Math 122L - Brief Review of Prerequisites

1. If

   (a) $f(x)$ is an even function, then $f(-x) = \boxed{f(x)}$

   (b) $f(x)$ is an odd function, then $f(-x) = \boxed{-f(x)}$

2. What is the **definition** of the derivative of f at x?

   Solution: $\boxed{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$

3. Use the **definition** of the derivative to to derive $f'(x)$ for $f(x) = \frac{1}{\sqrt{1+x}}$.

   Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h} \cdot \frac{\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}}{\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}} \\
&= \lim_{h\to 0} \frac{\frac{1}{1+x+h} - \frac{1}{1+x}}{h\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)} = \lim_{h\to 0} \frac{\frac{-h}{(1+x+h)(1+x)}}{h\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)} \\
&= \frac{\frac{-1}{(1+x)^2}}{\frac{2}{\sqrt{1+x}}} = \boxed{\frac{-1}{2\,(1+x)^{3/2}}}
\end{aligned}$$

4. Find the derivative of each of the following functions (you need not use the defintion here):

   (a) $f(x) = x\sin(e^x)$

   Solution: $f'(x) = \boxed{(1)\sin(e^x) + x\Big(\cos(e^x)e^x\Big)}$

   (b) $f(x) = \frac{2^x}{1+x^3}$

   Solution: $f'(x) = \boxed{\frac{2^x\ln(2)(1+x^3) - 2^x(3x^2)}{(1+x^3)^2}}$

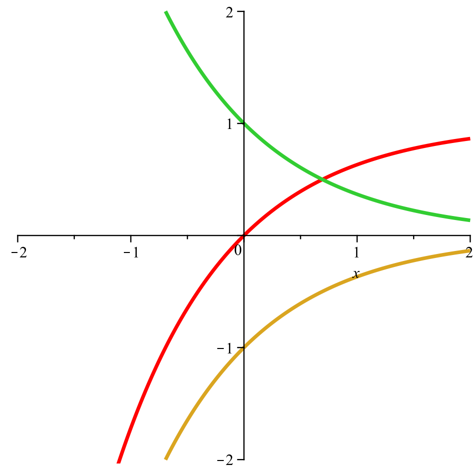
   (c) $f(x) = \arctan(x\cos(x))$

   Solution: $f'(x) = \boxed{\frac{1}{1+\Big(x\cos(x)\Big)^2} \cdot \Big((1)\cos(x) + x(-\sin(x))\Big)}$

   (d) $f(x) = e^{\arcsin(x^2)}$

   Solution: $f'(x) = \boxed{e^{\arcsin(x^2)} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x}$

5. On the graph below, identify which graph is f, f' and f''. Explain how you know.



Solution: Labeling from top to bottom on the left-hand side of the graph:
$f'(x)$
$f(x)$
$f''(x)$

6. Find the line tangent to the function $f(x) = xe^{kx}$ at $x = 0$. Assuming that $k > 0$, does the linear approximation underestimate or overestimate xe^{kx} near 0? Explain your answer carefully.
Solution: $f'(0) = 1$ which means the tangent line has slope 1. The tangent line must pass through $(0, f(0)) = (0, 0)$. Thus, the tangent line is $\boxed{T(x) = x}$.
$f''(0) = 2k > 0$, meaning that the graph of $f(x)$ is concave up. Thus, $T(x)$ $\boxed{\textit{underestimates}}$ $f(x)$ near 0.

7. The table below gives the values of the functions $f(x)$ and $g(x)$ at specified values of x.

x	1	2	3	4	5
f(x)	0	3	6	8	2
g(x)	1	4	5	2	0

(a) Using the table, estimate the value of the derivative of $f(g(x))$ at $x = 2$.
Solution: $f'(g(2))g'(2) = f'(4)g'(2) \approx \left(\frac{f(5)-f(3)}{2}\right)\left(\frac{g(3)-g(1)}{2}\right) = (-2)(2) = \boxed{-4}$

(b) Using the table, estimate the value of the derivative of $g(f(x))$ at $x = 2$.
Solution: $g'(f(2))f'(2) = g'(3)f'(2) \approx \left(\frac{g(4)-g(2)}{2}\right)\left(\frac{f(3)-f(1)}{2}\right) = (-1)(3) = \boxed{-3}$

8. Suppose $P(t)$ is the monthly payment, in dollars, on a mortgage which will take t years to pay off. What are the units of $P'(t)$? Is $P'(t)$ positive or negative? Explain.
Solution:

- units: $\boxed{\$ \textit{ per year}}$
- sign: $P'(t)$ is $\boxed{\textit{negative}}$ because as you increase the number of years needed to pay off the mortgage, the monthly payment decreases.

