'Hospital

How to define $\int_c^\infty f(x)dx$ c is discontinuous point.

$\int_c^d f(x) + \int_d^\infty f(x)dx$, as d is arbitrary.

A Comparison Test for Improper Integrals: [Related to that of finite integrals]

Comparison Theorem: Suppose f, g are continuous with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent

$\left(\int_a^\infty f(x) \geq \int_a^\infty g(x)\,dx\right)$

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Ex 1: $\int_0^\infty e^{-x^2}\,dx$.

Solution: $\underset{(t>1)}{\int_0^t e^{-x^2}\,dx} = \int_0^1 \underset{\underset{1}{\wedge\!|}}{e^{-x^2}}\,dx + \int_1^t \underset{\underset{e^{-x}}{\wedge\!|}}{e^{-x^2}}\,dx$

$\leq \int_0^\infty e^{-x^2}\,dx \leq 1 + \int_1^\infty e^{-x}\,dx$ — convergent

Ex 2: $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ convergent or divergent?

Answer: $\frac{1+e^{-x}}{x} \geq \frac{1}{x} \Longrightarrow \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(t) - 0 \Longrightarrow$ divergent

$\therefore \int \frac{1+e^{-x}}{x}$ divergent.

Ex3: $f(x)$ is continuous, positive that $\int_0^\infty f(x)dx$ converge, a is a positive number. Do ~~se~~ $\int_0^\infty f(ax)\,dx$ converge?

Answer: Yes!

$$\int_0^t f(ax)dx = \int_0^t \frac{1}{a} f(ax)\, a\,dx = \frac{1}{a}\int_0^t f(ax)\,d(ax) \qquad y = ax$$

$$= \frac{1}{a}\int_0^{at} f(y)\,dy$$

$$\therefore \lim \int_0^t f(ax)\,dx = \lim \frac{1}{a} \underbrace{\int_0^{at} f(y)\,dy}_{\text{conge}} \Rightarrow \text{convergent.}$$