Practice with Problem Solving

1. Suppose f is a function such that $f(x+y) = f(x)+f(y)+x^2y+xy^2$ and $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 1$.

   (a) Find $f(0)$.

   (b) Find $f'(0)$.

   (c) Find $f'(x)$.

2. Evaluate $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \ln\left(\sqrt[n]{\left(1+\dfrac{k}{n}\right)^2}\right)$

3. Let $f(x)$ have one zero, at $x = 3$, and suppose $f'(x) < 0$ for all x and that

$$\int_0^3 f(t)\ dt = -\int_3^5 f(t)\ dt$$

   Define $F(x) = \displaystyle\int_0^x f(t)\ dt$ and $G(x) = \displaystyle\int_1^x F(t)\ dt$.

   (a) Find the zeros of $F(x)$.

   (b) Find all critical points of $F(x)$ and classify each as a local min, a local max, or neither.

   (c) How many zeros does $G(x)$ have?

   (d) Find all critical points of $G(x)$ and classify each as a local min, a local max, or neither.

4. Determine whether, for a differentiable function f, $\dfrac{d}{dx}\left(\displaystyle\int_0^x f(t)\ dt\right)$ and $\displaystyle\int_0^x \left(\dfrac{d}{dt}f(t)\right)\ dt$ are always equal, sometimes equal, or never equal. Explain.

5. If $f(x)$ is an even function, and $g(y)$ is defined by $g(y) = \displaystyle\int_0^y f(x)\ dx$, decide whether g is even, odd, or neither. Justify your answer mathematically (not by example!).

6. Evaluate $\lim\limits_{h\to 0} \dfrac{\int_{x^2}^{(x+h)^2} \sqrt{1+t^2}\ dt}{h}$.