1 The Bridges of Konigsberg

- The city of Konigsberg consisted of two sides of the Pregel River and two large islands, all connected to each other by seven bridges. Is it possible to walk across each of the bridges once and only once?

- Euler noticed that the shapes of the islands and the river do not change the answer to this question.

- The only important information from the picture is connectivity. This means that we should be able to represent the bridges and land masses as merely a set of points and lines.

- Draw a set of points and lines that convey the important information from the map of Konigsberg. In other words, we want to represent the information on the map as a set of objects and connections between them.
What you have just drawn is called a graph, or a finite set of vertices (the points/objects) and edges (the lines/connections).

The number of edges connected to a vertex $v$ is called the degree of the vertex, denoted $\deg(v)$.

- Example: What is $\deg(v)$ in the following graph? What is $\deg(w)$?

- For which values of $n$ can we draw a graph on $n$ vertices where each vertex has degree 3?
  - Can you draw a graph on four vertices where each vertex has degree 3?
  - Can you draw a graph on five vertices where each vertex has degree 3?

- **Degree-Sum Theorem** (or The Handshaking Theorem): Let $G$ be a graph with $n$ vertices $\{v_1, v_2, \ldots, v_n\}$ and $m$ edges. Then

  $$\sum_{i=1}^{n} \deg(v_i) = 2m.$$

  - Why does this make sense?

  - Can a graph have an odd number of vertices of odd degree?
– So, to answer our question from earlier, can you draw a graph on five vertices where each vertex has degree 3?

• We say a vertex \( v \) is **adjacent** to a vertex \( w \) if they are connected by an edge.
  – Example: Is \( v \) adjacent to \( w \)? Is \( v \) adjacent to \( x \)?

![Graph Example]

• A set of distinct edges \( e_1e_2\cdots e_k \) is called a **walk** if we can trace these edges in the graph without picking up our pencil.
  – In other words, the endpoint of edge \( e_i \) is the starting point of edge \( e_{i+1} \).
  – Example: Draw a walk from \( v \) to \( w \).

![Walk Example]

• A **closed walk** is a walk \( e_1e_2\cdots e_k \) that starts and ends at the same vertex.
• If a walk uses all edges of a graph \( G \), it is called an **Eulerian walk**.
  – Example: Is there a closed Eulerian walk in the graph below? If so, draw one.
• If a walk does not go through any vertex twice, we call it a **path**.
  
  – Example: Is there a closed path in the graph below? If so, draw one.

![Graph](image)

• Suppose we want to see if there is a way to walk across each of the bridges once and only once, and end up where we started. Interpret this problem (let’s call it Problem 1) in terms of our new definitions. **Problem 1**:

A graph $G$ is **connected** if for any two vertices $x$ and $y$ of $G$, we can find a path from $x$ to $y$ along edges of $G$.

  – Draw two graphs: one that is connected and one that is not.

• **Theorem 1**: Let $G$ be a connected graph. Then $G$ has a closed Eulerian walk if and only if all vertices of $G$ have even degree.

  – Theorem 1 is an “if and only if” statement. This means that we can interpret the theorem as two separate statements:
    1. If all vertices of a connected graph $G$ have even degree, then $G$ has a closed Eulerian walk.
    2. If a connected graph $G$ has a closed Eulerian walk, then all vertices of $G$ have even degree.

  – So, if one side of the statement holds, the other must hold.
  – It also means that if one side of the statement does not hold, then the other side cannot hold.
• What does this theorem tell us about the answer to Problem 1?

• Suppose we remove the requirement to start and end at the same place. Interpret this problem (problem 2) in terms of our definitions.

  **Problem 2:**

• We can use the following theorem to help us determine if our problem has a solution:

  **Theorem 2:** Let $G$ be a connected graph. Then $G$ has an Eulerian walk starting at vertex $x$ and ending at vertex $y$ if and only if $\deg(x)$ and $\deg(y)$ are odd, and all other vertices of $G$ have even degree.

• What does this tell us about the answer to problem 2?
2 Vertex colorings

2.1 Coloring maps

- Suppose we are given a map, and we want to know the minimal number of colors required to color the map so that no two bordering countries are the same color.

- Let’s look at the map given below. How can we represent this as a graph?

- Coloring countries on our map is now the same as coloring vertices on our graph. How can we translate the requirement “no two bordering countries are the same color” as a requirement on the coloring of the vertices?

2.2 Colorings and the chromatic number

- A coloring of a graph $G$ is an assignment of a color to each vertex so that no two adjacent vertices are the same color.

- A $k$-coloring of $G$ is a coloring that uses exactly $k$ colors.

  - Example: Determine whether the following are colorings of the graphs.
– Example: Give an example of a 3-coloring of the following graph.

– Example: Is there a 2-coloring of the following graph? If so, give an example. If not, explain why.

– What is the smallest number $k$ such that there exists a $k$-coloring of the graph above?

