8.27.2015. First Part: Relative Growth Rate

We all know that how to use derivative to compute the velocity from distance function $(X(t))$

$$\boxed{X'(t) = \frac{dX(t)}{dt} = V(t)}$$

Same way, we may compute other "speed".

- How fast is the bond/stock price changing?
- How fast is some value changing? (How fast is something changing when something else changing)

We'll see several Example.

Ex1: Air is being pumped into a spherical balloon so that the volume increase at the rate of $100\ cm^3/s$. How fast is the radius of the balloon changing when $r = 50\ cm$?

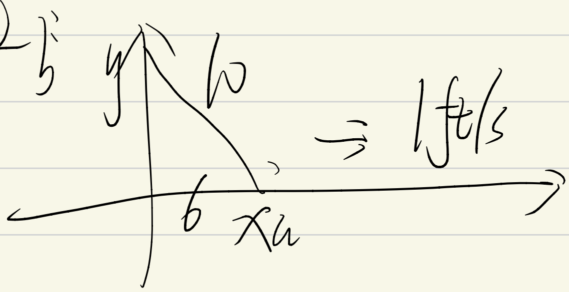
Solution: $\frac{dV}{dt} = 100$ find $\frac{dr}{dt}$

Recall: $V = \frac{4}{3}\pi r^3$ by chain rule

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 400\pi \frac{dr}{dt}$$

$$r = 10 \Rightarrow \frac{dr}{dt} = \frac{1}{400\pi}$$

Ex2:



How fast does the top slide down

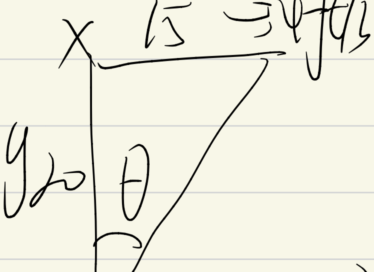
Ans: $x^2+y^2=100$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{3}{4}\ \text{ft/s}$$

(Think: How to do it in physics?)

Ex3: (tree alarms) $\frac{d\theta}{dt}$? (A man walk at $V = 4$ft/s)



$y\tan\theta = x$, $\frac{dx}{dt} = V = 4$ft/s
$y=20$

$$20 \cdot \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{16}{125}\ \text{rad/s} = 0.128\ \text{rad/s}$$

8.27.2019.   L'Hospital Rule (chapter 4.5)   Tao Tang.

Question: How to compute the limit of $\frac{f(x)}{g(x)}$?

$\lim_{x\to 0} \frac{x}{x}$; $\lim_{x\to 0} \frac{x+1}{x+2}$

what about $f(x), g(x) \to 0/\infty$ at the same time?

$\boxed{\frac{0}{0} / \frac{\infty}{\infty} = 1?\ 0?\ \infty?}$

Example:

(1) $\lim_{x\to\infty} \frac{2x}{x^2}$

(2) $\lim_{x\to\infty} \frac{2x}{3x}$

(3) $\lim \frac{x}{3\ln x}$

General How to compute it?

Theorem (L'Hospital Rule)

f, g which are differentiable (at x_0) I an interval ①, and

$\lim_{x\to x_0} f(x) = 0/\pm\infty$; $\lim_{x\to x_0} g(x) = 0/\pm\infty$ ②, and ($g'(x)\neq 0$ all x in $I\setminus x_0$)

$\lim \frac{f'(x)}{g'(x)}$ exsits ③

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Pf: (case when $\lim_{x\to c} f(x) = 0$)

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f(x)-0}{g(x)-0}$$

$$= \lim_{x\to c}\frac{f(x)-f(c)}{g(x)-g(c)}$$

$$= \lim_{x\to c}\frac{\left(\frac{f(x)-f(c)}{x-c}\right)}{\frac{g(x)-g(c)}{x-c}} = \frac{\lim\left(\frac{f(x)-f(c)}{x-c}\right)}{\lim\left(\frac{g(x)-g(c)}{x-c}\right)}$$

$$= \frac{f'(c)}{g'(c)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

□

Examples.

① $\lim\limits_{x\to 1}\dfrac{\ln(x)}{x-1} = \dfrac{(\ln(x))'}{(x-1)'} = \lim \dfrac{\frac{1}{x}}{1} = 1.$

② $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x} = \lim\dfrac{(\sin(x))'}{(x)'} = \dfrac{\cos x}{1} = 1$

③ $\lim\limits_{x\to\infty}\dfrac{e^x}{x^2} = \lim\dfrac{(e^x)'}{(x^2)'} = \lim\dfrac{e^x}{2} = \infty$ (L'Hôpital rule)

(Taylor series)

well 以后讲.

④ $\lim\limits_{x\to 0}\dfrac{x}{e^x} = \dfrac{0}{1} \neq \lim\limits_{x\to 0}\dfrac{1}{e^x} = \dfrac{(x)'}{}$

(why?)

fail.

caveat: $f(x)\to 0,\ g(x)\to 0$ together!

⑤ $\lim\limits_{x\to\infty}\dfrac{x}{e^x} = \lim\dfrac{1}{e^x} = 0$

⑥ $x\cdot\sin\left(\dfrac{\pi}{x}\right) = \lim\limits_{x\to\infty}\dfrac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \lim\dfrac{\cos\left(\frac{\pi}{x}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}}$

$= \lim \pi\cdot\cos\dfrac{\pi}{x} = \pi$

⑦ $\lim_{x\to1}\left(\frac{x}{x-1}-\frac{1}{\ln(x)}\right)$ (do algebra first)

$$=\lim\frac{x\ln(x)-x+1}{(x-1)\ln(x)}$$

$$=\lim\frac{1+\ln(x)-1}{\ln(x)+\frac{x-1}{x}}$$

$$=\lim_{x\to1}\frac{\ln(x)}{\ln(x)+1-\frac{1}{x}}=\lim\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{2}$$

(2)

⑧ $\lim\left(1+\frac{2}{x}\right)^x$

(1) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e\Rightarrow\lim\left(\left(1+\frac{2}{x}\right)^{\frac{x}{2}}\right)^2=e^2.$

(2) take $\ln(y)\Rightarrow x\cdot\ln\left(1+\frac{2}{x}\right)=\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}}=2$

(3) What is the limit ($\frac{f(x)}{g(x)}=\infty/0$?)

$y = \sqrt{x}$, $y = \ln(x)$?

$y = e^x$ $e^{\ln y} = e^{2x+1}$.

How to? ① draw picture

② L'Hospital Rule!

(限 1 time always)

Counter example. (When can't use L'Hospital?)

$f(x) = x + \sin(x)$

$g(x) = x$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x + \sin(x)}{x} = 1 + \lim_{x \to \infty} \left(\frac{1 + \cos(x)}{1} \right)$$

$$= \boxed{\lim_{x \to \infty} (1 + \cos x)}$$

the limit does not exist!!!

(1) Recall the criterion 1, 2, 3, 4,
Check them, then use it.

(2) do algebra, change the form.

A Story: L'Hospital Rule?
Bernoulli's Rule?