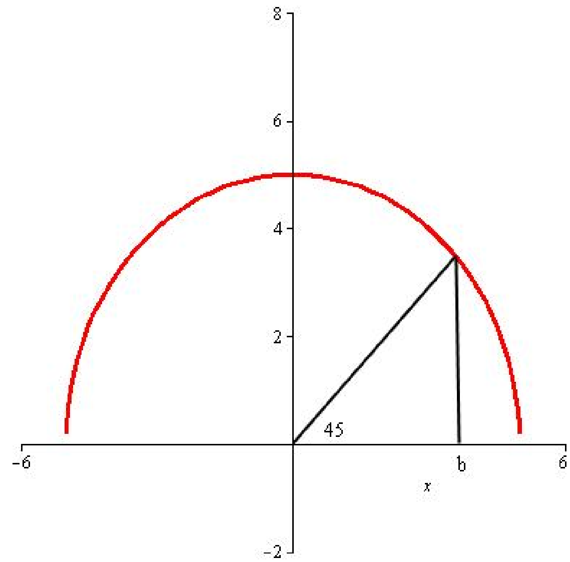
Riemann Sums

Consider the graph of $f(x) = \sqrt{25 - x^2}$ shown below:



1. What is the value of b?

   Answer: $b = 5\cos(45^o) = \boxed{\dfrac{5}{\sqrt{2}}}$

2. Use geometry to find the exact area of the region under $f(x)$ between $x = 0$ and $x = b$.

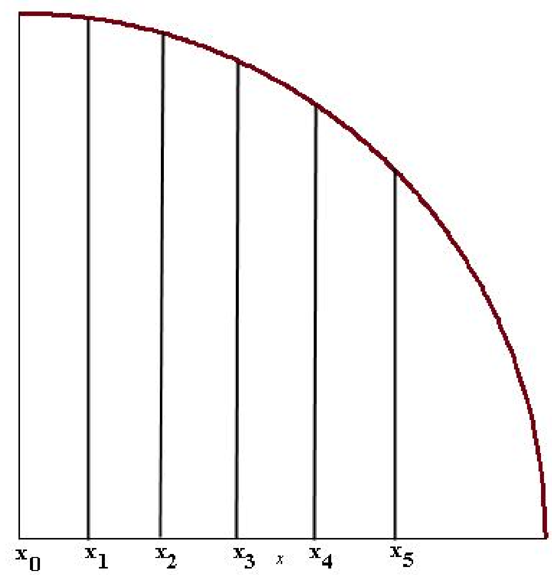
   Answer: Area=Area of (triangle + sector) $= \dfrac{1}{2}\left(\dfrac{5}{\sqrt{2}}\right)^2 + \dfrac{1}{8}\pi(5)^2 = \dfrac{25}{4} + \dfrac{25}{8}\pi \approx \boxed{16.0675}$

Our goal is to approximate this area using vertical rectangles with equal width. Suppose we want to use 5 rectangles.

3. What is the width of each rectangle?

   Answer: $\boxed{\Delta x = \dfrac{1}{\sqrt{2}}}$

4. Now let's let x_0, x_1, x_2, x_3, x_4, and x_5 represent the $x - values$ that pertain to the 6 edges of our rectangles:

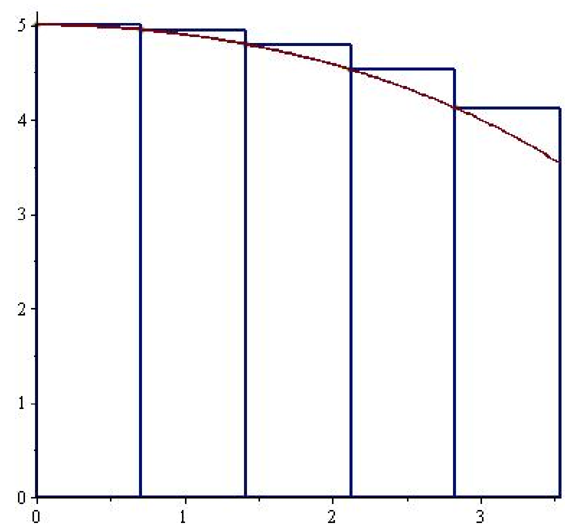


What are the values of x_0, x_1, x_2, x_3, x_4, and x_5?

Answer: $x_0 = \boxed{0}$, $x_1 = \boxed{\frac{1}{\sqrt{2}}}$, $x_2 = \boxed{\frac{2}{\sqrt{2}}}$, $x_3 = \boxed{\frac{3}{\sqrt{2}}}$, $x_4 = \boxed{\frac{4}{\sqrt{2}}}$, and $x_5 = \boxed{\frac{5}{\sqrt{2}}}$

Now we need to choose the height of our rectangles. For each rectangle, let's choose it's height to be where it's left edge intersects with $f(x)$. This is called the **Left Hand Sum** with five rectangles, or as we'll denote it, LHS(5).

