Summer Workshop in Mathematics

Application Problems (SWiM 2023)

Department of Mathematics, Duke University

Workshop Dates: June 10 - June 17

1 Instructions

Welcome SWiM 2023 applicant!

The problems below are the SWiM 2023 application problems. Solve as many problems as possible. These problems are meant to be challenging. They may take you days to solve them. If you can only solve a portion of the problems, you are still welcome to apply. Please write not only answers, but also proofs and partial solutions, partial results or ideas if you cannot completely solve the problem. (You will not receive credit for your answer if there is no work explaining why you have arrived at that answer.)

You can either type the solutions or write them up by hand and then scan them. We have included a problem from last year along with a sample solution for your reference. Please upload your solution as a PDF file attachment with the application and remember to upload your Cover Letter stating why you are interested in SWiM and what aspects of mathematics are fascinating to you. Moreover, you must include and sign the following pledge:

I pledge on my honor that I have solved the problems on my own without help of others, books, calculators, the internet, or other sources.

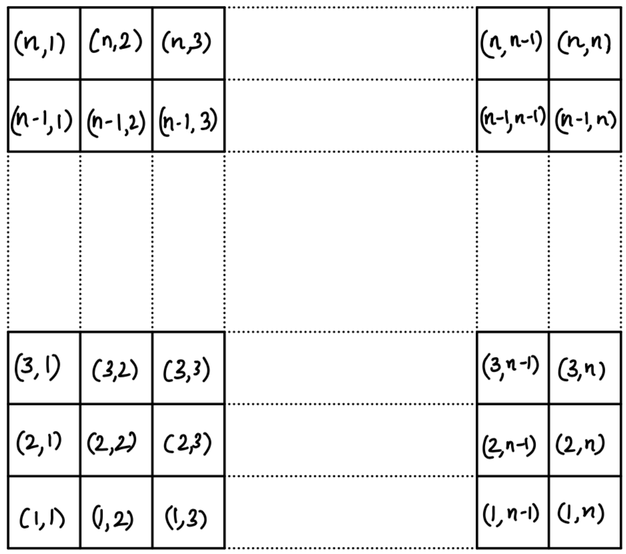
WARNING: Posting these problems on problem-solving websites is strictly forbidden. Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Good luck!

2 Application Problems

1. A bag contains n red marbles and one blue marble. Sophie and Ada draw marbles from this bag without replacement, alternating after each other. They do this till the blue marble is drawn and the player with the blue marble wins the game. Sophie gives Ada the choice whether she wants to start or not. Ada has a hunch that she might be better off if she starts; after all, she might succeed in the first draw. On the other hand, if her first draw yields a red marble, then Sophies's chances to draw the blue marble in her first draw are increased, because then one red marble is already removed from the bag. How should Ada decide in order to maximize her probability of winning?

2. Suppose there are n^2 unit squares as drawn below. Each square is indexed by two coordinates (r, c) where the left coordinate r is the row number and the right coordinate c is the column number. We need to colour each unit unit square in such a way that whenever $1 \leq p < q \leq n$ and $1 \leq r < s \leq n$, the three squares at $(p, r), (q, r)$ and (q, s) have different colours. What is the minimum number of colours required to colour the unit squares when there are n^2 squares? Your answer will be an expression in terms of n. Give a colouring for the case when $n = 3$ with minimum number of colours.



3. Let x, y, z, w be positive numbers with $x < y \leq z < w$ such that $xw = yz$ and $\sqrt{w} - \sqrt{x} \leq 1$. Show that x can be written as n^2 for some positive n.

3 Sample Question

The following is an example problem. It is of a similar difficulty as the others, and uses (as the others do) only high-school level mathematics to solve. We provide two examples of solutions, the first here and the second in the accompanying file. One solution is textbook-like, and the other more rambling. Both would receive full credit.

Consider polynomials $p(x)$ of degree two, with integer coefficients, and for which the x^2 term has coefficient 1. Find all such $p(x)$ for which there exists a polynomial $q(x)$ (of any degree) with integer coefficients such that $p(x)q(x)$ is a polynomial having all coefficients ± 1.

Solution: First note that $p(x)$ must be of the form $p(x) = x^2 + cx \pm 1$ where c is an integer.
For $c = \pm 1$, polynomial $p(x)$ has the required property by taking $q(x) = 1$.
For $c = 0$, polynomial $p(x)$ has the required property by taking $q(x) = x + 1$.

Suppose now that $|c| \geq 2$. Then $p(x)$ has 2 real roots, say x_1 and x_2 which are also roots of $p(x)q(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ with $c_i = \pm 1$. Thus

$$1 = \left| \frac{c_{n-1}}{x_i} + \cdots + \frac{c_0}{x_i^n} \right| \leq \frac{1}{|x_i|} + \cdots + \frac{1}{|x_i|^n} < \frac{1}{|x_i| - 1},$$

which implies $|x_1|, |x_2| < 2$. This rules out the cases $|c| \geq 3$ and the polynomials $p(x) = x^2 \pm 2x - 1$.
The remaining two polynomials $p(x) = x^2 \pm 2x + 1$ satisfy the condition for $q(x) = x \mp 1$.
Therefore, the 8 polynomials $p(x)$ with the desired property are

$$p(x) = x^2 \pm x \pm 1$$

$$p(x) = x^2 \pm 1$$

$$p(x) = x^2 \pm 2x + 1.$$

A more rambling solution to the example problem.