• The chromatic number of a graph $G$ is the smallest number $k$ such that there exists a $k$-coloring of $G$. We write this as $\chi(G)$.

– Example: Find $\chi(G)$ for the following graph.

• Interpret our map coloring problem in terms of the definitions we learned today:
• It turns out that every map can be colored with four or fewer colors:
  
  – **The Four Color Theorem**: Every map can be colored with four or fewer colors.
  – In other words, \( \chi(G) \leq 4 \) for any graph \( G \) that comes from a map (these are called planar graphs, and some of you will study these for your project).
  – Kenneth Appel and Wolfgang Haken proved this theorem in 1976, but much of their proof was done by checking thousands of special cases on a computer. This proof is very unsatisfying, and mathematicians still want to see a more elegant proof!

2.3 Bipartite graphs

• A graph is called **bipartite** if it has a 2-coloring.
  
  – Example: Is the following graph bipartite? Why or why not?

![Graph Example](image1)

– Example: Are the following graphs bipartite?

![Graph Example](image2)
• A **cycle** in a graph is a path that starts and ends at the same vertex.

• Suppose we have a cycle in a graph $G$ of **length** $n$, meaning there are $n$ edges in the cycle.
  
  – What is the relationship between the number of vertices and the number of edges in a cycle?

  – A cycle can have even or odd length. What do you think is true about the relationship between the length of a cycle and whether or not the cycle is bipartite? (Think of the examples we did above.)

  – **Theorem:**

• This means that the following theorem is true:

• **Theorem:** A graph $G$ is bipartite if and only if it does not contain a cycle of odd length.

• Example: Suppose Alicia, Bob, Chelsea, Diego, Edward, and Fred were hired for an internship. The company wants to split them into two teams for a team-building activity so that no person knows any other person on their team. Suppose Alice knows Bob, Chelsea, and Diego. Edward knows Bob and Chelsea. Fred knows Alice and Diego. Is this possible? Prove your answer. The number of people on each team does not matter.
2.4 Complete graphs

- A graph $G$ is complete if every pair of vertices of $G$ is adjacent (connected by an edge).

- We denote the complete graph on $n$ vertices by $K_n$.
  - Example: Draw $K_1$ through $K_5$. Which ones can you draw so that no edges intersect, except at vertices?

- Example: Is $K_3$ bipartite?

- Example: Is $K_4$ 3-colorable?

- Is $K_n$ $(n - 1)$-colorable?

- Is $K_n$ $n$-colorable? Why or why not?

- Can a graph $G$ be $(n - 1)$-colorable if it contains a copy of $K_n$?

- Example: Is the following graph 3-colorable? Prove your answer.
3 The chromatic polynomial

- Suppose you’re hosting a party, and you’ve decided you want to place your guests at tables so that no person knows any other person at the table. How many ways are there to separate the group into four different tables? Rephrase this problem in terms of graph theory.

- What if we want to know how many ways there are to separate the group into five, six, or seven tables? Instead of computing these values separately, we would like to instead come up with a function that tells us the answer for any number of tables.

- The **chromatic polynomial** $P_k(G)$ for a graph $G$ gives a formula for the number of possible $k$-colorings of a graph (assume we’ve already chosen the $k$ colors).

- This is a polynomial with variables in $k$, i.e., something like

  $$P_k(G) = k^5 + 4k^3 + 2k + 7,$$

  and if you plug in any positive integer value for $k$, say, 4, it will tell you the number of 4-colorings of the graph $G$.

- Let $G$ be a path (not closed) of length six. How many different 4-colorings of $G$ are there? (Let’s say the colors are red, green, blue, and yellow.)

- How many 7-colorings of $G$ are there?

- How many $k$-colorings of $G$ are there? (Your answer is $P_k(G)$, the chromatic polynomial of $G$.)
• What is $P_k(G)$, where $G$ is a path of length $n$?

• Use your answer above to give the number of 10-colorings of a path of length 8.

• What is $P_k(K_5)$?

• What is $P_k(K_n)$?

• Find the chromatic polynomial for a cycle of length four (caution: you will have to consider two cases!).
3.1 Deletions and contractions

- Pick an edge $e$ in your graph $G$. Then $G - e$ is the graph $G$ with edge $e$ removed.

- Example: For the following graph $G$ with labeled edge $e$, draw $G - e$.

![Graph](image)

- $G \setminus e$ is the graph $G$ with the edge $e$ contracted; i.e., we shrink edge $e$ down to a single vertex.

- Example: For the following graph $G$ with labeled edge $e$, draw $G \setminus e$.

![Graph](image)

- Sometimes the following theorem is helpful for computing the chromatic polynomial:

- **Theorem**: $P_k(G) = P_k(G - e) - P_k(G \setminus e)$.

- Example: Find the chromatic polynomial for the following graph using the theorem above:

![Graph](image)
- Example: First find the chromatic polynomial directly (without using the theorem above), and then find the chromatic polynomial by using the theorem above, with the edge $e$ selected. Show that your answers are the same.