5. Draw LHS(5) for this example.
   Answer:

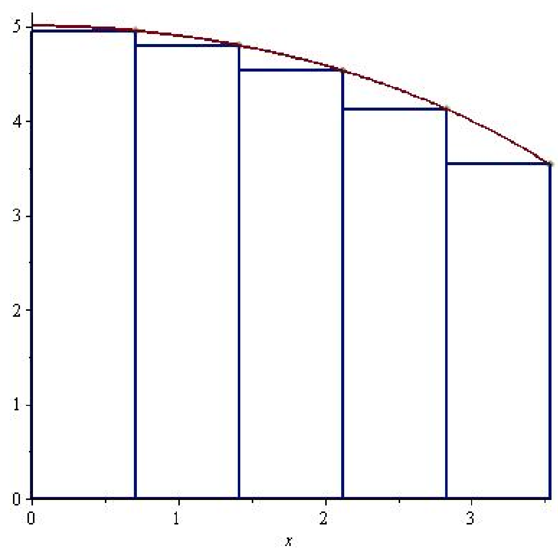


6. To approximate the area under $f(x)$ from $x = 0$ to $x = b$, we can add up the area of these 5 rectangles. Do this now.
   Answer: Area $\approx \frac{1}{\sqrt{2}}\left(f(0) + f\left(\frac{1}{\sqrt{2}}\right) + f\left(\frac{2}{\sqrt{2}}\right) + f\left(\frac{3}{\sqrt{2}}\right) + f\left(\frac{4}{\sqrt{2}}\right)\right)$

$$\approx \frac{1}{\sqrt{2}}\left(\sqrt{25}+\sqrt{\frac{49}{4}}+\sqrt{\frac{46}{4}}+\sqrt{\frac{41}{4}}+\sqrt{\frac{34}{4}}\right)$$
$$\approx \boxed{16.5437}$$

Suppose, instead, for each rectangle we choose it's height to be where it's *right* edge intersects with $f(x)$. This is called the **Right hand Sum** with five rectangles, or as we'll denote it, RHS(5).

7. Draw RHS(5) for this example.
   Answer:



8. To approximate the area under $f(x)$ from $x = 0$ to $x = b$, we can add up the area of these 5 rectangles. Do this now.
   Answer: Area $\approx \frac{1}{\sqrt{2}}\left(f\left(\frac{1}{\sqrt{2}}\right)+f\left(\frac{2}{\sqrt{2}}\right)+f\left(\frac{3}{\sqrt{2}}\right)+f\left(\frac{4}{\sqrt{2}}\right)+f\left(\frac{5}{\sqrt{2}}\right)\right)$
$$\approx \frac{1}{\sqrt{2}}\left(\sqrt{\frac{49}{4}}+\sqrt{\frac{46}{4}}+\sqrt{\frac{41}{4}}+\sqrt{\frac{34}{4}}+\sqrt{\frac{25}{4}}\right)$$
$$\approx \boxed{15.508}$$

Sigma Notation

- Sigma notation is used to concisely write a sum: $\displaystyle\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$
  - i is the index, and a_i is the ith term
  - $i = 1$ indicates that we start with that value
  - n indicates the last value of i

Examples:

1. $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$

2. $\sum_{k=1}^{4} k^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$

3. $\sum_{m=1}^{4} 3^m = 3^1 + 3^2 + 3^3 + 3^4 = 120$

4. $\sum_{n=6}^{8} n = 6 + 7 + 8 = 21$

5. $\sum_{r=1}^{4} (-1)^r = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = 0$

- This notation is especially useful when we are writing a long sum, such as
  $\sum_{i=1}^{100} i = 1 + 2 + \cdots + 100,$
  or for sums of variable length, such as
  $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$

Properties of $\sum$-Notation

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

   - Ex: $$\begin{aligned}\sum_{i=1}^{n} 2i &= 2(1) + 2(2) + \cdots + 2(n) \\ &= 2(1 + 2 + \cdots + n) = 2\sum_{i=1}^{n} i\end{aligned}$$

2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

   - Ex: $$\begin{aligned}\sum_{i=1}^{n} \left(i + i^2\right) &= \left(1 + 1^2\right) + \left(2 + 2^2\right) + \cdots + \left(n + n^2\right) \\ &= (1 + 2 + \cdots + n) + \left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \sum_{i=1}^{n} i + \sum_{i=1}^{n} i^2\end{aligned}$$

- There is more than one way to represent a sum in Σ-notation

- Write $1 + 2 + \cdots + 10$ in Σ-notation

  $$\sum_{i=1}^{10} i \qquad \sum_{i=0}^{9} (i+1) \qquad \sum_{i=2}^{11} (i-1)$$

  are just a few possibilities

Notation:

- $\displaystyle\sum_{i=1}^{10} i = \sum_{k=1}^{10} k = 1 + 2 + \cdots + 10$
- What is $\displaystyle\sum_{i=1}^{10} k$? $k + k + \cdots + k = 10k$
- Be careful and consistent with notation

General Notation for Riemann Sums:

Suppose you want to approximate the area under the curve $f(x)$ from $x = a$ and $x = b$ using n rectangles. Then the width of the rectangles is $\Delta x = \dfrac{b-a}{n}$. Let x_k represent the x-value at the right edge of the kth rectangle. Then $x_k = a + k\Delta x$.

- $LHS(n) = \left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right)\Delta x = \displaystyle\sum_{k=0}^{n-1} f(x_k)\Delta x$
- $RHS(n) = \left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\Delta x = \displaystyle\sum_{k=1}^{n} f(x_k)\Delta x$