- The polynomials $p(x)$ must be of the form x^2+mx+n, where m and n are integers.
- A corresponding $q(x)$ of degree k,

$$q(x) = c_0 x^k + c_1 x^{k-1} + \dots + c_k \quad (\text{all } c_k \text{ integer})$$

must satisfy that $\underline{p(x)\,q(x)}$ has coeffs. ± 1, i.e. in

$$c_0 x^{k+2} + (c_0 m + c_1) x^{k+1} + (c_0 n + c_1 m + c_2) x^k + \dots$$

$$+ \dots + (c_{k-\ell} n + c_{k-\ell+1} m + c_{k-\ell+2}) x^\ell + \dots$$

$$\dots + (c_{k-2} n + c_{k-1} m + c_k) x^2 + (c_{k-1} n + c_k m) x + c_k n.$$

all coefficients should be ± 1.

This means that both c_0 and $c_k n$ should be ± 1; together with the requirement that n and c_k must be integer, this means $c_0 = \pm 1$, $c_k = \pm 1$ and $n = \pm 1$.

- If we have a solution for p and q, then p and $(-q)$ will also be OK. We can use this freedom of choice to make c_0 always positive. So we have $c_0 = 1$.
- Now we get that $m + c_1$ (the coeff of x^{k-1} in the product pq) must equal ± 1.

  Similarly $mc_k + nc_{k-1}$ (the coeff of x in pq) must equal ± 1, which means (after multiplying with c_k) that $m + nc_{k-1}c_k$ must equal ± 1.

  How attack this?
- In any case we have shown that $n = (-1)^j$ with $j = 0$ or 1.

Let's check whether we can find simple examples.

- $m = 0$ $\qquad p(x) = x^2 + (-1)^j$

  $\Rightarrow$ <u>OK</u>: $\quad (x^2 + (-1)^j)(x+1)$

  $$= x^3 + x^2 + (-1)^j x + (-1)^j$$

  $\Rightarrow q(x) = x+1$ has the desired property.

- $m = (-1)^\ell$ $\qquad p(x) = x^2 + (-1)^\ell x + (-1)^j$

  $q(x) = 1$ has the desired property!

- $m = 2(-1)^\ell$, $n = 1$ $\qquad p(x) = x^2 + 2(-1)^\ell x + 1$

  $$= (x + (-1)^\ell)^2$$

  $\Rightarrow$ set $q(x) = x - (-1)^\ell$

  Then $p(x)\,q(x) = (x + (-1)^\ell)(x^2 - 1)$

  $$= x^3 + (-1)^\ell x^2 - x - (-1)^\ell$$

  $\Rightarrow$ OK.

- What for $p(x) = x^2 + 2x - 1$?

  $\hookrightarrow$ this has roots $-1 \pm \sqrt{2}$

  $$p(x)\,q(x) = x^{K+2} + a_1 x^{K+1} + a_2 x^K + \dots + a_{K+1} x + (-1)^j$$

  must have those two roots as well.

  $$\Rightarrow (-1-\sqrt{2})^{K+2} = -a_1(-1-\sqrt{2})^{K+1} - a_2(-1-\sqrt{2})^K \dots - a_{K+1}(-1-\sqrt{2}) + (-1)^j$$

  now the right hand side satisfies

  $$|RHS| \le \sum_{\ell=0}^{K+1} |1+\sqrt{2}|^\ell \qquad (\text{since all } |a_j| \le 1)$$

  $$= \frac{(1+\sqrt{2})^{K+2} - 1}{\sqrt{2}} < (1+\sqrt{2})^{K+2} = |LHS|$$

$\Rightarrow$ impossible !

$\Rightarrow$ $x^2 + 2x - 1$ is not an acceptable $p(x)$.

Same argument for $x^2 - 2x - 1$

- Maybe we can make this more general?

$x^2 + mx + (-1)^\ell$ with $|m| \geq 3$

roots : $\dfrac{-m \pm \sqrt{m^2 - 4(-1)^\ell}}{2}$. Set $X = -\left(\dfrac{m + \sqrt{m^2 - 4(-1)^\ell}}{2}\right)$

set X to be the root with largest absolute value, $|X| = \dfrac{|m| + \sqrt{m^2 - 4(-1)^\ell}}{2}$.

If there is a valid candidate $q(x)$, then X must be a root of $p(x)q(x)$

$$\Rightarrow \quad X^{K+2} + a_1 X^{K+1} + a_2 X^K + \dots + a_{K+1} X + (-1)^j = 0$$

with $|a_1| = |a_2| = \dots = |a_{K+1}| = 1$.

$$\Rightarrow |X|^{K+2} \leq \sum_{\ell=0}^{K+1} |X|^\ell = \frac{|X|^{K+2} - 1}{|X| - 1}$$

this would be impossible if $|X| - 1 > 1$, or $|X| > 2$.

We have $|X| = \dfrac{|m|}{2} + \dfrac{1}{2}\sqrt{m^2 - 4(-1)^\ell} \geq \dfrac{|m|}{2} + \dfrac{1}{2}\sqrt{m^2 - 4}$.

For $|m| \geq 4$, $\dfrac{|m|}{2} \geq 2$, so $|X| > 2$.

For $|m| = 3$, $|X| \geq \dfrac{3}{2} + \dfrac{1}{2}\sqrt{5} > \dfrac{3}{2} + 1 > 2$.

$\Rightarrow$ all these cases are excluded.

In the end, these are the only possible $p(x)$:

$$x^2 + (-1)^j \qquad j = 0 \text{ or } 1$$

$$x^2 + (-1)^l x + (-1)^j \qquad j, l \in \{0, 1\},$$

$$x^2 + 2(-1)^l x + 1 \qquad l = 0 \text{ or } 1.$$