Practice with Problem Solving Solutions

1. Suppose f is a function such that $f(x+y) = f(x)+f(y)+x^2y+xy^2$ and $\lim_{x\to 0}\frac{f(x)}{x} = 1$.

   (a) Find $f(0)$.

   Solution: Let $x = 0$ and $y = 0$: Then
   $f(x+y) = f(x) + f(y) + x^2y + xy^2 \implies f(0+0) = f(0) + f(0) \implies f(0) = 2f(0) \implies f(0) = \boxed{0}$.

   (b) Find $f'(0)$.

   Solution: $f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h} = \boxed{0}$

   (c) Find $f'(x)$.

   Solution: $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h}$

   $= \lim_{h\to 0}\frac{f(h)}{h} + \lim_{h\to 0} x^2 + xh = \boxed{1 + x^2}$

2. Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\sqrt[n]{\left(1+\frac{k}{n}\right)^2}\right)$

   Solution: $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\sqrt[n]{\left(1+\frac{k}{n}\right)^2}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\left(1+\frac{k}{n}\right)^2\right)\frac{1}{n}$

   $= \int_1^2 \ln(x^2)\, dx = 2\int_1^2 \ln(x)$

   $u = \ln(x) \qquad dv = \ dx$

   $du = \frac{1}{x}dx \qquad v = x$

   $= 2\int_1^2 \ln(x) = 2x\ln(x)\Big|_1^2 - 2\int_1^2 (1)\, dx = 2\Big(x\ln(x) - x\Big)\Big|_1^2 = (4\ln(4) - 4) - (2\ln(1) - 2)$

   $= \boxed{4\ln(4) - 2}$

3. Let $f(x)$ have one zero, at $x = 3$, and suppose $f'(x) < 0$ for all x and that

$$\int_0^3 f(t)\, dt = -\int_3^5 f(t)\, dt$$

   Define $F(x) = \int_0^x f(t)\, dt$ and $G(x) = \int_1^x F(t)\, dt$.

(a) Find the zeros of $F(x)$.

Solution: Note that $\int_0^3 f(t)\ dt + \int_3^5 f(t)\ dt = 0$

$F(x) = 0$ at $\boxed{\text{x=0}}$ and $\boxed{\text{x=5}}$

(b) Find all critical points of $F(x)$ and classify each as a local min, a local max, or neither.

Solution: $F'(x) = f(x) \Longrightarrow F'(x) = 0$ when $\boxed{\text{x=3}}$. Because $f(x) < 0$, $f(x) > 0$ when $x < 3$ and $f(x) < 0$ when $x > 3$. Thus, $F(x)$ has a $\boxed{\text{local max at x=3}}$.

(c) How many zeros does $G(x)$ have?

Solution: $\boxed{\text{3 zeros}}$. $G(x)$ is zero at $x = 1$. The graph of $F(x)$ is concave down with zeros at 0 and 5. So $G(x)$ must also have a root at some value of $x < 0$ and some value of $x > 5$.

(d) Find all critical points of $G(x)$ and classify each as a local min, a local max, or neither.

Solution: $G'(x) = F(x)$, so $G'(x) = 0$ at $x = 0$ and $x = 5$. $G''(0) = f(0) > 0$ and $G''(5) = f(5) < 0$, so $G(x)$ has a $\boxed{\text{local min at x=0}}$ and a $\boxed{\text{local max at x=5}}$.

4. Determine whether, for a differentiable function f, $\frac{d}{dx}\left(\int_0^x f(t)\ dt\right)$ and $\int_0^x \left(\frac{d}{dt} f(t)\right)\ dt$ are always equal, sometimes equal, or never equal. Explain.

Solution: $\boxed{\text{Sometimes equal}}$: when $f(0) = 0$.

5. If $f(x)$ is an even function, and $g(y)$ is defined by $g(y) = \int_0^y f(x)\ dx$, decide whether g is even, odd, or neither. Justify your answer mathematically (not by example!).

Solution: $g(-y) = \int_0^{-y} f(x)\ dx = -\int_{-y}^0 f(x)\ dx = -\int_0^y f(x)\ dx$, because $f(x)$ is even. Thus, $g(-y) = -g(y)$, so $g(y)$ is $\boxed{\text{odd}}$.

6. Evaluate $\lim_{h\to 0} \frac{\int_{x^2}^{(x+h)^2} \sqrt{1+t^2}\ dt}{h}$.

Solution: Could use definition of the derivative or L'hopital's Rule:

$$= \lim_{h\to 0} \sqrt{1 + \left((x+h)^2\right)^2} \cdot (2(x+h)) = 2x\sqrt{1+x^4}$$