9. Let

$$f(x) = \begin{cases} c^x + x & \text{if } x < 1 \\ x^c + 2 & \text{if } x = \geq 1 \end{cases}$$

Answer the following without using a graphing calculator.

(a) Define what it means for a function, $g(x)$, to be continuous at the point $x = a$.
<u>**Solution:**</u> $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$

(b) What value(s) of c make $f(x)$ continuous? Explain.
<u>**Solution:**</u> $\lim_{x\to 1^-} f(x) = c+1$ and $\lim_{x\to 1^+} f(x) = 3$. In order for $f(x)$ to be continuous at $x = 1$, we need $\lim_{x\to 1^{-1}} f(x) = \lim_{x\to 1^{+1}} f(x) \implies \boxed{c = 2}$ Note that for this value of c, $f(x)$ is also continuous for $x \neq 1$.

(c) Define what it means for a function, $g(x)$, to be differentiable at the point $x = a$.
<u>**Solution:**</u> $\lim_{h\to 0^-} \dfrac{g(a+h) - g(a)}{h} = g(a) = \lim_{x\to 0^+} \dfrac{g(a+h) - g(a)}{h}$, where $g'(a)$ is finite.

(d) For this value(s) of c, is $f(x)$ differentiable? Explain.
<u>**Solution:**</u> $\lim_{h\to 0^-} \dfrac{f(x+h) - f(x)}{h} = \dfrac{d}{dx}(2^x + x) = 2^x \ln(2) + 1$ and
$\lim_{h\to 0^+} \dfrac{f(x+h) - f(x)}{h} = \dfrac{d}{dx}\left(x^2 + 2\right) = 2x$
Because these are not equal, $\boxed{\text{f(x)is not differentiable.}}$

10. The position of a particle (in centimeters) at time t (in seconds) is $s(t) = \dfrac{1}{3}t^3 - 5t^2 + 24t$.

(a) When is the particle at rest?
<u>**Solution:**</u> $v(t) = s'(t) = t^2 - 10t + 24 = (t-4)(t-6) \implies v(t) = 0$ at $t = 4$ and $t = 6$ seconds.

(b) When is the particle moving to the right?
<u>**Solution:**</u> $v(t) > 0 \iff (t-4)(t-6) > 0 \iff 0 \leq t < 4$ and $6 < t$.

(c) When is the particle speeding up?
<u>**Solution:**</u> $a(t) = v'(t) = 2t - 10 \implies a(t) > 0$ when $t > 5$. So $s(t)$ is concave down on $0 \leq t \leq 5$ and concave up on $t > 5$. The particle is speeding up (to the left) on $4 < t < 5$ and to the right on $t > 6$.

(d) Find the total distance traveled by the particle over the interval $0 \leq t \leq 10$.
<u>**Solution:**</u> Here we need to account for the distance the particle travels to the left on $4 < t < 6$. The total distance traveled is
$s(4) + |s(6) - s(4)| + (s(10) - s(6)) = \dfrac{112}{3} + \dfrac{4}{3} + \dfrac{112}{3} = 76.$

11. Find the following limits, or state that they do not exist, noting that a, b, c, and d are constants greater than 1. Make sure to justify your answers (not with a calculator).

(a) $\lim_{x\to\infty} \dfrac{4a^{-x} + 2b}{3c + d^{-2x}} = \boxed{\dfrac{2b}{3c}}$

(b) $\lim_{x\to c^-} \dfrac{|x-c|}{2x - 2c} = \boxed{\dfrac{-1}{2}}$

(c) $\lim_{x \to -a} \frac{x^2 - a^2}{(x)(x+a)} = \lim_{x \to -a} \frac{(x-a)(x+a)}{(x)(x+a)} = \lim_{x \to -a} \frac{x-a}{(x)} = \boxed{2}$